p-ADIC FAMILIES AND GALOIS REPRESENTATIONS FOR GSp(4) AND GL(2)

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Abstract. In this brief article we prove local-global compatibility for holomorphic Siegel modular forms with Iwahori level. In previous work we proved a weaker version of this result (up to a quadratic twist) and one of the goals of this article is to remove this quadratic twist by different methods, using p-adic families. We further study the local Galois representation at p for nonregular holomorphic Siegel modular forms. Then we apply the results to the setting of modular forms on GL(2) over a quadratic imaginary field and prove results on the local Galois representation ℓ, as well as crystallinity results at p.

Introduction

Let π be an irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ such that π_∞ is a holomorphic discrete series representation, and such that the functorial lift of π to GL(4) (whose existence is guaranteed by [22]) is a cuspidal representation. Then for every prime number ℓ there exists a continuous Galois representation $\rho_{\pi,\ell} : G_\mathbb{Q} \to \mathrm{GL}(4, \overline{\mathbb{Q}}_\ell)$ such that $L^S(\pi, \mathrm{spin}, s - \frac{3}{2}) = L^S(\rho_{\pi,\ell}, s)$ for a finite set S of "bad" primes (cf. [18], [13], [23], [22], [16]; for more details see §1).

This article is concerned, among others, with the local Galois representations $\rho_{\pi,\ell}|_{G_{\mathbb{Q}_p}}$ (via the associated Weil-Deligne representation) when $p \in S$, the cases $p \neq \ell$ and $p = \ell$ being interrelated.

Theorem A. *For π as above, if $\ell \neq p > 2$ and π_p is Iwahori-spherical, then*

$$\mathrm{WD}(\rho_{\pi,\ell}|_{G_{\mathbb{Q}_p}})^{\mathrm{ss}} \cong \iota \,\mathrm{rec}(\pi_p \otimes ||^{-3/2})^{\mathrm{ss}}$$

where $\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$, and rec *is the local Langlands correspondence for* GSp(4)*. If, moreover, π_p is assumed to be tempered or generic, then*

$$\mathrm{WD}(\rho_{\pi,\ell}|_{G_{\mathbb{Q}_p}})^{\text{Fr-ss}} \cong \iota \,\mathrm{rec}(\pi_p \otimes ||^{-3/2})$$

A previous result of the author's (Theorem 1.2) obtained the local-global compatibility result of Theorem A potentially up to a quadratic twist, via the doubling method and local converse theorems. We prove Theorem A using a different approach: knowing that local-global compatibility is satisfied up to a potentially quadratic twist allows one to move between the p-adic and the ℓ-adic Galois representations and the potential quadratic twist is removed using Kisin's work on crystalline periods on eigenvarieties.

We would like to remark that although Theorem A is obtained for Iwahori spherical representations, there are methods to deduce local-global compatibility in general from this setting, using strong base change, which is not yet available for non globally generic representations on GSp(4). Such a method was successfully used, e.g., by [3], based on the analogous result for Iwahori level representations in [2], the idea being that one can use solvable base change to reduce the ramification to the case of Iwahori level. One does have base change results for GSp(4) using functorial lifts to GL(4), but they do not suffice since such functorial lifts are not proven to be strong.

One may wonder what the relevance of these arguments is in the context of Arthur's book(s) on functorial transfer between matrix groups. Arthur's functorial transfers do not give a local identification of L-parameters, but character identities; to get from these character identities to L-parameters one needs to overcome nontrivial technical difficulties. The proof of Theorem A can be thought of as bypassing these technical difficulties.

The second result of this article concerns regular algebraic cuspidal representations π of $\mathrm{GL}(2, \mathbb{A}_K)$ where K is an imaginary quadratic field. Assuming that the central character of π is base changed from $\mathbb{Q}$, for a prime ℓ there exists a continuous Galois representation $\rho_{\pi,\ell} : G_K \to \mathrm{GL}(2, \overline{\mathbb{Q}}_\ell)$ such that $L^S(\pi, s - \frac{1}{2}) = L^S(\rho_{\pi,\ell}, s)$ for a finite set of places S (cf. [9], [19], [4]). The following theorem is the main result of the author's doctoral thesis ([10]), and answers a question posed by Andrew Wiles:

Theorem B. *Let π be as above, and let $v \notin S$ be a place of K. If $v = p$ is inert, assume that the Satake parameters of π_v are distinct; if $p = v \cdot v^c$ is split, assume that the four Satake parameters of π_v and π_{v^c} are distinct. Then $\rho_{\pi,p}|_{G_{K_v}}$ is a crystalline representation.*

The condition on the Satake parameters being distinct is structural to the argument; in fact, one doesn't even obtain that the representation is Hodge-Tate without this assumption. In the case when π is ordinary at p, this result followed from [20].

Our final result concerns the ℓ-adic local representations associated to π:

Theorem C. *Let π be as above, and let v be a place of K such that $K_v/\mathbb{Q}_p$ is unramified and π_v (and π_{v^c}, if v is split) are Iwahori-spherical. Then for $\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ we have $\mathrm{WD}(\rho_{\pi,\ell}|_{G_{K_v}})^{\mathrm{ss}} \cong \iota\, \mathrm{rec}(\pi_v|\,|^{-1/2})^{\mathrm{ss}}$.*

We would like to remark that, since strong cyclic base change is available for GL(2), one could extend this result in general, assuming that Theorem A is extended to totally real fields. This would require two ingredients: one is a weak functorial lift from GSp(4) to GL(4) over totally real fields, which should follow from the work of Arthur, and two an extension of Kisin's results on crystalline periods to extensions of $\mathbb{Q}_p$.

The article is organized as follows: in §1 we list previous results for GSp(4) over $\mathbb{Q}$ and GL(2) over K; in §2 we describe p-adic families of holomorphic Siegel modular forms, and generic classical points in such families. In §3 we deduce information about the local Galois representations for both regular and nonregular holomorphic Siegel modular forms. Finally, in §4 we study the local Galois representations attached to regular algebraic cuspidal representations on $\mathrm{GL}(2, \mathbb{A}_K)$.

After the completion of the research presented in this article the author was made aware of two recent preprints of C.P. Mok, one generalizing the author's results on families of Siegel modular forms to Siegel-Hilbert modular forms, and another, studying Galois representations attached to Siegel-Hilbert modular forms, whose results at $\ell = p$ are extensions of the author's thesis.

1. Notations and Known Results

We begin by recalling some notation. If K is a number field, $\mathbb{A}_K$ is the ring of adèles. The group GSp(4) consists of 4×4 matrices such that $g^t J g = \lambda(g) J$ where $\lambda(g)$ is the multiplier character and $J = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}$, while $\mathrm{Sp}(4) = \ker \lambda$. The spin representation spin : $\mathrm{GSp}(4) \to \mathrm{GL}(4)$ gives the spin L-function $L^S(\pi, \mathrm{spin}, s)$, while the standard representation std : $\mathrm{GSp}(4) \to \mathrm{GL}(5)$ gives the standard L-function $L^S(\pi, \mathrm{std}, s)$, the latter being defined at all places, using the doubling method. A Frobenius semisimple Weil-Deligne representation is a pair (r, N) of a semisimple continuous Galois representation $r : G_K \to \mathrm{GL}(V)$ and a nilpotent matrix $N \in \mathrm{End}(V)$ such that $r(g) \circ N = |\operatorname{rec}^{-1}(g)|_K N \circ r(g)$; given a continuous ℓ-adic Galois representation of G_{K_v} where $v \mid p$ one obtains an associated Weil-Deligne representation via Grothendieck's ℓ-adic monodromy theorem, while given a continuous p-adic Galois representation one obtains a Weil-Deligne representation using p-adic Hodge theory.

Let us now recall the result on the existence of Galois representations attached to Siegel modular forms.

Theorem 1.1. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ such that π_∞ is a holomorphic discrete series representation, and such that the functorial lift of π to $\mathrm{GL}(4)$ is cuspidal. Then for every prime number ℓ there exists a continuous Galois representation $\rho_{\pi,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}(4, \overline{\mathbb{Q}}_\ell)$ and a finite set S of places such that $L^S(\pi, \mathrm{spin}, s) = L^S(\rho_{\pi,\ell}, s)$. In particular, if $p \notin S$ then $\rho_{\pi,\ell}$ is unramified at p. Moreover, $\rho_{\pi,p}$ is crystalline at p, and the characteristic polynomial of Φ_{cris} equals the characteristic polynomial of Frob_p acting on $\rho_{\pi,\ell}$ for $\ell \neq p$.*

The compatible system of ℓ-adic representations was first constructed by Taylor in [18], where he was able to deduce the theorem for S of density 0. As stated, the theorem was finalized by Laumon in [13] and Weissauer in [23]. When π_∞ is a discrete series representation which is not holomorphic, but such that π is globally generic, the construction of the Galois representation is due to [16]. Weissauer's results on global L-packets [22] provides an alternative construction of the Galois representations in the theorem. Finally, the result on Φ_{cris} was first proven by Urban in [20] studying Hecke operators in the boundary of Shimura varieties for $\mathrm{GSp}(4)$, but can also be deduced from Sorensen's construction.

In [11] we proved that if for $\ell \neq p > 2$ the local representation π_p is a constituent of an induced representation from the Borel subgroup then local-global compatibility is satisfied up to semisimplification and a quadratic twist.

Theorem 1.2. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ such that π has a cuspidal lift to $\mathrm{GL}(4, \mathbb{A}_{\mathbb{Q}})$ and such that π_∞ is a holomorphic discrete series representation. Let p be a finite place such that π_p is a subrepresentation of an representation induced from a Borel subgroup. If π_p is assumed to be either tempered or generic, then for every $\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ we have*

$$\mathrm{WD}(\rho_{\pi,\ell}|_{G_{\mathbb{Q}_p}})^{\text{Fr-ss}} \cong \iota \operatorname{rec}(\pi_p \otimes |\lambda|^{-3/2}\eta)$$

where rec *is the local Langlands correspondence for* $\mathrm{GSp}(4)$ *defined by Gan and Takeda in* [7], *λ is the multiplier character, and η is either trivial, or a quadratic character. If π_p is not assumed to be tempered or generic, the above equality of Weil-Deligne representations holds up to semisimplification.*

This was proven in [11] using local converse theorems for the standard γ-factors for $\mathrm{Sp}(4)$ arising from the doubling method in conjunction with Casselman's proof of multiplicity one for GL(2) using the global functional equation, as well as Sorensen's result on local-global compatibility for globally generic Siegel modular forms ([16]). The reason the possibily quadratic character η appears is the following: the standard representation loses information about the multiplier character, but we may recover the square of the multiplier character using the determinant. We do not go into the details of the proof of this result, as the proof of Theorem A is via a different method.

If in the cuspidal representation π the infinite component π_∞ is a holomorphic limit of discrete series, one can still associate continuous ℓ-adic Galois representations $\rho_{\pi,\ell}$ to π; they were constructed using p-adic congruences (families) by Taylor ([17]), obtaining local-global compatibility at unramified primes for $\ell \neq p$ (at almost all places; the compatibility at all but finitely many places was later completed by Laumon [13] and Weissauer [23]).

Finally, let us recall the existence of Galois representations attached to regular algebraic cuspidal automorphic representations of $\mathrm{GL}(2,\mathbb{A}_K)$ where K is a quadratic imaginary field.

Theorem 1.3. *Let π be a regular algebraic cuspidal automorphic representations of $\mathrm{GL}(2,\mathbb{A}_K)$ where K is a quadratic imaginary field, such that the central character χ_π has the property that $\chi_\pi = \chi_\pi^c$, where c is complex conjugation. Then for each prime ℓ there exists a continuous Galois representation $\rho_{\pi,\ell} : G_K \to \mathrm{GL}(2,\overline{\mathbb{Q}}_\ell)$ and a finite set S of places of K such that if $v \notin S$ then $\rho_{\pi,\ell}$ is unramified at v and $L^S(\pi,s) = L^S(\rho_{\pi,\ell}, s-1/2)$. The finite set S consists of the infinite places and finite places v such that either $K_v/\mathbb{Q}_p$ is ramified, or one of π_v and π_{v^c} is ramified.*

It is useful to summarize the construction of these Galois representations. There are three possibilities. Either $\pi \otimes \delta \cong \pi$ for some quadratic character δ, in which case π is the automorphic induction of a character of the splitting field of δ, and the Galois representation and its properties follow from global class field theory; or $\pi\otimes\nu \cong (\pi\otimes\nu)^c$ for some character ν, in which case $\pi \otimes \nu$ is a base change from $\mathbb{Q}$, and the Galois representation and its properties follow from the theory over $\mathbb{Q}$; or in the remaining cases, we may use that there is an accidental isomorphism $(\mathrm{GL}(2,\mathbb{A}_K) \times \mathbb{A}_\mathbb{Q}^\times)/\{(xI_2, N_{K/\mathbb{Q}}(x)) | x \in \mathbb{A}_K^\times\} \cong \mathrm{GSO}(V_K, \mathbb{A}_\mathbb{Q})$ where V_K is a four dimensional quadratic vector space over $\mathbb{Q}$ such that the signature of $V_K \otimes \mathbb{R}$ is $(3,1)$. Then for sufficiently many finite order characters μ and suitable choices of lifts $\widehat{\pi\otimes\mu}$ of $\pi\otimes\mu$ from $\mathrm{GSO}(V_K,\mathbb{A}_\mathbb{Q})$ to $\mathrm{GO}(V_K,\mathbb{A}_\mathbb{Q})$, the theta transfer $\Theta(\widehat{\pi\otimes\mu})$ from $\mathrm{GO}(V_K,\mathbb{A}_\mathbb{Q})$ to $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$ is an irreducible cuspidal automorphic representation Π^μ such that Π^μ_∞ is a holomorphic limit of discrete series with Harish-Chandra weight $(k-1,0)$, where $k \geq 2$ is the integer such that the Langlands parameter of π_∞ is $z \mapsto \begin{pmatrix} z^{1-k} & \\ & \overline{z}^{1-k} \end{pmatrix}$. Then one can recover the Galois representation $\rho_{\pi,\ell}$ such that for every chosen μ one has $\rho_{\Pi^\mu,\ell} = \mathrm{Ind}_K^\mathbb{Q}(\rho_{\pi,\ell}\otimes\mu)$. Presupposing the existence of Galois representations for holomorphic Siegel modular forms, this was achieved by Harris-Soudry-Taylor ([9]), Taylor ([19]) and Berger-Harcos ([4]).

2. p-adic Families of Holomorphic Siegel Modular Forms

The proof of Theorem A will require switching from ℓ in Theorem 1.2 to p and using p-adic families of holomorphic Siegel modular forms. Rigid analytic families of finite slope overconvergent Siegel modular forms have now been constructed by several methods. The first one, due to Urban ([21] using overconvergent cohomology), works for regular forms, the second, due to the author ([10, Chapter 3]) generalizes the work of Kisin and Lai in the context of Hilbert modular forms, while a third one is due to Andreatta, Iovita and Pilloni ([1]). In particular, we have ([10, Proposition 4.2.6]):

Theorem 2.1. *Let π be a cuspidal representation of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ such that π_∞ is a holomorphic (limit of) discrete series of Harish-Chandra parameter κ. Assume also that π has nontrivial invariant under the group $U_{00}(Np)$, where $U_{00}(Np)$ contains matrices $\equiv \begin{pmatrix} I_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix}$ (mod Np). Then there exists a one-dimensional rigid analytic neighborhood $\mathcal{W}$ of κ and a rigid family $\mathcal{E}$ over $\mathcal{W}$, parametrizing systems of Hecke eigenvalues attached to finite-slope overconvergent holomorphic Siegel modular forms of level $\Gamma_{00}(Np)$. Moreover, there exists an analytic Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(4, \mathcal{O}_{\mathcal{E}})$ and a dense set of classical points f_t on $\mathcal{E}$ with weight $\kappa + p^t(p-1, p-1)$ such that the specialization of ρ at f_t is the Galois representation attached to f_t.*

Remark. We would like to remark that the rigid variety in Theorem 2.1 was obtained using $\mathbb{Z}_p$-exponents of a specific Eisenstein series, in the style of Coleman-Mazur and Kisin-Lai, and thus it is necessarily one-dimensional. Moreover, p-power exponents of the Hasse invariant times the original Siegel modular form provide the dense set of classical points which, crucially, also are of the same level, since the Hasse invariant has level 1.

According to the previous remark we observe that if the local representation π_p is an unramified principal series (in other words, if the Siegel modular eigenform in π is old at p) then the dense set of classical points f_t on the eigenvariety contains Siegel modular forms which are old at p. In effect, this says that the generic classical points converging to π are unramified at p. We would like a similar statement for any π_p of Iwahori level. Indeed, if π_p is Iwahori then by [6, Proposition 6.4.7] then the generic *very classical* points on the eigenvariety will be old at p.

3. Galois Representations for Siegel Modular Forms

Our first application of the existence of the one dimensional eigenvariety described in Theorem 2.1 is to the study of crystallinity of the p-adic Galois representations attached to nonregular holomorphic Siegel modular forms which are unramified at p. The following theorem is based on [10, Chapter 4].

Theorem 3.1. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ such that π_∞ is the limit of discrete series representation of Harish-Chandra parameter $(k, 0)$ and such that the smooth representation π_p of $\mathrm{GSp}(4, \mathbb{Q}_p)$ is an unramified principal series with Satake parameters $\alpha, \beta, \gamma, \delta$. Then $\dim_{\mathbb{Q}_p} D_{\mathrm{cris}}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \geq \#\{\alpha, \beta, \gamma, \delta\}$. In particular, if the Satake parameters are distinct, the p-adic Galois representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is crystalline.*

Proof. Let α be one of the Satake parameters and let f_α be one of the p-stabilizations of the eigenform f in π. The form f_α has level $\Gamma_{00}(Np)$ and is old at p, and is an eigenform of the $U_{p,1}$ operator with eigenvalue α. (To see why such p-stabilizations exist for Siegel modular forms see [10, pp. 21-22].) Also let f_n be the holomorphic Siegel modular forms of level $(k, 0) + p^n(p-1, p-1)$ for $n >> 0$ appearing in fE^{p^n}, where E is the Hasse invariant; let $f_{\alpha,n}$ be a p-stabilization of f_n such that $f_{\alpha,n}$ corresponds to the classical points on the eigencurve converging to f_α in Theorem 2.1.

Then $f_{\alpha,n}$ will have the same level as f_n which is unramified at p. Therefore, $f_{\alpha,n}$ generates an automorphic representation $\pi_{\alpha,n}$ which is unramified at p. But then $\rho_{\pi_{\alpha,n},p}|_{G_{\mathbb{Q}_p}}$ will be crystalline, and by Theorem 1.1, one may find an eigenvalue α_n of Φ_{cris} such that $\alpha_n \equiv \alpha \pmod{p^n}$. In fact, α_n is the $U_{p,1}$-eigenvalue of $f_{\alpha,n}$.

We now make recourse to Kisin's result on analytically varying crystalline periods on eigenvarieties [12, Corollary 5.15]. The result in question states that, since the infinitely many points $f_{\alpha,n}$ lie on a one-dimensional rigid variety and are thus dense, the fact that $\dim D_{\mathrm{cris}}(\rho_{\pi_n,p}|_{G_{\mathbb{Q}_p}})^{\Phi_{\mathrm{cris}}=\alpha_n} \geq 1$ implies that $\dim D_{\mathrm{cris}}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}})^{\Phi_{\mathrm{cris}}=\alpha} \geq 1$. Repeating this argument for each of the four Satake parameters will lead to the theorem. ☐

Our second application is to the study of ramified Galois representations attached to Iwahori level regular holomorphic Siegel modular forms.

Theorem A. *For π a cuspidal representation of $\mathrm{GSp}(4,\mathbb{A}_{\mathbb{Q}})$, not CAP, and such that π_∞ is a holomorphic discrete series, if $\ell \neq p > 2$ and π_p is Iwahori-spherical, then*

$$\mathrm{WD}(\rho_{\pi,\ell}|_{G_{\mathbb{Q}_p}})^{\mathrm{ss}} \cong \iota\,\mathrm{rec}(\pi_p \otimes ||^{-3/2})^{\mathrm{ss}}$$

where $\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$. If, moreover, π_p is assumed to be tempered or generic, then

$$\mathrm{WD}(\rho_{\pi,\ell}|_{G_{\mathbb{Q}_p}})^{\text{Fr-ss}} \cong \iota\,\mathrm{rec}(\pi_p \otimes ||^{-3/2})$$

Proof. Let π_p be the smooth Iwahori-spherical representation of $\mathrm{GSp}(4,\mathbb{Q}_p)$ in the automorphic representation π. The representation π_p, according to the Sally-Tadić classification ([15]), falls into one of six classes of representations, and is a quotient of an unramified induction from the Borel subgroup. Let X be the set of $U_{p,1}$-eigenvalues of the p-stabilizations of π. Then in the eigenvariety the Siegel modular forms f_n will generically be unramified at p, and thus by the same argument as in the proof of Theorem 3.1 it follows that the crystalline Frobenius acting on $D_{\mathrm{cris}}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}})$ will have elements of X as eigenvalues.

Let π^g be the globally generic cuspidal automorphic form weakly equivalent to π whose existence is guaranteed by [22] (and used in the proof of Theorem 1.2). Then $\pi_p \cong \pi_p^g \otimes \eta$. By [3, Theorem A] it follows, since $\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}})^{\text{Fr-ss}} \cong \iota\,\mathrm{rec}(\pi_p \otimes |\lambda|^{-3/2}\eta)$ has crystalline periods, that the quadratic character η is unramified. (Note that we are allowed to use the aforementioned result since functorial lifts of regular cuspidal automorphic representations to GL(4) are readily Shin-regular.) If nontrivial, it must be that $\eta(p) = -1$, and thus it follows that if α is a crystalline period, then $\alpha\eta(p) = -\alpha$ is also a crystalline period.

To show this cannot happen we must use the explicit description of the reciprocity map for Iwahori-spherical representations, that can be found in [14, §A.5], as well as [3, Theorem A] (although, since we already know we are using Iwahori-level forms, it would be enough to use [2, Theorem A]). Writing E_{ij} for the 4×4 matrix with 1 at position (i,j) and 0 elsewhere, let $N_1 = E_{23}, N_2 = E_{14}, N_3 = N_1 + N_2, N_4 = E_{12} - E_{34}$ and $N_5 = N_1 + N_4$. The following is a summary of the classes of Iwahori-spherical representations, the semisimple L-parameter, and the possible monodromy matrices, contained in [14, Table A.7]:

Class	$\mathrm{rec}^{\mathrm{ss}}$	N
I	$\chi_1\chi_2\sigma, \chi_1\sigma, \chi_2\sigma, \sigma$	0
II	$\chi^2\sigma, \nu^{1/2}\chi\sigma, \nu^{-1/2}\chi\sigma, \sigma$	$0, N_1$
III	$\nu^{1/2}\chi\sigma, \nu^{-1/2}\chi\sigma, \nu^{1/2}\sigma, \nu^{-1/2}\sigma$	$0, N_4$
IV	$\nu^{3/2}\sigma, \nu^{1/2}\sigma, \nu^{-1/2}\sigma, \nu^{-3/2}\sigma$	$0, N_1, N_4, N_5$
V	$\nu^{1/2}\sigma, \nu^{1/2}\xi\sigma, \nu^{-1/2}\xi\sigma, \nu^{-1/2}\sigma$	$0, N_1, N_2, N_3$
VI	$\nu^{1/2}\sigma, \nu^{1/2}\sigma, \nu^{-1/2}\sigma, \nu^{-1/2}\sigma$	$0, N_1, N_3$

The case of class I is treated already in Theorem 1.1. Writing $s = \sigma(p), c = \chi(p)$ and noting that $\xi(p) = -1$ the possible crysalline eigenvalues on rec are the following

Class	Eigenvalues
II	$c^2s, p^{-1/2}cs, s$ or $c^2s, p^{1/2}cs, p^{-1/2}cs, s$
III	$p^{-1/2}cs, p^{-1/2}s$ or $p^{1/2}cs, p^{-1/2}cs, p^{1/2}s, p^{-1/2}s$
IV	$p^{-3/2}s$ or $p^{1/2}s, p^{-3/2}s$ or $p^{3/2}s, p^{-1/2}s, p^{-3/2}s$ or $p^{3/2}s, p^{1/2}s, p^{-1/2}s, p^{-3/2}s$
V	$-p^{-1/2}s, p^{-1/2}s$ or $p^{1/2}s, -p^{-1/2}s, p^{-1/2}s$ or $-p^{1/2}s, -p^{-1/2}s, p^{-1/2}s$ or $p^{1/2}s, -p^{1/2}s, -p^{-1/2}s, p^{-1/2}s$
VI	$p^{-1/2}s, p^{-1/2}s$ or $p^{1/2}s, p^{-1/2}s, p^{-1/2}s$ or $p^{1/2}s, p^{1/2}s, p^{-1/2}s, p^{-1/2}s$

The only class where there exists the possibility that the set of crystalline eigenvalues is closed under negation is Vd, in which case the Galois representation is isomorphic to its quadratic twist and so there is nothing to prove. □

4. Galois Representations for GL(2) over Quadratic Imaginary Fields

We will study on the one hand the p-adic Galois representation $\rho_{\pi,p}$ at places $v \notin S$, and on the other hand the ℓ-adic Galois representation $\rho_{\pi,\ell}$ at certain places $v \in S$. Theorem B is the main result of the author's doctoral thesis, and is [10, Theorem 5.3.1].

Theorem B. *Let π be a regular algebraic cuspidal representation of $\mathrm{GL}(2, \mathbb{A}_K)$ such that the central character of π is base changed from $\mathbb{Q}$, and let $v \notin S$ be a place of K. If $v = p$ is inert, assume that the Satake parameters of π_v are distinct; if $p = v \cdot v^c$ is split, assume that the four Satake parameters of π_v and π_{v^c} are distinct. Then $\rho_{\pi,p}|_{G_{K_v}}$ is a crystalline representation.*

Proof. Let's start with $p = v$ inert. Choose μ such that μ_v is trivial and such that $\rho_{\pi,p} \otimes \mu \ncong (\rho_{\pi,p} \otimes \mu)^c$. Then $\mathrm{Ind}_K^{\mathbb{Q}}(\rho_{\pi,p} \otimes \mu)|_{G_{\mathbb{Q}_p}} = \mathrm{Ind}_{K_v}^{\mathbb{Q}_p}(\rho_{\pi,p}|_{G_{K_v}})$. But $D^*_{\mathrm{cris}}(\rho_{\Pi^\mu,p}|_{G_{K_p}}) \cong D^*_{\mathrm{cris}}(\rho_{\pi,p}|_{G_{K_v}})$ is a four dimensional $\mathbb{Q}_p$-vector space if and only if $D^*_{\mathrm{cris}}(\rho_{\pi,p}|_{G_{K_v}})$ is a two dimensional K_v-vector space (note that $K_v/\mathbb{Q}_p$ is unramified since $v \notin S$). Thus it is enough to show that $\rho_{\Pi^\mu,p}|_{G_{K_v}}$ is crystalline.

Similarly, in the case $p = v \cdot v^c$ split, choose μ such that μ_v and μ_{v^c} are trivial. Then $\mathrm{Ind}_K^{\mathbb{Q}}(\rho_{\pi,p} \otimes \mu)|_{G_{\mathbb{Q}_p}} = \rho_{\pi,p}|_{G_{K_v}} \oplus \rho_{\pi,p}|_{G_{K_{v^c}}}$. Then $D^*_{\mathrm{cris}}(\rho_{\Pi^\mu,p}|_{G_{K_v}}) \cong D_{\mathrm{cris}}(\rho_{\pi,p}|_{G_{K_v}}) \oplus D_{\mathrm{cris}}(\rho_{\pi,p}|_{G_{K_{v^c}}})$ is crystalline if and only if both $\rho_{\pi,p}|_{G_{K_v}}$ and $\rho_{\pi,p}|_{G_{K_{v^c}}}$ are crystalline. Again, it is enough to show that $\rho_{\Pi^\mu,p}|_{G_{K_v}}$ is crystalline.

Let α_v and β_v be the Satake parameters of π_v. If $p = v$ is inert then the Satake parameters of Π_p^μ are $\pm\sqrt{\alpha_v}, \pm\sqrt{\beta_v}$, which are all distinct; if $p = v \cdot v^c$ then the Satake parameters of Π_p^μ are $\alpha_v, \beta_v, \alpha_{v^c}, \beta_{v^c}$ which are assumed to be distinct. Therefore, by Theorem 3.1 it follows that $\rho_{\Pi^\mu,p}|_{G_{K_v}}$ is crystalline. □

Finally, we apply the local-global compatibility result in Theorem A to the study of the ℓ-adic Galois representations in $\rho_{\pi,\ell}$ at certain bad places v.

Theorem C. *Let π be as above, and let v be a place of K such that $K_v/\mathbb{Q}_p$ is unramified and π_v (and π_{v^c}, if v is split) are Iwahori-spherical. Then for $\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ we have* $\mathrm{WD}(\rho_{\pi,\ell}|_{G_{K_v}})^{\mathrm{ss}} \cong \iota\,\mathrm{rec}(\pi_v||^{-1/2})^{\mathrm{ss}}$.

Proof. Choose μ as in the proof of Theorem B. Then Π^μ_p is Iwahori-spherical (this follows in the case of $p = v \cdot v^c$ split from [8, Theorem A.10 (i,iv,v,vi)], and in the case that $p = v$ is inert from [8, Theorem A.11 (iii,iv)]). Finally, finding congruences between Π^μ and regular holomorphic Siegel modular forms of (necessarily) Iwahori level, Theorem A gives $\mathrm{WD}(\rho_{\Pi^\mu,\ell}|_{G_{\mathbb{Q}_p}})^{\mathrm{ss}} \cong \iota\,\mathrm{rec}(\Pi_p||^{-3/2})^{\mathrm{ss}}$, since a limit of unramified characters stay an unramified character. Finally, an argument as in [4, §6] gives the required local-global compatibility. □

5. Concluding Remarks

The proof of Theorems 1.2 and A carry over to totally real fields as long as the following are assumed: first, a convenient reference for functorial transfer from GSp(4) to GL(4) is made available; this transfer should follow from the work of Arthur, or alternatively from the work of Wesselman; second, a generalization of Kisin's results on crystalline periods to finite extensions of $\mathbb{Q}_p$.

In Theorems 1.2 and A one downside is that for representations which are not tempered or generic one gets local-global compatibility up to semisimplification. A comparison of monodromy operators will likely follow from the program initiated by Caraiani in [5] to prove the Ramanujan-Petersson conjecture for $\mathrm{GL}(n)$.

As already mentioned, extending Theorem C can be approached using (strong) base change for GL(2) as well as patching arguments as in [3].

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