Graduate Algebra, Fall 2014
Lecture 34

Andrei Jorza

2014-11-19

3 Modules

3.3 Noetherian rings and modules (continued)

Theorem 1 (Hilbert basis theorem)**.** *If R is a Noetherian ring then $R[X]$ is a Noetherian ring. Inductively, $R[X_1, \ldots, X_n]$ is a Noetherian ring for all $n \geq 0$.*

Proof. Let I be an ideal of $R[X]$. We want to show that I is finitely generated as an $R[X]$-module but we'll do something better: we'll show that it is finitely generated as an R-module, i.e., there exist $M_i \in R[X]$ such that $I = RM_1 + \cdots + RM_n$ in which case $R[X]M_1 + \cdots + R[X]M_n = R[X]I = I$. We'll do this as follows: we will find a finitely generated R-module J containing I. Since R is a Noetherian ring and J is finitely generated it follows that J is Noetherian. But then every R-submodule of J (including I) is finitely generated over R and the conclusion follows.

For a polynomial $P(X) = a_0 + a_1X + \cdots + a_nX^n$ let $i(P) = a_n$. Let $i(I) = \{i(P)|P \in I\}$. Then $i(I)$ is an ideal. For $r \in R$ have $ri(P) = i(rP)$. If $\deg P \leq \deg Q$ then $i(P) + i(Q) = i(PX^{\deg Q - \deg P} + Q)$ and so $i(I)$ is an ideal of R. R is a Noetherian ring so $i(I) = (a_1, \ldots, a_n)$ is finitely generated and let $P_i \in R[X]$ such that $i(P_i) = a_i$. Let $r = \max \deg P_i$.

We choose the R-module J generated by $P_1, \ldots, P_n, 1, X, \ldots, X^{r-1}$ so $J = \sum RP_i + \sum_{i=0}^{r-1} RX^i$. Let's show that $I \subset J$.

Suppose $P(X) \in I$ of the form $P(X) = b_0 + b_1X + \cdots + b_mX^m$. We'll show that $P(X) \in J$ by induction on m. If $m < r$ the already $P \in J$. Suppose now that $m \geq r \geq \deg P_i$ for all i. Write $b_m = \sum u_ia_i$. Then $P(X) - \sum u_iP_i(X)X^{m-\deg P_i} \in I$ has degree $\leq m - 1$. By induction we deduce that $P(X) - \sum u_iP_i(X)X^{m-\deg P_i} \in J$ but then $P(X) \in J$ as desired.

Thus $I \subset J$ as an R-module and the argument from the beginning of the proof yields that I is finitely generated as an R and thus also as an $R[X]$-module. We deduce that $R[X]$ is a Noetherian ring. ☐

3.4 Modules over PIDs

In this section we'll prove the following theorem.

Theorem 2. *If R is a PID and M is a finitely generated module over R then there exists $r \geq 0$ and $x_1, \ldots, x_n \in R$ such that*

$$M \cong R^r \oplus R/(x_1) \oplus \cdots \oplus R/(x_n)$$

where $x_1 \mid \ldots \mid x_n$.

Corollary 3. *Specializing to $R = \mathbb{Z}$ and knowing that abelian groups are $\mathbb{Z}$-modules we get the classification of finitely generated abelian groups.*

3.4.1 Free modules

Definition 4. Suppose R is an integral domain and M is an R-module. The rank $\operatorname{rank}_R(M)$ of M is the largest number of R-linearly independent elements of M.

Lemma 5. *Any $n+1$ elements of R^n are linearly dependent and so $\operatorname{rank}_R(R^n) = R$.*

Proof. Pick $m_1, \ldots, m_{n+1} \in R^n \subset (\operatorname{Frac} R)^n$ which is a vector space. Thus v_i are $\operatorname{Frac} R$-linearly dependent and, clearing denominators, they are R-linearly dependent. □

Proposition 6. *If R is a PID and M is a submodule of the free module N of rank n. Then M is free of rank $m \leq n$.*

Proof. By induction on $\operatorname{rank}_R(N)$. For rank 1, every submodule of R is a necessarily principal ideal so it is free of rank 1 or 0. Suppose we know it for rank n and want to show it for rank $n+1$. Let $v_1, \ldots, v_{n+1}$ be a basis of $N \cong R^{n+1}$ and let $f : N \to R$ be the projection to Rv_{n+1}. Then $\ker f$ is free of rank n and so every submodule of $\ker f$ is free of rank $\leq n$. Thus $M \cap \ker f$ is free of rank $m \leq n$ and let $u_1, \ldots, u_m$ be a basis of $M \cap \ker f$.

To be continued. □