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Lecture 38

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3 Modules

3.6 Operations on modules I (continued)

3.6.3 Localization (continued)

Proposition 1. *Let $f : M \to N$ be an R-module homomorphism. Then f injective is a local property and in fact f is injective iff $f_\mathfrak{p}$ is injective iff $f_\mathfrak{m}$ is injective.*

Proof. From the exactness of localization proposition above we deduce that if f is injective then all localizations are. Suppose $f_\mathfrak{m}$ is injective for all $\mathfrak{m}$. Let $K = \ker f$ so $0 \to K \to M \to N$ is exact. But then $0 \to K_\mathfrak{m} \to M_\mathfrak{m} \to N_\mathfrak{m}$ is exact so $K_\mathfrak{m} = 0$ for all $\mathfrak{m}$ which implies $K = 0$ by the previous proposition. $\square$

Proposition 2. *Whether a ring is reduced is a local property: a ring R is reduced iff $R_\mathfrak{p}$ is reduced for all prime ideals $\mathfrak{p}$.*

Proof. Homework 10. $\square$

3.7 Functors and exactness

Definition 3. A **functor of modules** is an assignment attaching $M \mapsto F(M)$ attaching to a module M over a ring R another module $F(M)$ over some ring S such that if $f : M \to N$ is an R-module homomorphism then there exists an S-module homomorphism $F(f) : F(M) \to F(N)$ (the **covariant** case) or $F(f) : F(N) \to F(M)$ (the **contravariant** case).

Example 4. Fix an R-module P. Then $F(M) = \mathrm{Hom}_R(M, P)$ and $G(M) = \mathrm{Hom}_R(P, M)$ are R-modules. Moreover, setting $F(f) = f^*$ and $G(f) = f_*$ gives that F is a contravariant functor while G is a covariant functor.

Example 5. Suppose V is an $\mathbb{R}$-vector space. Fixing a basis (e_i) of V over $\mathbb{R}$, we may allow finite linear combinations of (e_i) with $\mathbb{C}$-coefficients to obtain a $\mathbb{C}$-vector space. Let $F(V)$ be this $\mathbb{C}$-vector space. I claim that F is a covariant functor. Suppose $f : V \to W$ is a $\mathbb{R}$-vector space homomorphism, given by $f(\sum r_i e_i) = \sum_{i,j} a_{i,j} r_i e_j$ ($r_i \in \mathbb{R}$ and the matrix entries $a_{i,j} \in \mathbb{R}$). Define the homomorphism $F(f) : F(V) \to F(W)$ by $F(f)(\sum c_i e_i) = \sum a_{i,j} c_i e_j$ ($c_i \in \mathbb{C}$ and $a_{i,j} \in \mathbb{R} \subset \mathbb{C}$).

Definition 6. A covariant functor F is **left-exact** (resp. **right-exact**) if for any sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ of R-modules exactness in the middle and on the left (resp. on the right) implies exactness in the middle and on the left (resp. on the right).

For a contravariant function G the definition is similar except on wants the sequence after applying G to be exact on the left/right if the original is exact on the left/right.

Example 7. The main proposition about Hom_R says that $M \mapsto \mathrm{Hom}_R(M, P)$ and $M \mapsto \mathrm{Hom}_R(P, M)$ are right-exact.

3.8 Operations on modules II

3.8.1 Tensor products

Proposition 8. *Let M and N be two R-modules. Let*

$$C = \bigoplus_{(m,n)\in M\times N} R(m,n)$$

be the R-module generated by all pairs $(m,n) \in M \times N$.

Let $D \subset C$ be the R-submodule generated by the elements $(m + rm', n) - (m,n) - r(m', n)$ and $(m, n + rn') - (m,n) - r(m,n')$ for $m, m' \in M$, $n, n' \in N$ and $r \in R$.

Denote by $M \otimes_R N$ be quotient R-module C/D and let $m \otimes n$ the image of (m,n) in $M \otimes_R N$. The module $M \otimes_R N$ is the **tensor product***.*

1. *The map $\pi : M \times N \to M \otimes_R N$ sending $(m,n) \mapsto m \otimes n$ is bilinear.*

2. *If P is an R-module and $f : M \times N \to P$ is a bilinear map then there exists a unique homomorphism $g : M \otimes_R N \to P$ such that $f = g \circ \pi$.*

Proof. (1): Easy to check.

(2): If f is bilinear then we get a homomorphism $F : C \to P$ defined by $F(\sum r_i(m_i, n_i)) = \sum r_i f(m_i, n_i)$. Then $D \subset \ker F$ and so F factors through $C/D = M \otimes_R N$ from, e.g., the first isomorphism theorem for groups. $\square$

Remark 1.

1. From the definition, if $r \in R$ and $m \otimes n \in M \otimes_R N$ then $(rm) \otimes n = m \otimes (rn)$.

2. Elements $m \otimes n \in M \otimes_R N$ are called pure tensors. The general element of $M \otimes_R N$ however is of the form
$$\sum r_i(m_i \otimes n_i)$$

Example 9.

1. Suppose m, n are coprime. Then $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$. Indeed, $n \in (\mathbb{Z}/m\mathbb{Z})^\times$ so $x \otimes y = (nn^{-1}x) \otimes y = (n^{-1}x) \otimes (ny) = (n^{-1}x) \otimes 0 = 0$.

2. $R \otimes_R M \cong M$. Consider the map $m \mapsto m \otimes 1$. This is injective as $m \otimes 1$ is never in the submodule D. Moreover, if $x = \sum r_i(s_i \otimes m_i) \in R \otimes_R M$ then $x = (\sum r_i s_i) \otimes m$ from the properties of $\otimes$ and so $R \otimes_R M$ is in the image of $m \mapsto m \otimes 1$.

3. $R[X] \otimes_R R[X] \cong R[X, Y]$.