Graduate Algebra, Fall 2014
Lecture 9

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2014-09-15

1 Group Theory

1.15 Group actions

Definition 1. A **group action** of a group G on a set X is any homomorphism from G to the group of permutations of X. I.e., to each $g \in G$ one associates a map $x \mapsto gx$ on X such that if $g, h \in G$ then $(gh)x = g(hx))$ and $1x = x$ for all $x \in X$.

Example 2.

1. The trivial action: G acts on X trivially, sending every g to the identity map.
2. The left regular action of G on itself is $g \mapsto (x \mapsto gx)$. The right regular action is $g \mapsto (x \mapsto xg)$.
3. Let S be a set and X the set of functions $G \to S$. Then G acts on X by $(gf)(x) = f(xg)$, also called the right regular action.
4. The conjugation action. G acts on itself sending g to the inner homomorphism $h \mapsto ghg^{-1}$. The conjugation action gives an action of G on any normal subgroup of G.
5. If X is the set of subgroups of G then the conjugation action of G on itself yields a conjugation action on X. Indeed, if H is a subgroup then gHg^{-1} is also a subgroup. The left and right regular actions of G on itself also give actions on X.
6. The group S_n acts on $\mathbb{C}^n$ by permuting coordinates.
7. For $R = \mathbb{Z}/p\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ the group $\mathrm{GL}(n, R)$ acts on R^n by left matrix multiplication.
8. If H is a subgroup of G then the left regular action of G on itself gives the action of G on G/H by $g \mapsto (xH \mapsto gxH)$. Similarly the right regular action of G on itself gives an action of G on $H\backslash G$.
9. The group $\mathrm{GL}(2, R)$ acts on $\mathbb{P}^1_R$ as follows: the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts by sending $z \in \mathbb{R} \cup \infty$ to $\frac{az+b}{cz+d} \in R \cup \infty$.
10. The group $\mathrm{GL}(n, R)$ acts on the set of k-dimensional sub-vector space of R^n by left matrix multiplication.
11. Let $k \geq 0$ and V_k be the set of polynomials $P(X, Y) \in \mathbb{C}[X]$ homogeneous of degree k. Then $\mathrm{GL}(2, \mathbb{C})$ acts on V_k as follows: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(X, Y) = P(aX + bY, cX + dY)$. This is called the k-th symmetric representation.

Definition 3. Suppose G acts on X. The **orbit** of $x \in X$ is the set $O(x) = \{gx | g \in G\}$.

Remark 1. Two orbits are either disjoint or coincide and the space X becomes a disjoint union of orbits of G acting on X.

Example 4. 1. The group $G = \mathbb{R}/\mathbb{Z}$ acts on $\mathbb{C}$ sending x to rotation by x: $x \mapsto (z \mapsto ze^{2\pi i x})$. This is a group action. Suppose $z \in \mathbb{C}$. Then the orbit of z contains all $ze^{2\pi i x}$ for all x and so $O(z) = \{w \in \mathbb{C} | |z| = |w|\}$ is a circle of radius $|z|$. Two such circles are either disjoint or coincide and of course $\mathbb{C}$ is a union of all these concentric circles.

(a) If a group G acts by conjugation on itself, the orbits are called **conjugacy classes**.

(b) The group $\mathrm{GL}(2, \mathbb{C})$ acts on the space X of 2×2 matrices with complex coordinates by conjugation: $g \mapsto (X \mapsto gXg^{-1})$. What are the orbits? The Jordan canonical form of a 2×2 matrix A is a matrix B of the form $\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ for $\alpha, \beta \in \mathbb{C}$ or $\begin{pmatrix} \alpha & 1 \\ & \alpha \end{pmatrix}$ for $\alpha \in \mathbb{C}$ such that $A = SBS^{-1}$ for some $S \in \mathrm{GL}(2, \mathbb{C})$. Thus every orbit on G on X, i.e., every conjugacy class, contains a matrix of this special form. Moreover, the only way for one orbit (conjugacy class) to contain two matrices of this special form is if the two matrices are $\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ and $\begin{pmatrix} \beta & \\ & \alpha \end{pmatrix}$. We thus get a complete enumeration of all the conjugacy classes of $\mathrm{GL}(2, \mathbb{C})$ acting on X.

Definition 5. Suppose G acts on X and $x \in X$. The **stabilizer** of x in G is the set $\mathrm{Stab}_G(x) = \{g \in G | gx = x\}$. It is a subgroup of G.

Example 6. 1. In the $\mathbb{R}/\mathbb{Z}$ acting on $\mathbb{C}$ by rotation there are two stabilizers: $\mathbb{R}/\mathbb{Z}$ when $z = 0$ and 0 if $z \neq 0$.

2. Suppose G acts by conjugation on itself. What is $\mathrm{Stab}_G(g)$? It is $\{h \in G | h \cdot g = g\}$ in other words $hgh^{-1} = g$ or $hg = gh$. This is called the **centralizer** of g in G, often denoted $C_G(g)$.

3. Suppose S_n acts on $\mathbb{C}^n$ by $\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Then $\mathrm{Stab}_G(x_1, \ldots, x_n) = \{\sigma \in S_n | \sigma(i) = i\}$. For example, $\mathrm{Stab}_{S_3}((1, 1, 0)) = \langle (12) \rangle$.

Theorem 7 (Class equation)**.** *Let G be a finite group acting on a finite set X.*

1. *$X = \sqcup O_i$ where the O_i are the orbits of G on X.*

2. *If $x \in X$ then $|O(x)| = [G : \mathrm{Stab}_G(x)]$.*

3. *In each orbit of G acting on X choose an element x_i. Then*

$$|X| = \sum [G : \mathrm{Stab}_G(x_i)]$$

4. *In each conjugacy class in G with more than one element select an element g_i. Then*

$$|G| = |Z(G)| + \sum [G : C_G(g_i)]$$

Corollary 8. *Let G be a finite group such that $|G| = p^m$ for $m > 0$. Then $Z(G) \neq 1$.*

Proof. From the class equation $|G| = |Z(G)| + \sum [G : C_G(g_i)]$ where $[G : C_G(g_i)] \neq 1$. But then $[G : C_G(g_i)] \mid |G| = p^m$ and so must be a power of p. We deduce that $|Z(G)|$ is divisible by p and thus is not 1. ☐

Proposition 9. *If $|G| = p^2$ then G is abelian.*

Proof. From the corollary $Z(G)$ is nontrivial and so $|Z(G)| = p$ or p^2. If p^2 then G is abelian. If p then $G/Z(G)$ has p elements and thus is cyclic. But then the homework implies that G must be abelian to begin with and so this case cannot happen. ☐