# On the Teaching of the Lumped Model for Unsteady Heat Conduction: Natural Convection Versus Forced Convection

**Antonio Campo[1]**

Abstract –Within the lumped model platform for unsteady heat conduction, the Biot number criterion between a solid body and a surrounding fluid requires that $Bi = \frac{\bar{h}}{k_s}\left(\frac{V}{S}\right) < 0.1$. Although not clearly stated, this holds true mostly for forced convection where the mean convective coefficient $\bar{h}$ is affected by the impressed fluid velocity. Conversely, when heat is exchanged by natural convection, the Biot number criterion involves a mean convective coefficient $\bar{h}$ that depends on the temperature difference between the body and the fluid. Consequently, the above-cited Biot number criterion must be modified to incorporate the variability of the mean convective coefficient $\bar{h}$ with the temperature difference. This situation gives rise to a new $Bi_{max} = \frac{\bar{h}_{max}}{k_s}\left(\frac{V}{S}\right) < 0.1$ where $\bar{h}_{max}$ is the maximum mean convective coefficient that, in the case of cooling, happens at the initial temperature $T_i$ and time t = 0. In this paper on engineering education, the exact mean temperature distribution T (t) is deduced for a case study wherein the solid body is a sphere being cooled by natural convection in quiescent air under the premises of the lumped model. A physics-based equivalence of $Bi_{max}$ interweaves the solid thermal conductivity, the fluid thermal conductivity and the extended Grashof number embracing the initial–to–fluid temperature difference.

Keywords – Lumped Model, Unsteady Heat Conduction, Natural Convection, Mean Convective Coefficient, Modified Biot Number Criterion

## NOMENCLATURE

| | |
|---|---|
| a, b | coefficients in eq. (9a) |
| Bi | Biot number for lumped body, $\frac{\bar{h}}{k_s}\left(\frac{V}{S}\right)$, dimensionless |
| $Bi_{max}$ | modified Biot number for lumped body with natural convection cooling, $\frac{\bar{h}_{max}}{k_s}\left(\frac{V}{S}\right)$, dimensionless |
| $c_p$ | specific heat capacity at constant pressure, J/kg.°C |
| $c_v$ | specific heat capacity at constant volume, J/kg.°C |
| D | diameter of sphere, m |
| g | gravitational acceleration, m/s$^2$ |
| $Gr_D$ | Grashof number for sphere, $g\beta\frac{\rho^2}{\mu^2}(T_s - T_\infty)\ D^3$, dimensionless |
| $Gr_{D,i}$ | extended Grashof number for sphere, $g\beta\frac{\rho^2}{\mu^2}(T_i - T_\infty)\ D^3$, dimensionless |
| $\bar{h}$ | mean convective coefficient, W/m$^2$.°C |
| $\bar{h}_{max}$ | maximum mean convective coefficient, W/m$^2$.°C |
| k | thermal conductivity, W/m.°C |
| $\overline{Nu}_D$ | mean Nusselt number for sphere, $\frac{\bar{h}D}{k_f}$, dimensionless |
| Pr | Prandtl number, $\frac{\mu c_{p,f}}{k_f}$, dimensionless |
| R | radius of sphere, m |
| $Ra_D$ | Rayleigh number for sphere, $Gr_D$ Pr, dimensionless |
| $Ra_{D,i}$ | extended Rayleigh number for sphere, $Gr_{D,i}$ Pr, dimensionless |
| S | surface area, m$^2$ |
| t | time, s |
| T | mean temperature, °C |
| $T_i$ | initial temperature, °C |

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| | |
|---|---|
| $T_s$ | surface temperature, °C |
| $T_\infty$ | fluid temperature, °C |
| V | volume, $m^3$ |

**GREEK LETTERS**

| | |
|---|---|
| β | coefficient of volumetric thermal expansion, 1/K |
| θ | temperature excess, $T - T_\infty$, °C |
| μ | dynamic viscosity, N.s/$m^2$ |
| ρ | density, kg/$m^3$ |

**SUBSCRIPTS**

| | |
|---|---|
| f | fluid |
| s | solid |

## INTRODUCTION

The heat interaction between a solid body immersed in an infinite fluid at a different temperature normally occurs by either forced convection or natural convection [1-15]. In a solid body absent of internal heat generation, the conductive heat is dependent upon two resistances: (1) the internal conductive resistance inside the body and (2) the surface convective resistance between the body surface and the fluid. Within these possibilities, there are two limiting cases of importance. The first limiting case deals with negligible internal conductive resistance and the second limiting case deals with negligible surface convective resistance. The former case is associated with a small temperature difference between the center and the surface of the body and a large temperature difference between the body surface and the infinite fluid. Putting this statement in perspective, it connotes that during a cooling period, the solid body can be considered as a "lump" with nearly uniform temperature at any instant of time. In other words, the unsteady heat conduction takes place in a lumped body whose mean temperature decreases gradually with time. This rationale sets the groundwork for the primary assumption underlying the resourceful lumped model.

With regards to the teaching of unsteady heat conduction in regular or irregular solid bodies exposed to a neighboring fluid, textbooks on heat transfer explain the lumped model with various degrees of depth [1–15]. Conceptually, the lumped model subscribes to the notion that the surface convective resistance dominates the internal conductive resistance. This resistance imbalance is responsive to the so–called Biot number criterion Bi = $\frac{\bar{h}}{k_s}\left(\frac{V}{S}\right)$ < 0.1. More specifically, in practice the lumped model is satisfied by bodies of small size, and/or solids possessing large thermal conductivity, and/or convective environments owning weak mean convective coefficients. Qualitatively, the interior temperatures in a large plate, long cylinder and sphere are quasi–uniform at any given time because the surface–to–center temperature ratios lie within an approximate 5% error band [1-15].

Intriguingly, what all textbooks on heat transfer habitually omit to mention with regards to the lumped model is that the heat exchange between the body and the surrounding fluid has to be ensued by forced convection. Since forced convection is a linear mode of heat transfer (practically impervious to temperature changes), the Biot number criterion Bi = $\frac{\bar{h}}{k_s}\left(\frac{V}{S}\right)$ < 0.1 depends on a mean convective coefficient $\bar{h}$ that remains constant during the entire cooling period. On the contrary, when natural convection is the heat exchange mode, the heat transfer is nonlinear and the corresponding mean convective coefficient $\bar{h}$ does not stay constant; instead $\bar{h}$ varies with the instantaneous space–mean temperature. Thereby, to comply with the lumped model for natural convection, the Biot number criterion must be viewed through a different perspective. This forcibly implies that the Biot number criterion must be modified, being rewritten as $Bi_{max}$ = $\frac{\bar{h}_{max}}{k_s}\left(\frac{V}{S}\right)$ < 0.1 where $\bar{h}_{max}$ for cooling denotes the maximum mean convective coefficient occurring at the initial temperature $T_i$ and time t = 0. Regardless of the heat transfer mode (either forced or natural convection), $\bar{h}$ is evaluated from appropriate correlation equations for the mean Nusselt number which are tied up to the shape and orientation of the solid bodies [1–15].

The objective of this engineering education paper is two–fold. The first objective is to develop the exact mean temperature distribution T (t) for a case study involving unsteady heat conduction in a solid sphere cooled by natural convection in still atmospheric air. The second objective is to seek a material's alternative for the new Biot number criterion, which will be expressed in terms of the thermophysical properties of the solid and the air.

## NATURAL CONVECTION

Natural convection from a hot solid body to an extensive quiescent fluid induces an upward flow by heating a portion of the fluid in the vicinity of the body. The heated fluid expands, becomes less dense than the cooler fluid and rises due to gravitational buoyancy effects. Parallel to this, the cooler fluid descends cyclically to replace the space occupied by the heated fluid.

Application of dimensional analysis to natural convection heat transfer establishes a relation between the mean Nusselt number $\overline{Nu}$, the Rayleigh number Ra and the Prandtl number Pr, via the double–valued function [1–15]:

$$\overline{Nu} = f\,(\underbrace{Ra}_{\substack{\text{primary}\\\text{independent}\\\text{variable}}},\ \underbrace{Pr}_{\substack{\text{secondary}\\\text{independent}\\\text{variable}}}) \tag{1}$$

For the estimation of $\overline{Nu}$, modern correlation equations have been developed by Churchill and coworkers (Churchill [16]) using theoretical, numerical and experimental data for vertical plates, horizontal cylinders and spheres. Typical uncertainties in the prediction of $\bar{h}$ by correlation equations are within $\pm 10\%$ to $\pm 20\%$ margin (Holman [10]).

## LUMPED HEAT EQUATION

Consideration is given to a heated solid body at a uniform temperature $T_i$ immersed in a cold stagnant fluid at a different uniform temperature $T_\infty$ as shown in Figure 1. In general, the applicable lumped heat equation is

$$\rho c_{v,s} V \frac{dT}{dt} = -\bar{h} S (T - T_\infty), \qquad T(0) = T_i \tag{2}$$

which obeys the Biot criterion $\frac{\bar{h}}{k_s}\left(\frac{V}{S}\right) < 0.1$. The characteristic length corresponds to the volume–to–surface area ratio $\frac{V}{S}$ .

Let us take the solid sphere as a case study where $\frac{V}{S} = \frac{D}{6}$. Then, eq. (2) is particularized to

$$\rho c_{v,s} \frac{D}{6} \frac{dT}{dt} = -\bar{h}(T - T_\infty), \qquad T(0) = T_i \tag{3}$$

and since $V = \frac{R}{3}$, the Bi criterion becomes $\frac{\bar{h} R}{k_s} < 0.3$.

The mean Nusselt number $\overline{Nu}_D$ for a solid sphere cooled in a natural convection environment is obtained from the correlation equation due to Churchill [10]

$$\overline{Nu}_D = 2.0 + 0.589 \frac{Ra_D^{1/4}}{f(Pr)} \quad \text{for } Ra_D < 10^{11} \tag{4}$$

where f (Pr) is the so–called "universal" Prandtl number function

$$f(Pr) = \left[ 1 + \left( \frac{0.469}{Pr} \right)^{9/16} \right]^{4/9} \tag{4a}$$

Here, the intervening thermophysical properties of the fluid are evaluated at the film temperature $T_f = \frac{1}{2}(T_s + T_\infty)$.

At this stage, it is worth pointing out that the solid sphere constitutes an interesting configuration. First, the mean Nusselt number $\overline{Nu}_D$ in eq. (4) has a specifiable conductive lower bound $\overline{Nu}_D \rightarrow 2$ as $Ra_D \rightarrow 0$ by virtue of a combination of factors, such as high viscosity $\mu \rightarrow \infty$, small density $\rho \rightarrow 0$, or small diameter $D \rightarrow 0$.

Isolating the mean natural convective coefficient $\bar{h}$ in eq. (4), $\bar{h}$ is expressed in terms of primitive quantities by the two–term expression:

$$\bar{h} = 2\frac{k_f}{D} + 0.589 \frac{k_f}{D^{1/4}} \left( g\beta \frac{\rho^2}{\mu^2} \right)^{1/4} g(\mathrm{Pr}) (T - T_\infty)^{1/4} \tag{5}$$

where the new grouping $g(\mathrm{Pr}) = \frac{1}{\mathrm{Pr}^{3/4}}$ is introduced for convenience. It is recognizable in eq. (5) that $\bar{h}$ entails to a nonlinear, single–valued function of the temperature difference $(T - T_\infty)$. As time evolves ($t > 0$), the contribution of the second term in eq. (5) weakens, and the mean temperature T gradually decreases from the initial temperature $T_i$ to the equilibrium fluid temperature $T_\infty$. For large time $t \rightarrow \infty$, the second term in eq. (5) vanishes resulting in the lowest value of $\bar{h}$, that is $\bar{h}_{min} = 2\frac{k_f}{D}$. Under these circumstances, this signifies that the transfer of heat from a hot solid sphere to a cold fluid happens by pure conduction through a stagnant layer of fluid coating the sphere. For visualization purposes, Figure. 2 displays the variation of $\bar{h}$ with temperature T for the natural convection cooling of an aluminum sphere in still air where the diameter D is the parameter. In numbers, for a large diameter D = 0.5 m and $T_\infty$ = 29 °C, $\bar{h}_{max}$ = 5 W/m$^2$. °C, and $\bar{h}_{min}$ = 0.1 W/m$^2$.°C (there is a large factor of 50 between the two $\bar{h}$'s), whereas for a small diameter D = 0.01 m, $\bar{h}_{max}$ = 21.6 W/m$^2$.°C and $\bar{h}_{min}$ = 5.2 W/m$^2$.°C (there is a small factor of 4 between the two $\bar{h}$'s).

Further inspection of eq. (5) indicates that the largest value of $\bar{h}$, say $\bar{h}_{max}$, happens at the initial temperature $T = T_i$ when t = 0. Consequently, $\bar{h}_{max}$ is represented by

$$\bar{h}_{max} = 2\frac{k_f}{D} + 0.589 \frac{k_f}{D^{1/4}} \left( g\beta \frac{\rho^2}{\mu^2} \right)^{1/4} g(\mathrm{Pr}) (T_i - T_\infty)^{1/4} \tag{5a}$$

Introducing eq. (5) into eq. (3) delivers the following nonlinear differential equation of first order

$$\rho c_{v,s} D \frac{dT}{dt} = -12\frac{k_f}{D}(T - T_\infty) - 3.53 \frac{k_f}{D^{1/4}} \left( g\beta \frac{\rho^2}{\mu^2} \right)^{1/4} g(\mathrm{Pr}) (T - T_\infty)^{5/4}, \quad T(0) = T_i \tag{6}$$

Now, to comply with the lumped model for a sphere, the applicable Biot criterion must be rewritten as

$$\mathrm{Bi}_{max} = \frac{\bar{h}_{max} R}{k_s} < 0.3 \tag{7}$$

where $\bar{h}_{max}$ being linked to $T = T_i$ , t = 0 is determined from eq. (5a). Upon defining the temperature excess

$$\theta = T - T_\infty \tag{8}$$

eq. (6) can be homogenized into the compact form

$$\frac{d\theta}{dt} + a\,\theta + b\,\theta^{5/4} = 0, \qquad \theta(0) = \theta_i \tag{9}$$

where the pair of coefficients "a" and "b" are computed from:

$$a = \frac{12}{D^2}\left(\frac{1}{\rho c_{v,s}}\right)k_f \quad and \quad b = \frac{3.53}{D^{5/4}}\left(\frac{1}{\rho c_{v,s}}\right)\left[k_f\left(g\beta\frac{\rho^2}{\mu^2}\right)^{1/4} g(\Pr)\right] \tag{9a}$$

In here, it is seen that "a" and "b", vary inversely proportional with the sphere diameter D.

In principle, eq. (9) belongs to the general class of nonlinear differential equations

$$\frac{d\theta}{dt} + a\,\theta + b\,\theta^n = 0, \quad n \neq 1 \tag{10}$$

which is named Bernoulli equation (Polyanin and Zaitsev [17]). The exact analytic solution of eq. (9) delivers the following mean temperature distribution T (t):

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = \left[1 + \frac{b}{a}\exp\left(\frac{a}{4}t\right) - \frac{b}{a}\right]^4$$
(11)

where "a" and "b" are taken from eq. (9a). The reader should notice that the structure of this exact solution for natural convection with variable $\bar{h}$ is quite different from the structure of the exact solution for forced convection with constant $\bar{h}$,

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = \exp\left(-\frac{3\bar{h}}{\rho c_{v,s} R}t\right) \tag{12}$$

## ALTERNATE MODIFIED BIOT CRITERION

At this point, let us move back to the prevalent Biot criterion for a sphere. Combining eqs. (5a) and (7), the Biot criterion can be re-expressed in a convenient alternate manner by way of the dimensionless inequality

$$\frac{k_f}{k_s} + 0.295\frac{k_f}{k_s}D^{3/4}\left(g\beta\frac{\rho^2}{\mu^2}\right)^{1/4} g(\Pr)(T_i - T_\infty)^{1/4} < 0.3 \tag{13a}$$

The interpretation of this inequality suggests that the region of validity for the lumped heat equation (2) embodies the interplay between the sphere diameter D (a geometric quantity) and various thermophysical properties, such as 1) the solid thermal conductivity $k_s$, 2) three fluid thermophysical properties $k_f$, ρ, μ as well as 3) the prescribed initial–to–air temperature difference $T_i - T_\infty$.

Next, isolating the solid–to–fluid thermal conductivity ratio $\frac{k_s}{k_f}$ in eq. (13a) results in

$$\frac{k_s}{k_f} > 3.333\left[1 + 0.295D^{3/4}\left(g\beta\frac{\rho^2}{\mu^2}\right)^{1/4} g(\Pr)(T_i - T_\infty)^{1/4}\right] \tag{13b}$$

Its equivalent abbreviated form is

$$\frac{k_s}{k_f} > 3.333 + 0.983\,g(\Pr)Gr_{D,i}^{1/4} \tag{13c}$$

Here, the subscript 'i' in the extended Grashof number $Gr_{D,i}$ signals to the initial–to–fluid temperature difference $T_i - T_\infty$ and g(Pr) is found in eq. (4a). When eq. (7) is compared against eq. (13c), it is crystal clear that despite $k_s$ being common to both equations, $\bar{h}_{max}$ (the derived quantity) is supplanted by the fluid thermal conductivity $k_f$ (a primitive quantity), accompanied by the two dimensionless groups characterizing a natural convection environment, i.e., $Gr_{D,i}$ and Pr.

## PRACTICAL EXAMPLE

To put things in engineering perspective, let us analyze a solid sphere of small diameter D at an initial temperature $T_i$ immersed in extensive quiescent air at a different temperature $T_\infty$. The question that needs to be addressed is: What solid materials make the lumped model acceptable for the solid sphere when natural convection is the heat transfer mechanism?

For air at an ambient temperature of 20°C, the thermal conductivity stays around $k_f$ = 0.028 W/m. °C and Pr = 0.71 (Incropera and DeWitt [12]). From eq. (13c), it is imminent that the thermal conductivity of the solid $k_s$ has to satisfy the inequality

$$k_s > 0.087 + 0.018 Gr_{D,i}^{1/4} \tag{14}$$

Obviously, two practical limits for $Gr_{D,i}$ are physically plausible. First, in the lower conductive limit for vanishing $Gr_{D,i} \to 0$, the answer is simply $k_s$ > 0.087 W/m.C (nearly zero!!!). Second, setting the upper convective limit for a large $Gr_{D,i}$ at the critical laminar–turbulent value $Gr_{D,i,cr} \approx 10^9$, it turns out that $k_s$ > 3.3 W/m.°C (a small number!!!). Upon consulting the Tables of Thermophysical Properties in [10], the interpretation of the two numbers 0.087 W/m.°C and 3.3 W/m.C provides the sought answer right away. As a reference, the smallest thermal conductivity of a metal is $k_s$ = 12 W/m.°C for Nichrome (80% Ni, 20% Cr). As a consequence, all metals fulfill the lumped model requirement for a sphere/air ensemble with natural convection cooling in quiescent air.

Despite that the above analysis was geared toward a solid sphere exposed to air, notwithstanding the simple analysis can be easily extended to a large plate or long cylinder when articulated with the three basic fluids: air, water and oil. Thereby, the nature of the permissible solid materials can be readily obtained for each body/fluid combination is a straightforward manner.

## CONCLUSIONS

A practical example has been delineated for a sphere undergoing natural convection cooling in air and the corresponding temperature-time distribution has been obtained. Beginning with the new modified Biot criterion written as $Bi_{max} = \frac{\bar{h}_{max}}{k_s}\left(\frac{V}{S}\right) < 0.1$, and using an appropriate mean Nusselt number correlation equation for ambient air at 20°C, the $Bi_{max}$ inequality translates into the alternate inequality $k_s > 0.087 + 0.018\,Gr_{D,\,i}^{\ 1/4}$ where $k_s$ is the thermal conductivity of the solid, and $Gr_{D,i}$ stands for the extended Grashof number accounting for the largest temperature difference, $T_i - T_\infty$, i.e., the initial–to–fluid temperature difference. From a practical perspective, the main conclusion that can be drawn from this engineering education paper is that all metals fulfill the lumped model requirement for the sphere/air ensemble. Exploiting this finding, it turns out that no calculations have to be made for the Biot number criterion.

Needless to say, similar inequalities for $k_s$ linked to $Gr_{D,i}$ can be easily constructed for the large plate and long cylinder immersed in standard fluids, such as air, water and oil.

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## FIGURE CAPTIONS

Figure 1 – A solid sphere immersed in an infinite fluid at rest

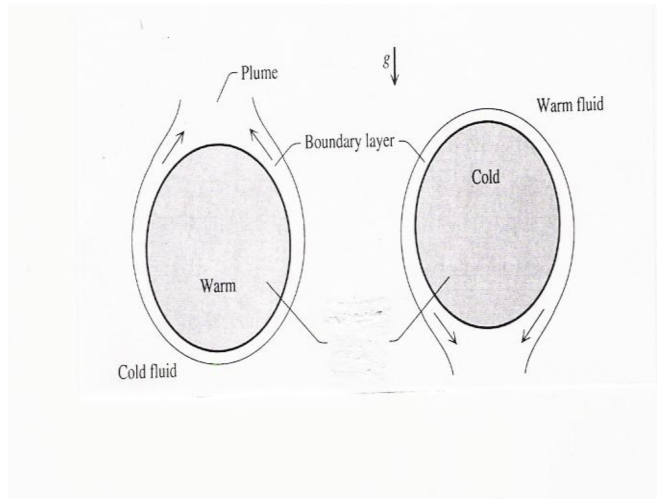


Figure 2 – Variation of the mean convective coefficient $\bar{h}$ (W/m$^2$.°C) with temperature T (°C) for an aluminum sphere undergoing natural convection cooling in quiescent air at 29°C.

The sphere diameter D is a parameter varying from 0.01m to 0.5m