Graduate Algebra, Fall 2014
Lecture 36

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3 Modules

3.4 Modules over PIDs (continued)

3.4.2 Finitely generated modules

Proof of the theorem. Suppose M is a finitely generated module. Then there exists an R-module homomorphism $f : R^n \twoheadrightarrow M$. Then $\ker f \subset R^n$ is a submodule and so, by the previous proposition, $\ker f \cong R^m$. Moreover, we can find a basis $R^n = \oplus R y_i$ and $a_1 \mid \ldots \mid a_m \in R$ such that $a_1 y_1, \ldots, a_m y_m$ is a basis of $\ker f$, i.e., $\ker f = \oplus R a_i y_i$.

Thus we get the exact sequence $0 \to R^m \xrightarrow{i} R^n \xrightarrow{f} M \to 0$ and so $M \cong \operatorname{coker} i = R^n/R^m = \oplus R y_i / \oplus R a_i y_i \cong R^{n-m} \oplus \bigoplus R/(a_i)$ as desired. □

Proposition 1. *Suppose R is a PID and M a finitely generated module of the form $R^r \oplus \bigoplus R/(a_i)$ such that $a_1 \mid \ldots \mid a_n$. Then $\operatorname{Ann}_R(M) = (a_n)$ and $\operatorname{Ann}_R(M)$ is called the characteristic ideal of M.*

Proof. Very easy. □

3.5 Nakayama's lemma

Recall that if M and N are R-modules then $\operatorname{Hom}_R(M, N)$ has the structure of an R-module. If, moreover, $M = N$ then $\operatorname{End}_R(M) := \operatorname{Hom}_R(M, M)$ in fact has the structure of a ring. Indeed, we can define "multiplication" of endomorphisms as composition. The ring $R \hookrightarrow \operatorname{End}_R(M)$ via $r \mapsto r\,\mathrm{id}$.

Proposition 2 (Nakayama's lemma)**.** *Let M be a finitely generated R-module and I an ideal of R.*

1. *If $f : M \to M$ is an R-module homomorphism such that $f(M) \subset IM$ then f satisfies an equation*
$$f^n + a_{n-1} f^{n-1} + \cdots + a_0 = 0$$
where $a_i \in I$. The equation is taken in the ring $\operatorname{End}_R(M)$.

2. *If $IM = M$ then there exists $x \in 1 + I$ such that $xM = 0$.*

3. *If I is an ideal contained in the Jacobson radical of R and M is finitely generated such that $M = IM$ then $M = 0$.*

Proof. (1): Let $M = \sum R m_i$. Then $f(m_i) \in IM = \sum I m_i$ so we write $f(m_i) = \sum a_{i,j} m_j$ with $a_{i,j} \in I$. Let $A = (a_{i,j})$ in which case we have $f - A I_n$ acts trivially on $\sum R m_i$. This implies that $\det(f - A I_n) = 0$ and this determinant, expanded, yields the desired equation.

(2): Apply the first part to $f = \mathrm{id}$ and take $x = 1 + \sum a_i$. Then $f^k = f \circ f \circ \cdots \circ f = \mathrm{id}$ and $a_0 \in R \subset \operatorname{End}_R(M)$ is in fact $a_0\,\mathrm{id}$ so the result follows.

(3): Pick $1 + x \in 1 + I$ such that $(1 + x)M = 0$. Since $x \in I \subset J(R)$ it follows that $1 + x$ is a unit and so $M = (1 + x)^{-1}(1 + x)M = 0$. □

Example 3. Take $R = \mathbb{Z}$ and M finitely generated over $\mathbb{Z}$ in which case the next section shows that $M \cong R^r \oplus \bigoplus \mathbb{Z}/(n_i)$. If $I = (m)$ then $IM = M$ iff $(m, \prod n_i) = 1$ and $r = 0$. Finding $x \in \mathbb{Z}$ such that $xM = 0$ is equivalent to $[n_1, \ldots, n_k] \mid x$ so the previous result implies that there exists $x \equiv 1 \pmod{m}$ such that $[n_1, \ldots, n_k] \mid x$ which is clear from the fact that $[n_1, \ldots, n_k] \in (\mathbb{Z}/m\mathbb{Z})^\times$.

Problem 4. On the homework you will use Nakayama's lemma to show that if I is an ideal in a Noetherian ring then $\cap I^n = 0$ which has topological implications on completed rings.

3.6 Operations on modules I

3.6.1 Annihilators

Definition 5. Let M be an R-module. Then $\mathrm{Ann}_R(M) = \{r \in R | rM = 0\}$ is the **annihilator** of M.

Example 6.

1. If I is an ideal $\mathrm{Ann}_R(R/I) = I$.
2. If R is an integral domain and I is an ideal then $\mathrm{Ann}_R(I) = 0$.

Lemma 7. $\mathrm{Ann}_R(M + N) = \mathrm{Ann}_R(M) \cap \mathrm{Ann}_R(N)$.

3.6.2 Homs

Proposition 8. *R is a commutative ring.*

1. *If $f : M \to M'$ is a homomorphism then get homomorphism $f^* : \mathrm{Hom}_R(M', N) \to \mathrm{Hom}_R(M, N)$ and $f_* : \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, M')$.*
2. *The sequence of R-modules $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact if and only if for every R-module N the sequence*
$$0 \to \mathrm{Hom}_R(M'', N) \xrightarrow{g^*} \mathrm{Hom}_R(M, N) \xrightarrow{f^*} \mathrm{Hom}_R(M', N)$$
is exact.
3. *The sequence of R-modules $0 \to N' \xrightarrow{f} N \xrightarrow{g} N''$ is exact if and only if for every R-module M the sequence*
$$0 \to \mathrm{Hom}_R(M, N') \xrightarrow{f_*} \mathrm{Hom}_R(M, N) \xrightarrow{g_*} \mathrm{Hom}_R(M, N'')$$
is exact.

Proof. (1): $f^*(\phi) = \phi \circ f$ and $f_*(\phi) = f \circ \phi$ are R-module homomorphisms.

(2): Suppose the initial sequence is exact. First, we show that g^* is injective. If $g^*(\phi) = 0$ then $\phi \circ g = 0$ but g is surjective so $\phi = 0$ as desired. Suppose now that $\phi \in \ker f^*$, we want $\phi \in \mathrm{Im}\, g^*$. So $f^*(\phi) = 0$ so $\phi \circ f = 0$. But then $\mathrm{Im}\, f \subset \ker \phi$ and so $\ker g = \mathrm{Im}\, f \subset \ker \phi$ which implies that ϕ factors through $M/\ker g \cong M''$ by exactness. Thus $\phi \in \mathrm{Im}\, g^*$ as desired.

Now suppose the resulting sequence is exact for all N. If g is not surjective take $N = \mathrm{coker}\, g$ in which case if $\pi : M'' \to \mathrm{coker}\, g$ is the natural projection map then $\pi \neq 0$ and $g^*(\pi) = 0$ yielding a contradiction. Take $N = \mathrm{coker}\, f$ and $\pi : M \to \mathrm{coker}\, f$ the natural projection. Then $f^*(\pi) = 0$ and so $\pi = g^*(\psi)$ for some $\psi : M'' \to \mathrm{coker}\, f$, i.e., $\pi = \psi \circ g$. But this implies that $\ker g \subset \ker \pi = \mathrm{Im}\, f$ as desired.

(3): Similar to (2). □

Definition 9. The R-module P is **injective** if injectivity of f implies surjectivity of f^*. It is **projective** if surjectivity of g implies surjectivity of g_*.

Example 10. Consider $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$. Let $M = \mathbb{Z}/2\mathbb{Z}$. Then f is injective but $f^* : \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is visibly the 0 map so M is not injective. Moreover $g^* : \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is in fact $g_* : 0 \to \mathbb{Z}/2\mathbb{Z}$ which is not surjective. (We used that there exist not homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$.) Thus M is also not projective.