Graduate Algebra, Fall 2014
Lecture 41

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3 Modules

3.9 Operations on modules II (continued)

3.9.2 Flatness (continued)

Theorem 1. *Let R be a commutative ring and S a multiplicatively closed subset containing 1. Let M be an R-module.*

1. $S^{-1}M \cong S^{-1}R \otimes_R M$.
2. $S^{-1}R$ *is a flat* R*-module.*
3. *If* N *is an* R*-module then* $S^{-1}M \otimes_{S^{-1}R} S^{-1}N \cong S^{-1}(M \otimes_R N)$.
4. *Flatness is a local property:* M *is flat over* R *iff* $M_\mathfrak{p}$ *is flat over* $R_\mathfrak{p}$ *for all prime ideals* $\mathfrak{p}$ *iff* $M_\mathfrak{m}$ *is flat over* $R_\mathfrak{m}$ *for all maximal ideals* $\mathfrak{m}$.

Proof. (1) and (2) last time.

(3): $S^{-1}R$ is an $(R, S^{-1}R)$-bimodule and so

$$\begin{aligned} S^{-1}M \otimes_{S^{-1}R} S^{-1}N &\cong (M \otimes_R S^{-1}R) \otimes_{S^{-1}R} S^{-1}N \\ &\cong M \otimes_R (S^{-1}R \otimes_{S^{-1}R} S^{-1}N) \\ &\cong M \otimes_R (S^{-1}R \otimes_R N) \\ &\cong S^{-1}R \otimes_R (M \otimes_R N) \\ &\cong S^{-1}(M \otimes_R N) \end{aligned}$$

(4): We deduce that if M is flat over R then $S^{-1}M$ is flat over $S^{-1}R$. It remains to check that if $M_\mathfrak{m}$ is flat over $R_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ then M is in fact flat over R. Suppose $N \to N'$ is an injection morphism of R-modules. Then we know that $N_\mathfrak{m} \to N'_\mathfrak{m}$ is injective by the local properties of injective maps. Thus $M_\mathfrak{m} \otimes_{R_\mathfrak{m}} N_\mathfrak{m} \to M_\mathfrak{m} \otimes_{R_\mathfrak{m}} N'_\mathfrak{m}$ is injective as $M_\mathfrak{m}$ is flat over $R_\mathfrak{m}$. But by the above this is a map $(M \otimes_R N)_\mathfrak{m} \to (M \otimes_R N')_\mathfrak{m}$ which is injective for all maximal $\mathfrak{m}$. Again by the locality of injections we deduce that $M \otimes_R N \to M \otimes_R N'$ is injective and so M is flat over R. □

Proposition 2. *Let R be a commutative ring and M an R-module. Suppose $I \otimes_R M \to M$ is injective for all ideals I.*

1. *If* $J \subset F \cong R^n$ *is a submodule of the free module* F *then* $M \otimes_R J \to M \otimes_R F$ *is injective.*
2. *If* $J \subset F$ *is a submodule of a free module* F *(not necessarily finitely generated) then* $M \otimes_R J \to M \otimes_R F$ *is injective.*

3. M is flat over R.

Proof. (1): We do this by induction on n. The base case $n = 1$ is the hypothesis. Consider $R \subset R^n$ as the first summand and let $I = \{r \in R | (r, 0, \ldots, 0) \in J\}$. Then get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & J & \longrightarrow & J/I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R & \longrightarrow & F & \longrightarrow & F/R & \longrightarrow & 0 \end{array}$$

where all vertical maps are injective. Tensoring with M gives

$$\begin{array}{ccccccc} I \otimes_R M & \longrightarrow & J \otimes_R M & \longrightarrow & J/I \otimes_R M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ M & \longrightarrow & F \otimes_R M & \longrightarrow & F/R \otimes_R M & \longrightarrow & 0 \end{array}$$

and the inductive hypothesis implies that the left and right vertical maps are injective. The map $M \to F \otimes_R M \cong M^n$ is injective (inclusion in the first coordinate) so simple diagram chasing implies that the middle vertical map is also injective.

(2): Suppose $J \otimes_R M \to F \otimes_R M$ is not injective with $\sum \iota(f_i) \otimes m_i = 0$ in $F \otimes_R M$ for $f_i \in J$, where $\iota : J \hookrightarrow F$. For a basis a_i of F over R write $f_i = \sum r_{i,j} a_j$. Let F' be the submodule generated by the a_j appearing in the f_i. Let $J' = J \cap F'$. Then $J' \otimes_R M \to F' \otimes_R M$ is injective by the previous part and so $\sum f_i \otimes m_i$ must vanish.

(3): Suppose $f : N \to N'$ is an injection of R-modules such that $N \otimes_R M \to N' \otimes_R M$ is not injective. Let F be a free R-module such that $F \twoheadrightarrow N'$ and let K be the kernel: $0 \to K \to F \xrightarrow{\pi} N \to 0$.

Let $J = \pi^{-1}(f(N))$. Define $J \to N$ by sending x to the unique preimage of $\pi(x) \in f(N)$. This is a homomorphism of R-modules and the map $J \to N$ has kernel K. Thus we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & J & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & K & \longrightarrow & F & \xrightarrow{\pi} & N' & \longrightarrow & 0 \end{array}$$

Tensoring with M get

$$\begin{array}{ccccccc} M \otimes_R K & \longrightarrow & M \otimes_R J & \longrightarrow & M \otimes_R N & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle 1\otimes\iota} & & \downarrow{\scriptstyle 1\otimes f} & & \\ M \otimes_R K & \longrightarrow & M \otimes_R F & \xrightarrow{1\otimes\pi} & M \otimes_R N' & \longrightarrow & 0 \end{array}$$

Since the left vertical map is surjective for the right vertical map to be injective it suffices to show that the middle vertical map is injective (from the homework). This follows from the previous part. □

Theorem 3. *Let R be an integral domain and M an R-module.*

1. *If M is flat over R then M is torsion-free, i.e.,* $\mathrm{Ann}_R(M) = 0$.

2. *If R is a PID and M is torsion-free then M is flat over R.*

Proof. (1): Suppose M is flat but $0 \neq a \in \mathrm{Ann}_R(M)$. Then multiplication by a is injective $R \to R$ but multiplication by a in $M \to M$ is the zero map. Thus $\mathrm{Ann}_R(M) = 0$.

(2): Now suppose M is torsion-free. If $I = (a)$ is an ideal then $R \to I$ given by $x \mapsto xa$ is an isomorphism of R-modules and so $M \cong I \otimes_R M$. Thus M must be flat by the previous proposition.

□

Example 4. Let $A \in M_{n \times n}(R)$ where R is a commutative ring. Consider $M = R^n$ as a module over $R[X]$ via $P(X) \cdot m := P(T)m$. Then M is not flat over R. Indeed, let $P(X)$ be the characteristic polynomial of the matrix A. Then $P(A) = 0$ by Cayley-Hamilton and so $P(X) \cdot m = 0$ for all $m \in M$. Since M is then torsion we deduce it is not flat.

Example 5. Let R be any commutative ring. Then the ideal $I = (x, y)$ is not flat over $R[x, y]$. Indeed, if it were, then $I \otimes_R I \to I$ given by multiplication would be injective. But $x \otimes y - y \otimes x$ maps to 0 and is no 0 in $I \otimes_R I$. (Careful, $x \otimes y \neq (xy) \otimes 1$ because even if $y \in R$ so in principle can be moved from one side to the other, $1 \notin I$ so the RHS is not in $I \otimes_R I$.