1 Basics

Definition 1. Let $F$ be a field. A noncommutative ring $A$ is said to be a finite dimensional $F$-algebra if $\dim_F A < \infty$ and it is equipped with a ring homomorphism $F \rightarrow Z(A)$ taking $1$ to $1$.

• By a finite dimensional module $M$ over $A$ we mean a finite dimensional left $A$-module.
• The algebra $A$ is said to be simple if its only two-sided ideals are $0$ and $A$.
• A is said to be a division algebra if $A - \{0\}$ is a group under multiplication in the algebra.
• A finite dimensional $A$-module $M$ is simple if its only $A$-submodules are $0$ and $M$.

Definition 2. If $(A, +, \times)$ is an $F$-algebra define the opposite $F$-algebra $(A, +, \times^{op})$ where the set is $A$, the addition $+$ is the same as in $A$ but multiplication is $a \times^{op} b = b \times a$.

Lemma 3 (Schur). 1. If $M$ and $N$ are simple $A$-modules and $f \in \text{Hom}_A(M, N)$ then either $f = 0$ or $f$ is an isomorphism.
2. If $M$ is simple then $\text{End}_A(M)$ is a division algebra.

Proof. Note that $\ker f \subset M$ and $\text{Im } f \subset N$ so either $\ker f = 0$ or $\ker f = M$ and either $\text{Im } f = 0$ or $\text{Im } f = N$.

Lemma 4. Let $M$ be a finite-dimensional $A$-module. The following are equivalent:
1. $M = N_1 \oplus \cdots \oplus N_r$ where $N_i$ are simple.
2. $M = \sum N_i$ with simple $N_i \subset M$.
3. If $N \subset M$ then there exists $P \subset M$ such that $M = N \oplus P$.
4. If $N \subset M' \subset M$ there exists $P \subset M'$ such that $M' = N \oplus P$.

Proof. 1 implies 2 is vacuous.
2 implies 3: choose a maximal set of simple submodules $Q_1, \ldots, Q_r \subset M$ such that $N + Q_1 + \cdots + Q_r = N \oplus Q_1 \oplus \cdots \oplus Q_r$. If $N \oplus Q_1 \oplus \cdots \oplus Q_r \neq M$ choose a simple $Q_{r+1} \subset M$ such that $Q_{r+1} \not\subset N \oplus Q_1 \oplus \cdots \oplus Q_r$.
Since $Q_{r+1}$ is simple it follows that $N \oplus Q_1 \oplus \cdots \oplus Q_r \cap Q_{r+1} = 0$ so $N + Q_1 + \cdots + Q_{r+1} = N \oplus Q_1 \oplus \cdots \oplus Q_{r+1}$ contradicting the maximality of $r$.
3 implies 4: If $M = N \oplus Q$ then $M' = N \oplus (Q \cap M')$.
4 implies 1: choose $N \subset M$ a simple submodule. Then $M = N \oplus P$ and inductively we get the required decomposition.

Definition 5. When the equivalent conditions of the previous lemma hold the module $M$ is said to be semisimple.

Note 6. It is left as an exercise that a semisimple module decomposes uniquely (up to reordering) as a direct sum of simple submodules.
Corollary 7. Semisimplicity is preserved under direct sums and passage to quotients and submodules.

Corollary 8. If $A$ is a semisimple $A$-module then all finite dimensional $A$-modules are semisimple. In that case $A$ is said to be a (left) semisimple ring.

Proof. If $M$ is any finite dimensional $A$-module then $A^r \to M$ for some $r$ and $M$ must be semisimple. \hfill \Box

Corollary 9. Let $M$ be a finite dimensional $A$ module such that the action of $A$ on $M$ is faithful, i.e., if for $a \in A$ we have $am = 0$ for all $m \in M$ then $a = 0$. If $M$ is semisimple then $A$ is semisimple.

Corollary 10. If $A$ is simple as a ring, i.e., there are no nontrivial two-sided ideals, then it is left semisimple. 

Proof. Let $M \subset A$ a simple left $A$-submodule. Then $\sum_{a \in A} Ma \subset A$ is a two-sided ideal so $\sum_{a \in A} Ma = A$. Since $A$ is then a semisimple $A$-module it follows that $A$ is a semisimple ring. \hfill \Box

2 Structure of Algebras

Lemma 11 (Wedderburn). If $A$ is a finite dimensional semisimple $F$-algebra then

$$A \cong M_{n_1 \times n_1}(D_1) \oplus \cdots \oplus M_{n_r \times n_r}(D_r)$$

where $D_i$ are division algebras over $F$.

Any algebra of the above form is semisimple and the expression is unique up to reordering. Moreover, the semisimple modules over $A$ are $D_i^{n_i}$ where the action is given by matrix multiplication.

Proof. Write $A = N_1^{n_1} \oplus \cdots \oplus N_r^{n_r}$ with $N_i$ pairwise nonisomorphic simple modules. Then $\text{End}_A(A) = \oplus \text{End}_A(N_i^{n_i}) = \oplus M_{n_i \times n_i}(D_i)$ where $D_i = \text{End}_A(N_i)$ is a division algebra. Have a natural map $A^{op} \cong \text{End}_A(A)$ given by $a \mapsto \langle b \mapsto ba \rangle$ and so

$$A \cong \oplus M_{n_i \times n_i}(D_i^{op})$$

Remark 12. The above lemma shows that $A$ is simple if and only if $A = M_n(D)$ where $D$ is a division algebra.

Corollary 13. If $A$ is a semisimple $F$-algebra and $M$ and $N$ are finite dimensional $A$ modules then $M \cong N$ if and only if $\text{Tr}a|_{M} = \text{Tr}a|_{N}$ for all $i \geq 0$ and $a \in A$. If $F$ has characteristic $0$ then it is enough to check $\text{Tr}a|_{M} = \text{Tr}a|_{N}$ for all $a$.

Proof. Let $M \cong \oplus P_i^{s_i}$ and $N \cong \oplus P_i^{t_i}$ where the $P_i$ are nonisomorphic simple $A$-modules. Clearly $M \cong N$ if and only if $s_i = t_i$ for all $i$, if and only if $\dim e_i M = \dim e_i N$ for $i = 1, \ldots, r$ where $e_i$ is the projector onto $P_i$: $e_i^2 = e_i$, $e_i = 1$ on $P_i$ and $e_i = 0$ on $P_j \neq P_i$. Then $\text{Tr} e_i|_M = \langle \dim P_i \rangle$ for all $j$ and the condition on traces becomes $(s_i \dim P_i) = (t_i \dim P_i)$ for all $j$.

If $F$ has characteristic $0$ then $\text{Tr} e_i|_M = s_i \dim P_i$ and $\text{Tr} e_i|_N = t_i \dim P_i$ so if $\text{Tr} e_i|_M = \text{Tr} e_i|_N$ then $s_i = t_i$ for all $i$. If $F$ has positive characteristic then the condition on traces implies that $(1 + x)^{s_i \dim P_i} = (1 + x)^{t_i \dim P_i}$ which implies that $s_i = t_i$ for a variable $x$. \hfill \Box

Definition 14. The $F$-algebra $A$ is said to be a central simple algebra if it is a simple finite dimensional algebra such that $F \cong Z(A)$.

Lemma 15 (Jacobson density theorem). Let $A$ be a finite dimensional $F$-algebra and let $M$ be a simple $A$-module. Let $D = \text{End}_A(M)$ (a division algebra by Schur’s lemma). Let $n_1, \ldots, n_r \in M$ be linearly independent over $D$ and let $a_1, \ldots, a_r \in M$. Then there exists $a \in A$ such that $am_i = a_i$ for all $i$. (In other words, “$A$ is close to $\text{End}_D(M)$.”)
That central algebra. Let \( B = A(m_1, \ldots, m_r) \oplus P \) and \( \text{End}_A(M^r) = M_{r \times r}(D) \) so there exists \( h \in M_{r \times r}(D) \) which is projection to \( A(m_1, \ldots, m_r) \). Then

\[
\begin{align*}
 f + \cdots + f(m_1, \ldots, m_r) &= (n_1, \ldots, n_r) \\
 f + \cdots + f(h(m_1, \ldots, m_r)) &= h(f + \cdots + 1)(m_1, \ldots, m_r) \\
 &= h(n_1, \ldots, n_r)
\end{align*}
\]

so \( h(n_1, \ldots, n_r) \in A(m_1, \ldots, m_r) \) and the conclusion follows. \( \square \)

**Lemma 16.** Let \( A \) be a central simple \( K \)-algebra. Then \( A \otimes_K A^{op} \cong \text{End}_K(A) \cong M_{n \times n}(K) \) where \( n = \dim_K A \).

**Proof.** \( A \otimes_K A^{op} \) acts on \( A \) with a left \( \otimes \) right action so get \( A \otimes_K A^{op} \to \text{End}_K(A) \). Let \( f \in \text{End}_K A \) and let \( a_1, \ldots, a_n \) be a basis of \( A \) as a \( K \)-vector space. Apply the Jacobson density theorem to the \( A \otimes_K A^{op} \)-module \( A \). We may do this because \( A \) is a simple \( A \otimes_K A^{op} \)-module. We get that there exists \( c \in A \otimes_K A^{op} \) such that \( ca_i = f(a_i) \) for all \( i \). Therefore \( c \) maps to \( f \) so \( A \otimes_K A^{op} \to \text{End}_K(A) \). A dimension comparison shows that this linear map is an isomorphism. \( \square \)

**Corollary 17.** If \( A \) is a central simple \( K \)-algebra and \( B \) is any simple \( K \)-algebra then \( A \otimes_K B \) is a simple \( K \)-algebra.

**Proof.** Let \( a_1, \ldots, a_n \) be a basis of \( A/K \). For \( i = 1, \ldots, n \) find \( c_i \in A \otimes_K A^{op} \) with \( c_i(a_j) = \delta_{ij} \). Let \( I \) be a two-sided ideal of \( A \otimes_K B \). If \( \sum a_j \otimes b_j \in I \) then \( \sum c_i(a_j) \otimes b_j \in I \) so \( 1 \otimes b_j \in I \cap K \otimes_K B \), where \( I \cap K \otimes B \) is a two-sided ideal of \( B \). Since \( B \) is simple, either \( I \cap K \otimes B = 0 \), in which case \( b_j = 0 \) so \( I = 0 \), or \( I \cap K \otimes B = K \otimes B \) in which case 1 \( \in I \) so \( I = A \otimes B \). \( \square \)

**Corollary 18.** Let \( A \) and \( B \) be central simple \( K \)-algebras. Then \( A \otimes_K B \) is also central simple.

**Proof.** That \( A \otimes B \) is simple follows from the previous corollary. Let \( a_i \) be a basis of \( A/K \) and let \( \sum a_i \otimes b_i \in Z(A \otimes_K B) \). For any \( b \in B \) we have \( (1 \otimes b)(\sum a_i \otimes b_i) = \sum a_i \otimes (bb_i - b_i b) = 0 \). Therefore \( bb_i = b_i b \) for all \( b \) so \( b_i \in Z(B) = K \). Thus \( \sum a_i \otimes b_i \in Z(A \otimes_K K) \otimes Z(A) = K \). \( \square \)

### 3 The Brauer Group

**Definition 19.** Two central simple \( K \)-algebras \( A \) and \( B \) are equivalent if there exists a division algebra \( D \) and two nonnegative integers \( r \) and \( s \) such that \( A \cong M_{r \times r}(D) \) and \( B \cong M_{s \times s}(D) \). Let \( \text{Br}(K) \) be the set of central simple \( K \)-algebras up to equivalence.

**Lemma 20.** The set \( \text{Br}(K) \) becomes an abelian group under \( \otimes_K \).

**Proof.** The identity element is \( [K] \) and the inverse of \( A \) is \( A^{op} \): \([A]\otimes_K [A^{op}] = [A \otimes_K A^{op}] = [M_{n \times n}(K)] = [K] \). \( \square \)

**Definition 21.** For \( L/K \) a field extension there is a natural map \( \text{Br}(K) \to \text{Br}(L) \) given by \([A] \mapsto [A \otimes_K L]\). Let \( \text{Br}(L/K) = \ker(\text{Br}(K) \to \text{Br}(L)) \).

**Lemma 22** (Double centralizer theorem). Let \( A \) be a central simple \( K \)-algebra and let \( B \subset A \) be a \( K \)-subalgebra. Let \( C_A(B) = \{ c \in A | cb = bc, \forall b \in B \} \) be the centralizer of \( B \) in \( A \). Then

1. \( C_A(B) \) is simple.
2. \( \dim_K C_A(B) \dim_K B = \dim_K A \).
3. $C_A(C_A(B)) = B$.

Proof. Since $B \subseteq A$ it follows there exists $n$ and a division algebra $D$ such that $B \otimes_K A^{\text{op}} = M_{n \times n}(D)$ ($[B][A^{\text{op}}] = [K]$). Therefore there exists an integer $r$ such that $A \cong (D^r)^r$ as a $M_{n \times n}(D) = B \otimes_K A^{\text{op}}$ module. Note that $C_A(B) = \text{End}_{B \otimes_K A^{\text{op}}}(A)$ ($A \cong \text{End}_{A \otimes_K A^{\text{op}}}(A)$). But $\text{End}_{B \otimes_K A^{\text{op}}}(A) = M_{r \times r}(D^{\text{op}})$ which implies that $C_A(B)$ is simple, as matrix algebras are simple.

Also, $\dim_K C_A(B) = r^2 \dim_K D^{\text{op}} = r^2 \dim_K D$ and $\dim_K B \dim_K A = \dim_K (B \otimes_K A^{\text{op}}) = n^2 \dim_K D$. Therefore $\dim_K A = rn \dim_K D$ which implies the second part.

Finally, $B \subseteq C_A(C_A(B))$ and a dimension comparison implies isomorphism.

Corollary 23. Let $D/K$ be a division algebra. Then $\dim_K D$ is a square number and any maximal subfield of $D$ has dimension $\sqrt{\dim_K D}$.

Proof. Let $L \subseteq D$ be a maximal subfield. Then $C_D(L) \subseteq L$. If $L \neq C_D(L)$ choose $x \in C_D(L) - L$ in which case $L(x)$ is a commutative division algebra, so a field, which contradicts the choice of $L$. Therefore $L = C_D(L)$ and the previous lemma implies that $(\dim_K L)^2 = \dim_K D$.

Corollary 24. Let $A$ be a central simple $K$-algebra and let $L$ be a maximal subfield of $A$. Then $A \otimes_K L \cong M_{n \times n}(L)$ for some $n$, i.e., $[A] \in \text{Br}(L/K)$.

Proof. Let $L \subseteq C_A(L) \cong M_{r \times r}(D)$ for some division algebra $D$. Then $L \subseteq Z(C_A(L)) = D$ so $L \subseteq D$. Again, by maximality of $L$ we deduce that $L = D$ so $C_A(L) \cong M_{r \times r}(L)$, but this implies (as in the proof of the double centralizer theorem) that $L \otimes_K A^{\text{op}} \cong M_{n \times n}(L)$.