Graduate Algebra, Fall 2014
Lecture 40

Andrei Jorza

2014-12-08

3 Modules

3.9 Operations on modules II (continued)

3.9.2 Flatness (continued)

Proposition 1. *Let R be a commutative ring.*

1. *Every finitely generated free module is flat over R.*
2. *Every free module is flat over R.*
3. *Every projective R module is flat over R.*

Proof. (1): If $M = R^n$ then $M \otimes_R N \cong N^n$ and if $f : N \hookrightarrow N'$ then $1 \otimes f = f \oplus \cdots \oplus f$ is clearly injective.

(2): Suppose F is free over R and $g : M \to M'$ is injective but $F \otimes_R M \to F \otimes_R M'$ is not injective, with $\sum f_i \otimes m_i$ in the kernel, i.e., $\sum f_i \otimes g(m_i) = 0$. Let a_i be a basis of F over R and write $f_i = \sum r_{i,j} a_j$ with only finitely many a_j appearing among the f_i. Let $F' \subset F$ be the free submodule generated by all the a_j in the f_i-s. Then $\sum f_i \otimes m_i \in F' \otimes_R M$. Since $F' \otimes_R M \to F' \otimes_R M'$ is injective and $\sum f_i \otimes g(m_i) = 0$ in $F' \otimes_R M'$ it follows that $\sum f_i \otimes m_i$ is 0 in $F' \otimes_R M$ and so also in $F \otimes_R M$.

(3): If M is projective then it is a direct summand of a free module. If $M \oplus N = F$ for F free and $T \to T'$ is injective then $F \otimes_R T \to F \otimes_R T'$ is injective and so $(M \otimes_R T) \oplus (N \otimes_R T) \to (M \otimes_R T') \oplus (N \otimes_R T')$ is injective. But the necessarily $M \otimes_R T \to M' \otimes_R T'$ is injective. □

Example 2. $R[X]$ is flat over R since it is free as an R-module.

Lemma 3 (Bimodules)**.** *Suppose R and S are rings, M is an R-module, P is an S-module and N is an (R,S)-module by which we mean that there are scalar multiplications $r \cdot m$ and $m \cdot s$ such that $r \cdot (m \cdot s) = (r \cdot m) \cdot s$. Then $M \otimes_R N$ is naturally an S-module and $N \otimes_S P$ is naturally an R-module and*

$$(M \otimes_R N) \otimes_S P \cong M \otimes_R (N \otimes_S P)$$

Proof. The S-module structure on $M \otimes_R N$ comes from N by letting $s \cdot \sum m_i \otimes n_i := \sum m_i \otimes (n_i \cdot s)$ and similarly for the R-module structure on $N \otimes_S P$.

The rest is an exercise. □

Theorem 4 (Pullbacks and pushforwards)**.** *Let $f : R \to S$ be a homomorphism of commutative rings, M an R-module and N an S-module. Recall that we obtain modules $f_*(M) = M \otimes_R S$ over S and $f^*(N)$ over R.*

1. *If M is flat over R then $f_*(M)$ is flat over S.*
2. *If S is flat as a module over R and N is flat over S then $f^*(N)$ is flat over R.*

Proof. (1): Suppose $g : N \to N'$ is an injection of S-modules. Need that $(M \otimes_R S) \otimes_S N \to (M \otimes_R S) \otimes_S N'$ is also injective. But this is just $M \otimes_R f_*(N) \to M \otimes_R f_*(N')$. By the flatness of M it suffices to check that $f^*(N) \to f^*(N')$ is injective, which is clear since the map doesn't change, we just interpret is as R-linear instead of S-linear.

(2): Note that $M \otimes_R f^*(N) = M \otimes_R N \cong (M \otimes_R S) \otimes_S N$. Since S is flat over R then $M \otimes_R S \hookrightarrow M' \otimes_R S$ and then injectivity of $M \otimes_R f^*(N) \to M' \otimes_R f^*(N)$ follows from the fact that N is flat over S. □

Example 5. An application of the previous. M is flat over R iff $M[X]$ is flat over $R[X]$.

Proof. Note that $M[X] \cong R[X] \otimes_R M$ and so if M is flat over R then the pushforward $M[X]$ is flat over $R[X]$.

Let $\pi : R[X] \to R \cong R[X]/(X)$. Then $\pi_*(M[X]) = M[X] \otimes_{R[X]} R \cong M[X]/XM[X] = M$ and so $M = \pi_*(M[X])$ is flat over R from the proposition. □

Theorem 6. *Let R be a commutative ring and S a multiplicatively closed subset containing 1. Let M be an R-module.*

1. *$S^{-1}M \cong S^{-1}R \otimes_R M$.*
2. *$S^{-1}R$ is a flat R-module.*
3. *If N is an R-module then $S^{-1}M \otimes_{S^{-1}R} S^{-1}N \cong S^{-1}(M \otimes_R N)$.*
4. *Flatness is a local property: M is flat over R iff $M_\mathfrak{p}$ is flat over $R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ iff $M_\mathfrak{m}$ is flat over $R_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$.*

Proof. (1): The map $S^{-1}R \times M \to S^{-1}M$ sending $\sum a_i/s_i \otimes m_i \mapsto \sum a_i m_i/s_i$ is bilinear and so factors through an R-linear homomorphism $S^{-1}R \otimes_R M \to S^{-1}M$. It is clearly surjective. Since $\sum a_i/s_i \otimes m_i = 1/\prod s_i \otimes \sum a_i \prod_{j \neq i} s_j m_i$ every element of $S^{-1}R \otimes_R M$ is of the form $(1/s) \otimes m$ for $s \in S$ and $m \in M$.

Suppose that $1/s \otimes m$ maps to 0 under the homomorphism. Then $m/s = 0$ and so $tm = 0$ for some $t \in S$. But then $1/s \otimes m = t/(ts) \otimes m = 1/(ts) \otimes tm = 0$ so the map is injective.

(2): Suppose $M \to N$ is injective. Need to show that $S^{-1}R \otimes_R M \to S^{-1}R \otimes_R N$ is injective, i.e., that $S^{-1}M \to S^{-1}N$ is injective. This we already proved.

(3) and (4) next time.

□

Example 7.

1. Let $M = R[x,y]/(xy-1)$ as a module over $R[x]$. Then M is flat over R. Indeed, if $S = \{x^n | n \geq 0\}$ then $R[x,y]/(xy-1) \cong R[x][x^{-1}] = S^{-1}R[x]$ is flat over $R[x]$ from the proposition.
2. Let $N = R[x,y]/(xy)$ as a module over $R[x]$. Then N is NOT flat because $0 \neq y \in N$ is torsion and flat modules over integral domains are torsion-free. Indeed, $xy = 0$ in N. Directly from the definition: multiplication by x is injective in $R[x]$ but not on M.