The abelianization of the level L mapping class group

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Abstract

We calculate the abelianizations of the level L subgroup of the genus g mapping class group and the level L congruence subgroup of the $2g \times 2g$ symplectic group for L odd and $g \geq 3$.

Historical note. I originally wrote this paper in March of 2008. Towards the end of that month, I gave a master class on the Torelli group at the University of Aarhus. That master class ended in a conference, and I had intended to speak about this paper at that conference. However, I learned that both Bernard Perron and Masatoshi Sato had proven similar theorems and intended to speak about them at the same conference! Sato was a graduate student and had actually proved somewhat better results (in particular, he could deal with $L = 2$), so I decided not to publish this paper. Sato's work appeared in [19], and Perron's work was sketched in [14]. See my later paper [18] for results for L not divisible by 4. Dealing with the case where L is divisible by 4 is still open.

1 Introduction

Let $\Sigma_{g,n}$ be an orientable genus g surface with n boundary components and let $\mathrm{Mod}_{g,n}$ be its *mapping class group*, that is, the group $\pi_0(\mathrm{Diff}^+(\Sigma_{g,n}, \partial\Sigma_{g,n}))$. This is the (orbifold) fundamental group of the moduli space of Riemann surfaces and has been intensely studied by many authors. For $n \in \{0, 1\}$, the action of $\mathrm{Mod}_{g,n}$ on $\mathrm{H}_1(\Sigma_{g,n}; \mathbb{Z})$ induces a surjective representation of $\mathrm{Mod}_{g,n}$ into the symplectic group whose kernel $\mathcal{I}_{g,n}$ is known as the *Torelli group*. This is summarized by the exact sequence

$$1 \longrightarrow \mathcal{I}_{g,n} \longrightarrow \mathrm{Mod}_{g,n} \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

For $L \geq 2$, let $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$ denote the *level L congruence subgroup* of $\mathrm{Sp}_{2g}(\mathbb{Z})$, that is, the subgroup of matrices that are equal to the identity modulo L. The pull-back of $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$ to $\mathrm{Mod}_{g,n}$ is known as the *level L subgroup* of $\mathrm{Mod}_{g,n}$ and is denoted by $\mathrm{Mod}_{g,n}(L)$. The group $\mathrm{Mod}_{g,n}(L)$ can also be described as the group of mapping classes that act trivially on $\mathrm{H}_1(\Sigma_{g,n}; \mathbb{Z}/L\mathbb{Z})$. It fits into an exact sequence

$$1 \longrightarrow \mathcal{I}_{g,n} \longrightarrow \mathrm{Mod}_{g,n}(L) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}, L) \longrightarrow 1.$$

In [6], Hain proved that the abelianization of $\mathrm{Mod}_{g,n}(L)$ consists entirely of torsion for $g \geq 3$ (an alternate proof was given by McCarthy in [12]). In this note, we compute this torsion for L odd.

To state our theorem, we need some notation. Denoting the $n \times n$ zero matrix by $\mathbb{O}_n$ and the $n \times n$ identity matrix by $\mathbb{I}_n$, let Ω_g be the matrix $\left(\begin{smallmatrix} \mathbb{O}_g & \mathbb{I}_g \\ -\mathbb{I}_g & \mathbb{O}_g \end{smallmatrix}\right)$ (we will abuse notation and let the entries of Ω_g lie in whatever ring we are considering at the moment). By definition, the group $\mathrm{Sp}_{2g}(\mathbb{Z})$ consists of $2g \times 2g$ integral matrices X that satisfy $X^t \Omega_g X = \Omega_g$. We will denote by $\mathfrak{sp}_{2g}(L)$ the additive group of all $2g \times 2g$ matrices A with entries in $\mathbb{Z}/L\mathbb{Z}$ that satisfy $A^t \Omega_g + \Omega_g A = 0$.

Our main theorem is as follows, and is proven in §4.

Theorem 1.1 (Integral H_1 of level L subgroups)**.** *For $g \geq 3$, $n \in \{0, 1\}$, and L odd, set $H(L) = \mathrm{H}_1(\Sigma_{g,n}; \mathbb{Z}/L\mathbb{Z})$. We then have an exact sequence*

$$0 \longrightarrow K \longrightarrow \mathrm{H}_1(\mathrm{Mod}_{g,n}(L); \mathbb{Z}) \longrightarrow \mathfrak{sp}_{2g}(L) \longrightarrow 0,$$

where $K = \wedge^3 H(L)$ if $n = 1$ and $K = (\wedge^3 H(L))/H(L)$ if $n = 0$.

Remark. The condition $g \geq 3$ is necessary, since in [12] McCarthy proves that if 2 or 3 divides L, then $\mathrm{Mod}_2(L)$ surjects onto $\mathbb{Z}$. A computation of $\mathrm{H}_1(\mathrm{Mod}_{2,n}(L); \mathbb{Z})$ (or even $\mathrm{H}_1(\mathrm{Mod}_{2,n}(L); \mathbb{Q})$) would be very interesting.

We now describe the sources for the terms in the exact sequence of Theorem 1.1. The kernel K comes from the *relative Johnson homomorphisms* of Broaddus-Farb-Putman [4]. For $\mathrm{Mod}_{g,n}(L)$, these are surjective homomorphisms

$$\tau_{g,1}(L) : \mathrm{Mod}_{g,1}(L) \longrightarrow \wedge^3 H(L)$$

and

$$\tau_g(L) : \mathrm{Mod}_g(L) \longrightarrow (\wedge^3 H(L))/H(L)$$

which are related to the celebrated Johnson homomorphisms on the Torelli group (see §3 and §4).

The cokernel $\mathfrak{sp}_{2g}(L)$ is the abelianization of $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$. Now, the isomorphism

$$\mathrm{H}_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \cong \mathfrak{sp}_{2g}(L)$$

can be deduced from general theorems of Borel on arithmetic groups (see [3, §2.5]); however, Borel's results are much more general than we need and it takes some work to derive the desired result from them. We instead imitate a beautiful argument of Lee-Szczarba [11], who prove that

$$\mathrm{H}_1(\mathrm{SL}_n(\mathbb{Z}, L); \mathbb{Z}) \cong \mathfrak{sl}_n(L)$$

for $n \geq 3$. Here $\mathrm{SL}_n(\mathbb{Z}, L)$ is the level L congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$ and $\mathfrak{sl}_n(L)$ is the additive group of $n \times n$ matrices with coefficients in $\mathbb{Z}/L\mathbb{Z}$ and trace 0. The proof of the following theorem is contained in §2.

Theorem 1.2 (Integral H_1 of $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$). *For $g \geq 3$ and L odd, we have*

$$\mathrm{H}_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \cong \mathfrak{sp}_{2g}(L).$$

Moreover, $[\mathrm{Sp}_{2g}(\mathbb{Z}, L), \mathrm{Sp}_{2g}(\mathbb{Z}, L)] = \mathrm{Sp}_{2g}(\mathbb{Z}, L^2)$.

Remark. It is unclear whether the hypothesis that L is odd is necessary for Theorems 1.1 or 1.2, but it is definitely used in both proofs.

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2 The abelianization of $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$

We will need the following notation.

Definition 2.1. For $1 \leq i, j \leq n$, let $\mathcal{E}^n_{i,j}(r)$ be the $n \times n$ matrix with an r at position (i, j) and 0's elsewhere. Similarly, let $\mathcal{SE}^n_{i,j}(r)$ be the $n \times n$ matrix with an r at positions (i, j) and (j, i) and 0's elsewhere.

Definition 2.2. For $1 \leq i, j \leq g$, denote by $\mathcal{X}^g_{i,j}(r)$ the matrix $\begin{pmatrix} \mathbb{I}_g & \mathbb{O}_g \\ \mathcal{SE}^g_{i,j}(r) & \mathbb{I}_g \end{pmatrix}$, by $\mathcal{Y}^g_{i,j}(r)$ the matrix $\begin{pmatrix} \mathbb{I}_g & \mathcal{SE}^g_{i,j}(r) \\ \mathbb{O}_g & \mathbb{I}_g \end{pmatrix}$, and by $\mathcal{Z}^g_{i,j}(r)$ the matrix $\begin{pmatrix} \mathbb{I}_g + \mathcal{E}^g_{i,j}(r) & \mathbb{O}_g \\ \mathbb{O}_g & \mathbb{I}_g - \mathcal{E}^g_{j,i}(r) \end{pmatrix}$.

Observe that $\mathcal{X}^g_{i,j}(L), \mathcal{Y}^g_{i,j}(L) \in \mathrm{Sp}_{2g}(\mathbb{Z}, L)$ for all $1 \leq i, j \leq g$ and that $\mathcal{Z}^g_{i,j}(L) \in \mathrm{Sp}_{2g}(\mathbb{Z}, L)$ for $1 \leq i, j \leq g$ with $i \neq j$. The following theorem forms part of Bass-Milnor-Serre's solution to the congruence subgroup problem for the symplectic group.

Theorem 2.3 (Bass-Milnor-Serre [1, Theorem 12.4, Corollary 12.5]). *For $g \geq 2$ and $L \geq 1$, the group $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$ is normally generated by*

$$\{\mathcal{X}^g_{i,j}(L) \mid 1 \leq i, j \leq g\} \cup \{\mathcal{Y}^g_{i,j}(L) \mid 1 \leq i, j \leq g\}.$$

Remark. We emphasize that the matrices $\mathcal{Z}^g_{i,j}(L)$ are not needed – the proof of [1, Lemma 13.1] contains an explicit formula for them in terms of the $\mathcal{X}^g_{i,j}$ and the $\mathcal{Y}^g_{i,j}$.

Using this, we can prove the following.

Lemma 2.4. *For $g \geq 3$ and L odd, we have* $\mathrm{Sp}_{2g}(\mathbb{Z}, L^2) < [\mathrm{Sp}_{2g}(\mathbb{Z}, L), \mathrm{Sp}_{2g}(\mathbb{Z}, L)]$.

Proof. We must show that each normal generator of $\mathrm{Sp}_{2g}(\mathbb{Z}, L^2)$ given by Theorem 2.3 is contained in $[\mathrm{Sp}_{2g}(\mathbb{Z}, L), \mathrm{Sp}_{2g}(\mathbb{Z}, L)]$. We will do the case of $\mathcal{X}^g_{i,j}(L^2)$; the other case is similar. Assume first that $i \neq j$. Since $g \geq 3$, there is some $1 \leq k \leq g$ so that $k \neq i, j$. The following matrix identity then proves the desired claim:

$$\mathcal{X}^g_{i,j}(L^2) = [\mathcal{X}^g_{i,k}(L), \mathcal{Z}^g_{k,j}(L)].$$

Now assume that $i = j$. Again, there exists some $1 \le k_1 < k_2 \le g$ so that $k_1, k_2 \ne i$. Also, since L is odd there exists some integer N so that $2N + L = 1$. We thus have

$$\mathcal{X}^g_{i,i}(L^2) = \mathcal{X}^g_{i,i}((2N + L)L^2) = \mathcal{X}^g_{i,i}(2NL^2) \cdot \mathcal{X}^g_{i,i}(L^3),$$

so the following matrix identities complete the proof:

$$\begin{aligned} \mathcal{X}^g_{i,i}(2NL^2) &= [\mathcal{X}^g_{i,k_1}(NL), \mathcal{Z}^g_{k_1,i}(L)], \\ \mathcal{X}^g_{i,i}(L^3) &= [\mathcal{X}^g_{k_1,k_1}(L), \mathcal{Z}^g_{k_1,i}(L)] \cdot [\mathcal{Z}^g_{k_2,i}(L), \mathcal{X}^g_{k_1,k_2}(L)]. \end{aligned}$$ □

Proof of Theorem 1.2. We begin by defining a function $\phi : \mathrm{Sp}_{2g}(\mathbb{Z}, L) \to \mathfrak{sp}_{2g}(L)$. Consider any matrix $X \in \mathrm{Sp}_{2g}(\mathbb{Z}, L)$. Write $X = \mathbb{I}_{2g} + LA$, and define

$$\phi(X) = A \qquad (\mathrm{mod}\ L).$$

We claim that $\phi(X) \in \mathfrak{sp}_{2g}(L)$. Indeed, by the definition of the symplectic group we have $X^t \Omega_g X = \Omega_g$. Writing $X = \mathbb{I}_{2g} + LA$ and expanding out, we have

$$\Omega_g + L(A^t\Omega_g + \Omega_g A) + L^2(A^t\Omega_g A) = \Omega_g.$$

We conclude that modulo L we have $A^t\Omega_g + \Omega_g A = 0$, as desired.

Next, we prove that ϕ is a homomorphism. Consider $X, Y \in \mathrm{Sp}_{2g}(\mathbb{Z}, L)$ with $X = \mathbb{I}_{2g} + LA$ and $Y = \mathbb{I}_{2g} + LB$. Thus $XY = \mathbb{I}_{2g} + L(A + B) + L^2AB$, so modulo L we have $\phi(XY) = A + B$, as desired.

The fact that ϕ is surjective is a fun exercise.

Observe now that $\ker(\phi) = \mathrm{Sp}_{2g}(\mathbb{Z}, L^2)$. Since $\mathfrak{sp}_{2g}(L)$ is abelian, this implies that $[\mathrm{Sp}_{2g}(\mathbb{Z}, L), \mathrm{Sp}_{2g}(\mathbb{Z}, L)] < \mathrm{Sp}_{2g}(\mathbb{Z}, L^2)$. Lemma 2.4 then allows us to conclude that $\ker(\phi) = \mathrm{Sp}_{2g}(\mathbb{Z}, L^2) = [\mathrm{Sp}_{2g}(\mathbb{Z}, L), \mathrm{Sp}_{2g}(\mathbb{Z}, L)]$, and the theorem follows. □

3 The Torelli group

We now review some facts about $\mathcal{I}_{g,n}$.

Definition 3.1. Let $n \in \{0, 1\}$. A *bounding pair* on $\Sigma_{g,n}$ is a pair $\{x_1, x_2\}$ of disjoint nonhomotopic nonseparating curves on $\Sigma_{g,n}$ so that $x_1 \cup x_2$ separates $\Sigma_{g,n}$. Letting T_γ denote the Dehn twist about a simple closed curve γ, the *bounding pair map* associated to a bounding pair $\{x_1, x_2\}$ is $T_{x_1}T_{x_2}^{-1}$.

Observe that if $\{x_1, x_2\}$ is a bounding pair, then $T_{x_1}T_{x_2}^{-1} \in \mathcal{I}_{g,n}$. Building on work of Birman [2] and Powell [15], Johnson proved the following.

Theorem 3.2 (Johnson, [7])**.** *For $g \ge 3$ and $n \in \{0, 1\}$, the group $\mathcal{I}_{g,n}$ is generated by bounding pair maps.*

Remark. In fact, under the hypotheses of this theorem Johnson later proved that finitely many bounding pair maps suffice [9]. This should be contrasted with work of McCullough-Miller [13] that says that for $n \in \{0, 1\}$, the group $\mathcal{I}_{2,n}$ is *not* finitely generated.

We will also need Johnson's computation of the abelianization of $\mathcal{I}_{g,n}$.

Theorem 3.3 (Johnson, [10]). *Let $g \geq 3$, and set $H = \mathrm{H}_1(\Sigma_g; \mathbb{Z}) \cong \mathrm{H}_1(\Sigma_{g,1}; \mathbb{Z})$. Then*

$$\mathrm{H}_1(\mathcal{I}_{g,1}; \mathbb{Z}) \cong \wedge^3 H \oplus (2\text{-}torsion)$$

and

$$\mathrm{H}_1(\mathcal{I}_g; \mathbb{Z}) \cong ((\wedge^3 H)/H) \oplus (2\text{-}torsion).$$

The maps

$$\tau_{g,1} : \mathcal{I}_{g,1} \longrightarrow \mathrm{H}_1(\mathcal{I}_{g,1}; \mathbb{Z})/(\text{2-torsion}) \cong \wedge^3 H$$

and

$$\tau_g : \mathcal{I}_g \longrightarrow \mathrm{H}_1(\mathcal{I}_g; \mathbb{Z})/(\text{2-torsion}) \cong (\wedge^3 H)/H$$

are known as the *Johnson homomorphisms* and have many remarkable properties. For a survey, see [8].

4 The abelianization of $\mathrm{Mod}_{g,n}(L)$

Partly to establish notation, we begin by recalling the statement of the 5-term exact sequence in group homology.

Theorem 4.1 (see, e.g., [5, Corollary VII.6.4]). *Let*

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a short exact sequence of groups and let R be a ring. There is then an exact sequence

$$\mathrm{H}_2(G; R) \longrightarrow \mathrm{H}_2(Q; R) \longrightarrow \mathrm{H}_1(K; R)_Q \longrightarrow \mathrm{H}_1(G; R) \longrightarrow \mathrm{H}_1(Q; R) \longrightarrow 0,$$

where $\mathrm{H}_1(K; R)_Q$ is the ring of co-invariants *of $\mathrm{H}_1(K; R)$ under the natural action of Q, that is, the quotient of $\mathrm{H}_1(K; R)$ by the ideal generated by $\{q(k) - k \mid q \in Q \text{ and } k \in K\}$.*

We will need a special case of a theorem of Broaddus-Farb-Putman that gives "relative" versions of the Johnson homomorphisms on certain "homologically defined" subgroups of $\mathrm{Mod}_{g,b}$. In our situation, the result can be stated as follows.

Theorem 4.2 (Broaddus-Farb-Putman, [4, Example 5.3 and Theorem 5.8]). *Fix $L \geq 2$, $g \geq 3$, and $n \in \{0, 1\}$. Set $H = \mathrm{H}_1(\Sigma_{g,n}; \mathbb{Z})$ and $H(L) = \mathrm{H}_1(\Sigma_{g,n}; \mathbb{Z}/L\mathbb{Z})$, and define X and $X(L)$ to equal H and $H(L)$ if $n = 0$ and to equal 0 if $n = 1$. Hence $(\wedge^3 H)/X$*

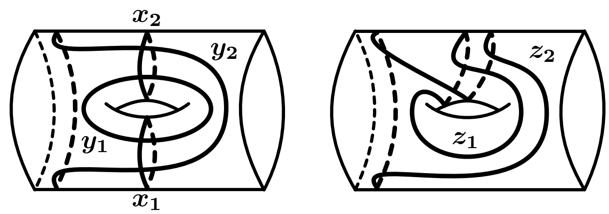

Figure 1: The crossed lantern relation $(T_{y_1}T_{y_2}^{-1})(T_{x_1}T_{x_2}^{-1}) = (T_{z_1}T_{z_2}^{-1})$

is the target for the Johnson homomorphism on $\mathcal{I}_{g,n}$. Then there exist homomorphisms $\tau_{g,n}(L) : \mathrm{Mod}_{g,1}(L) \to (\wedge^3 H(L))/X(L)$ that fit into the commutative diagram

$$\begin{array}{ccc} \mathcal{I}_{g,n} & \xrightarrow{\tau_{g,n}} & (\wedge^3 H)/X \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{g,n}(L) & \xrightarrow{\tau_{g,n}(L)} & (\wedge^3 H(L))/X(L) \end{array}$$

Here the right hand vertical arrow is reduction mod L.

We preface the proof of Theorem 1.1 with two lemmas. Our first lemma was originally proven by McCarthy [12, proof of Theorem 1.1]. We give an alternate proof. If G is a group and $g \in G$, then denote by $[g]$ the corresponding element of $\mathrm{H}_1(G;\mathbb{Z})$.

Lemma 4.3. *For $n \in \{0,1\}$, let $\{x_1, x_2\}$ be a bounding pair on $\Sigma_{g,n}$. Then $L[T_{x_1}T_{x_2}^{-1}] = 0$ in $\mathrm{H}_1(\mathrm{Mod}_{g,n}(L);\mathbb{Z})$.*

Proof. Embed $\{x_1, x_2\}$ in a 2-holed torus as in Figure 1. We will make use of the *crossed lantern relation* from [17]. Letting $\{y_1, y_2\}$ and $\{z_1, z_2\}$ be the other bounding pair maps depicted in Figure 1, this relation says that

$$(T_{y_1}T_{y_2}^{-1})(T_{x_1}T_{x_2}^{-1}) = (T_{z_1}T_{z_2}^{-1}).$$

Observe that for $i = 1, 2$ we have $z_i = T_{x_2}(y_i)$. The key observation is that for all $n \geq 0$ we have another crossed lantern relation

$$(T_{T_{x_2}^n(y_1)}T_{T_{x_2}^n(y_2)}^{-1})(T_{x_1}T_{x_2}^{-1}) = (T_{T_{x_2}^{n+1}(y_1)}T_{T_{x_2}^{n+1}(y_2)}^{-1}).$$

Since $T_{x_2}^L \in \mathrm{Mod}_{g,n}(L)$, we conclude that in $\mathrm{H}_1(\mathrm{Mod}_{g,n}(L);\mathbb{Z})$ we have

$$\begin{aligned}
[T_{y_1}T_{y_2}^{-1}] &= [T_{x_2}^L] + [T_{y_1}T_{y_2}^{-1}] - [T_{x_2}^L] = [T_{x_2}^L(T_{y_1}T_{y_2}^{-1})T_{x_2}^{-L}] = [(T_{T_{x_2}^L(y_1)}T_{T_{x_2}^L(y_2)}^{-1})] \\
&= [T_{x_1}T_{x_2}^{-1}] + [(T_{T_{x_2}^{L-1}(y_1)}T_{T_{x_2}^{L-1}(y_2)}^{-1})] \\
&= 2[T_{x_1}T_{x_2}^{-1}] + [(T_{T_{x_2}^{L-2}(y_1)}T_{T_{x_2}^{L-2}(y_2)}^{-1})] \\
&\vdots \\
&= L[T_{x_1}T_{x_2}^{-1}] + [T_{y_1}T_{y_2}^{-1}],
\end{aligned}$$

so $L[T_{x_1}T_{x_2}^{-1}] = 0$, as desired. $\square$

For the statement of the following lemma, recall that if a group G acts on a ring R, then the coinvariants of that action are denoted R_G.

Lemma 4.4. *For $L \geq 2$, define $H = \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ and $H(L) = \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L\mathbb{Z})$. Then*

$$(\wedge^3 H)_{\mathrm{Sp}_{2g}(\mathbb{Z},L)} \cong \wedge^3 H(L)$$

and

$$((\wedge^3 H)/H)_{\mathrm{Sp}_{2g}(\mathbb{Z},L)} \cong (\wedge^3 H(L))/H(L).$$

Proof. Letting $S = \{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for H, the groups $\wedge^3 H$ and $(\wedge^3 H)/H$ are generated by $T := \{x \wedge y \wedge z \mid x, y, z \in S \text{ distinct}\}$. Consider $x \wedge y \wedge z \in T$. It is enough to show that in the indicated rings of coinvariants we have $L(x \wedge y \wedge z) = 0$. Now, one of x, y, and z must have algebraic intersection number 0 with the other two terms. Assume that $x = a_1$ and $y, z \in \{a_2, b_2, \ldots, a_g, b_g\}$ (the other cases are similar). There is then some $\phi \in \mathrm{Sp}_{2g}(\mathbb{Z}, L)$ so that $\phi(b_1) = b_1 + La_1 = b_1 + Lx$ and so that $\phi(y) = y$ and $\phi(z) = z$. We conclude that in the indicated ring of coinvariants we have $b_1 \wedge y \wedge z = (b_1 + Lx) \wedge y \wedge z$, so $L(x \wedge y \wedge z) = 0$, as desired. $\square$

Remark. Lemma 4.4 would *not* be true if $\wedge^3 H$ were replaced by $\wedge^2 H$, as $\wedge^2 H$ contains a copy of the trivial representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Proof of Theorem 1.1. We will do the proof for $\mathrm{Mod}_{g,1}(L)$; the other case is similar. Let H and $H(L)$ be as in Theorem 4.2. Associated to the short exact sequence

$$1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathrm{Mod}_{g,1} \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}, L) \longrightarrow 1$$

is the 5-term exact sequence in homology given by Theorem 4.1. Theorem 3.3 says that

$$\mathrm{H}_1(\mathcal{I}_{g,1};\mathbb{Z}) \cong \wedge^3 H \oplus (2\text{-torsion})$$

and Theorem 1.2 says that $\mathrm{H}_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \cong \mathfrak{sp}_{2g}(\mathbb{Z}/L\mathbb{Z})$. The last 3 terms of our 5-term exact sequence are thus

$$(\wedge^3 H \oplus (\text{2-torsion}))_{\mathrm{Sp}_{2g}(\mathbb{Z},L)} \xrightarrow{i} \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); \mathbb{Z}) \longrightarrow \mathfrak{sp}_{2g}(\mathbb{Z}/L\mathbb{Z}) \longrightarrow 0.$$

Since L is odd, Lemma 4.3 together with Theorem 3.2 say that if

$$x \in (\wedge^3 H \oplus (\text{2-torsion}))_{\mathrm{Sp}_{2g}(\mathbb{Z},L)}$$

is 2-torsion then $i(x) = 0$. Moreover, Lemma 4.4 says that

$$(\wedge^3 H)_{\mathrm{Sp}_{2g}(\mathbb{Z},L)} \cong \wedge^3 H(L).$$

We thus obtain an exact sequence

$$\wedge^3 H(L) \xrightarrow{j} \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); \mathbb{Z}) \longrightarrow \mathfrak{sp}_{2g}(\mathbb{Z}/L\mathbb{Z}) \longrightarrow 0.$$

Theorem 4.2 then implies that j is an injection, and the proof is complete. □

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