Graduate Algebra, Fall 2014
Lecture 39

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3 Modules

3.8 Operations on modules II (continued)

3.8.1 Tensor products (continued)

Proposition 1. *Let M, N, P be R-modules. Then*

1. $M \otimes_R N \cong N \otimes_R M$.
2. $(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$ *and it's denoted* $M \otimes_R N \otimes_R P$.
3. $(M \oplus N) \otimes_R P \cong M \otimes_R P \oplus N \otimes_R P$.

Proof. The following maps can be checked to be well-defined on the tensor product and to yield isomorphisms: $x \otimes y \mapsto y \otimes x$, $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$, $(x \oplus y) \otimes z \mapsto x \otimes z \oplus y \otimes z$. ☐

Remark 1. Suppose M and N are R-modules but N also has the structure of an S-module. Then $M \otimes_R N$, a priori only an R-module, is also an S-module. Indeed define $s \cdot (m \otimes n) = m \otimes (sn)$.

Definition 2 (Extensions of scalars)**.** Suppose $f : R \to S$ is a homomorphism of rings and M is an R-module. Then S is both an R (via f) and an S-module and so $f_*(M) := M \otimes_R S$ is an S-module, called the pushforward.

Lemma 3. *Have a canonical isomorphism of R-modules* $\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$.

Proof. Write $\mathrm{Bil}_R(M \times N, P)$ for the R-module of R-bilinear maps from $M \times N$ to P. We already know from last time that $\mathrm{Hom}_R(M \otimes_R N, P) = \mathrm{Bil}_R(M \times N, P)$. Suppose $F \in \mathrm{Bil}_R(M \times N, P)$. Then $m \mapsto (n \mapsto F(m, n))$ is an element of $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ and the assignment is visibly an isomorphism. ☐

Theorem 4. *Suppose M is an R-module. The assignment $N \mapsto N \otimes_R M$ is a right-exact functor on R-modules. I.e., $- \otimes_R M$ is a covariant functor and if $0 \to N' \xrightarrow{i} N \xrightarrow{j} N'' \to 0$ is an exact sequence of R-modules and then $N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M \to 0$ is exact.*

Proof. First, suppose $f : N' \to N$ is a homomorphism. Then we need a homomorphism $N' \otimes_R M \to N \otimes_R M$. Define it by $f \otimes 1(\sum n_i \otimes m_i) = \sum f(n_i) \otimes m_i$. This produces the sequence $N' \otimes_R M \xrightarrow{i \otimes 1} N \otimes_R M \xrightarrow{j \otimes 1} N'' \otimes_R M \to 0$ that we need to check to be exact.

(More generally, if $f : M \to M'$ and $g : N \to N'$ are R-module homomorphisms then $f \otimes g : M \otimes_R N \to M' \otimes_R N'$ defined as $(f \otimes g)(\sum m_i \otimes n_i) = \sum f(m_i) \otimes g(n_i)$ yields an R-module homomorphisms.)

On the homework you have to show that $A \to B \to C \to 0$ is an exact sequence of R-modules iff for all R-modules P the sequence $0 \to \mathrm{Hom}_R(C, P) \to \mathrm{Hom}_R(B, P) \to \mathrm{Hom}_R(A, P)$ is exact. Therefore to check that $N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M \to 0$ is exact it suffices to check that for each P the sequence $0 \to \mathrm{Hom}_R(N'' \otimes_R M, P) \to \mathrm{Hom}_R(N \otimes_R M, P) \to \mathrm{Hom}_R(N' \otimes_R M, P)$ is exact. The previous lemma restates this as $0 \to \mathrm{Hom}_R(N'', \mathrm{Hom}_R(M,P)) \to \mathrm{Hom}_R(N, \mathrm{Hom}_R(M,P)) \to \mathrm{Hom}_R(N', \mathrm{Hom}_R(M,P))$ being exact which follows from the exactness of Hom applied to $\mathrm{Hom}(-, \mathrm{Hom}_R(M,P))$. ☐

Example 5. Let R be a commutative ring and M an R-module. If I is an ideal then $R/I \otimes_R M \cong M/IM$.

Proof. Consider the exact sequence $0 \to I \to R \to R/I \to 0$ which gives $M \otimes_R I \to M \otimes_R R \to M \otimes_R R/I \to 0$. Under $M \otimes_R R \cong M$ we have $M \otimes_R I \cong IM$ and the conclusion follows. ☐

3.8.2 Flatness

Definition 6. An R-module M is **flat over** R if the functor $- \otimes_R M$ is exact. The previous theorem implies that M is flat over R if for every injective homomorphism $f : N \to P$ of R-modules the map $f \otimes 1 : N \otimes_R M \to P \otimes_R M$ is also injective.

We will prove three main properties about flatness.

1. We will show that if M is flat over R and $f : R \to S$ then $f_*(M)$ is flat over S. Moreover, if S is flat over R and N is flat over S then $f^*(N)$ is flat over R.

2. We will show that $S^{-1}M \cong S^{-1}R \otimes_R M$ and so $S^{-1}R$ is flat over R. We will deduce from this that flatness is a local property, which is of crucial importance in algebraic geometry.

3. We will show that flat modules over integral domains are torsion-free. We will also show that over PIDs the converse is true: that torsion-free modules are free.