Graduate Algebra, Fall 2014
Lecture 37

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2014-12-01

3 Modules

3.6 Operations on modules I (continued)

3.6.2 Homs (continued)

Proposition 1. *Let R be a commutative ring.*

1. *Every free R-module is projective.*

2. *An R-module M is projective if and only if there exists an R-module N such that $M \oplus N$ is a free R-module. (We say M is a direct summand of a free module.)*

3. *If R is an integral domain and M is projective then M is torsion-free, i.e., $\mathrm{Ann}_R(m) = 0$ for all $m \in M$. (More generally, if R is not an integral domain and $rm = 0$ for $m \neq 0$ then r is a zero divisor.)*

4. *If R is a PID then a finitely generated R-module M is projective if and only if it is torsion-free.*

Proof. (1): We need to show that if $F = \oplus_{i \in I} R$ and $g : M \twoheadrightarrow N$ is surjective then $g_* : \mathrm{Hom}_R(F, M) \to \mathrm{Hom}_R(F, N)$ is also surjective. Suppose $\phi : F \to N$ is a homomorphism. Let $(g_i)_{i \in I}$ be a basis of F over R. Since g is surjective there exist $m_i \in M$ such that $g(m_i) = \phi(g_i)$. For $f = \sum f_i g_i \in F$ define $\psi(f) = \sum f_i m_i$. This is well defined as there is no relation between the g_i and is clearly a homomorphism. Moreover $g \circ \psi = \phi$ so $g_*(\psi) = \phi$ as desired.

(2): If M is projective, take a surjection $g : F \to M$ from a free module F. (E.g., $F = \oplus_{m \in M} R$.) Projectivity gives that $g_* : \mathrm{Hom}_R(M, F) \to \mathrm{Hom}_R(M, M)$ is surjective and so there exists $s : M \to F$ such that $g_*(s) = \mathrm{id}$, i.e., $g \circ s = \mathrm{id}$. Let $N = \ker g$, a submodule of F. Then s is injective and so $M \cong \mathrm{Im}\, s$. Finally, $\mathrm{Im}\, s \cap \ker g = 0$ because otherwise their composition would not be the identity. For every $f \in F$ we have $f - s(g(f)) \in \ker g = N$ and so $f \in M + N$. Thus $F = M \oplus N$.

Reciprocally, suppose $M \oplus N = F$ is free. Let $g : P \to P'$ be surjective. Want that $g_* : \mathrm{Hom}_R(M, P) \to \mathrm{Hom}_R(M, P')$ is also surjective. Let $f' : M \to P'$ and define $f' \oplus 0 : F \to P'$ sending N to 0. Since F is projective, there exists $h : F \to P$ such that $g \circ h = g_*(h) = f' \oplus 0$. Define $f : M \to P$ by restriction from $F = M \oplus N$ to M. Then $g_*(f) = f'$.

(3): If M is projective then M is a direct summand of a free module so it is torsion-free as any free module over an integral domain is torsion-free.

(4): Homework. □

Proposition 2. *For an integral ring R a module M is said to be divisible if for $m \in M$ and $r \neq 0 \in R$ there exists $m/r \in M$. (A divisible group is a divisible $\mathbb{Z}$-module.)*

1. *If M is injective then it is divisible.*

2. *If R is a PID and M is divisible then M is injective.*

Example 3. $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules.

3.6.3 Localization

Definition 4. Let R be a ring and $S \subset R$ a multiplicatively closed subset. Define $S^{-1}M$ the equivalence classes of fractions m/s with $m \in M$ and $s \in S$ under $m/s = n/r$ iff for some $t \in S$, $t(mr - ns) = 0$. Equivalently iff $\mathrm{Ann}_R(mr - ns) \cap S \neq \emptyset$.

If $f : M \to N$ is an R-module hom then $S^{-1}f : S^{-1}M \to S^{-1}N$ defined by $f(m/s) = f(m)/s$ is well-defined and gives an $S^{-1}R$-module hom.

Proposition 5. *Suppose $M \to N \to P$ is exact. Then $S^{-1}M \to S^{-1}N \to S^{-1}P$ is exact. In particular, $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$.*

Proof. If $gf = 0$ then $S^{-1}gS^{-1}f = 0$ so $\mathrm{Im}\, S^{-1}f \subset \ker S^{-1}g$. If $m/s \in \ker S^{-1}g$ then $g(m)/s = 0$ so $g(m)t = 0$ for some $t \in S$. But then $g(tm) = 0$ so $tm \in \mathrm{Im}\, f$ which means $m \in \mathrm{Im}\, S^{-1}f$. □

Definition 6. A property $\mathcal{P}$ of modules is said to be **local** if M has $\mathcal{P}$ iff $M_\mathfrak{p}$ have $\mathcal{P}$ for all prime ideals $\mathfrak{p}$.

Proposition 7. *Let M be an R-module. Then $M = 0$ is a local property and in fact $M = 0$ iff $M_\mathfrak{p} = 0$ iff $M_\mathfrak{m} = 0$.*

Proof. Suppose $M_\mathfrak{m} = 0$ for all $\mathfrak{m}$. Suppose $0 \neq m \in M$ and let $I = \mathrm{Ann}_R(m)$. Then $I \neq R$ is an ideal of R and let $\mathfrak{m}$ be a maximal ideal containing I. Let $m/1 \in M_\mathfrak{m}$. Since $M_\mathfrak{m} = 0$ it follows that for some $r \in R - \mathfrak{m}$ have $rm = 0$ which cannot be since $I \supset \mathrm{Ann}_R(m)$. □