HALF LIVES, HALF DIES AND THE SIGNATURES OF BOUNDARIES

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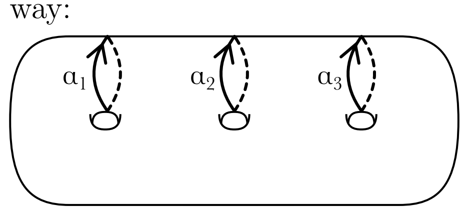
In this note we prove the well-known "half lives, half dies" theorem, and as an application prove that the signatures of boundaries are 0. Throughout, we fix a field $\mathbf{k}$ of characteristic not equal to 2.

0.1. **Easy example.** Consider a closed genus g surface Σ_g embedded in $\mathbb{R}^3$ in the usual way:



The surface Σ_g forms the boundary of a genus g handlebody $\mathcal{H}_g$ embedded in $\mathbb{R}^3$. The kernel L of the map $\mathrm{H}_1(\Sigma_g; \mathbf{k}) \to \mathrm{H}_1(\mathcal{H}_g; \mathbf{k})$ satisfies $L \cong \mathbf{k}^g$ with basis the curves $\{\alpha_1, \ldots, \alpha_g\}$ indicated above. The subspace L is a half-dimensional subspace on which the algebraic intersection pairing vanishes. The general half lives, half dies theorem generalizes this to boundaries of arbitrary odd-dimensional manifolds.

0.2. **Nondegenerate forms.** Its statement requires some preliminaries. Let V be a finite-dimensional vector space over $\mathbf{k}$. In this note, a *form* on V is a bilinear form $\omega(-,-)$ that is either symmetric or antisymmetric. Such an ω induces a map $V \to V^*$ taking $\vec{v} \in V$ to the map $\omega(\vec{v}, -)$ from V to $\mathbf{k}$, and we say that ω is *nondegenerate* if this map $V \to V^*$ is an isomorphism.

Example 0.1. If M^{2n} is a closed oriented $2n$-dimensional manifold, then by Poincaré duality the algebraic intersection pairing on $V = \mathrm{H}_n(M^{2n}; \mathbf{k})$ is a nondegenerate form. □

0.3. **Lagrangians.** Let V be a finite-dimensional vector space over $\mathbf{k}$ equipped with a nondegenerate form ω. For a subspace W of V, define

$$W^{\perp} = \{\vec{v} \in V \mid \omega(\vec{w}, \vec{v}) = 0 \text{ for all } \vec{w} \in W\}.$$

We say that W is a *Lagrangian* in V if $W^{\perp} = W$. We have the following lemma:

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Lemma 0.2. *Let V be a finite-dimensional vector space over $\mathbf{k}$ equipped with a nondegenerate form ω. Let L be a Lagrangian in V. Then there is a basis $\{\vec{a}_1, \vec{b}_1, \ldots, \vec{a}_g, \vec{b}_g\}$ for V with the following properties:*

(i) $\{\vec{a}_1, \ldots, \vec{a}_g\}$ is a basis for L.
(ii) For all $1 \leq i, j \leq g$, we have

$$\omega(\vec{a}_i, \vec{a}_j) = \omega(\vec{b}_i, \vec{b}_j) = 0 \quad \text{and} \quad \omega(\vec{a}_i, \vec{b}_j) = \delta_{ij}.$$

Proof. Pick a basis $\{\vec{a}_1, \ldots, \vec{a}_g\}$ for L. Since ω is nondegenerate, we can find some $\vec{b}_1 \in V$ with $\omega(\vec{a}_i, \vec{b}_1) = \delta_{i1}$ for all $1 \leq i \leq g$. If ω is antisymmetric, then we have $\omega(\vec{b}_1, \vec{b}_1) = 0$. If instead ω is symmetric, then this might not hold. However, for $c \in \mathbf{k}$ we have

$$\omega(\vec{b}_1 + c\vec{a}_1, \vec{b}_1 + c\vec{a}_1) = \omega(\vec{b}_1, \vec{b}_1) + 2c.$$

Since $\mathbf{k}$ does not have characteristic 2, we can replace $\vec{b}_1$ with $\vec{b}_1 + c\vec{a}_1$ for an appropriate value of c and ensure that $\omega(\vec{b}_1, \vec{b}_1) = 0$.

It is clear that $\{\vec{a}_1, \ldots, \vec{a}_g, \vec{b}_1\}$ is linearly independent. Again using the nondegeneracy of ω, we can find some $\vec{b}_2 \in V$ with $\omega(\vec{a}_i, \vec{b}_2) = \delta_{i2}$ for all $1 \leq i \leq g$ and with $\omega(\vec{b}_1, \vec{b}_2) = 0$. Just like above, we can add an appropriate multiple of $\vec{a}_2$ to $\vec{b}_2$ and ensure that $\omega(\vec{b}_2, \vec{b}_2) = 0$ as well.

Repeating this process, we obtain a set of vectors $\{\vec{a}_1, \vec{b}_1, \ldots, \vec{a}_g, \vec{b}_g\}$ satisfying (i) and (ii). It is clear that these vectors are linearly independent, so all that remains is to prove that they span V. Consider some $\vec{v} \in V$. By adding a linear combination of the $\vec{b}_i$ to $\vec{v}$, we can ensure that $\omega(\vec{a}_i, \vec{v}) = 0$ for all $1 \leq i \leq g$. Since the $\vec{a}_i$ are a basis for L, this implies that $\vec{v} \in L^{\perp}$. But since L is a Lagrangian we have $L^{\perp} = L$, so $\vec{v} \in L$ and we can write $\vec{v}$ as a linear combination of the $\vec{a}_i$, as desired. □

Corollary 0.3. *Let V be a finite-dimensional vector space over $\mathbf{k}$ equipped with a nondegenerate form ω. Let L be a Lagrangian in V. Then L is a half-dimensional subspace of V on which ω vanishes.*

Proof. Immediate from Lemma 0.2. □

0.4. **Half-lives, half dies.**

We now come to our main result.

Theorem 0.4 (Half-lives, half dies)**.** *Let M^{2n+1} be a compact oriented $(2n+1)$-dimensional manifold with boundary and let L be the kernel of the map $\mathrm{H}_n(\partial M^{2n+1}; \mathbf{k}) \to \mathrm{H}_n(M^{2n+1}; \mathbf{k})$. Then L is a Lagrangian with respect to the algebraic intersection form on $\mathrm{H}_n(\partial M^{2n+1}; \mathbf{k})$.*

By Corollary 0.3, this implies in particular that $\mathrm{H}_n(\partial M^{2n+1}; \mathbf{k})$ is even-dimensional and that L is a half-dimensional subspace of $\mathrm{H}_n(\partial M^{2n+1}; \mathbf{k})$ on which the algebraic intersection form vanishes.

Proof of Theorem 0.4. To simplify our notation, we will omit the coefficients $\mathbf{k}$ from all our homology groups. Let $\iota\colon \mathrm{H}_n(\partial M^{2n+1}) \to \mathrm{H}_n(M^{2n+1})$ be the map induced by the inclusion and let $\omega_\partial(-,-)$ be the algebraic intersection form on $\mathrm{H}_n(\partial M^{2n+1})$. There is also an algebraic intersection pairing

$$\omega_M\colon\ \mathrm{H}_n(M^{2n+1}) \times \mathrm{H}_{n+1}(M^{2n+1}, \partial M^{2n+1}) \to \mathbf{k}.$$

Poincaré–Lefschetz duality implies that ω_M is a perfect pairing between $\mathrm{H}_n(M^{2n+1})$ and $\mathrm{H}_{n+1}(M^{2n+1}, \partial M^{2n+1})$, i.e., it identifies one with the dual of the other. There is a boundary map $\partial\colon\ \mathrm{H}_{n+1}(M^{2n+1}, \partial M^{2n+1}) \to \mathrm{H}_n(\partial M^{2n+1})$, and our two algebraic intersection forms are related as follows: for all $a \in \mathrm{H}_n(\partial M^{2n+1})$ and $B \in \mathrm{H}_{n+1}(M^{2n+1}, \partial M^{2n+1})$, we have

$$\omega_M(\iota(a), B) = \omega_{\partial M}(a, \partial(B))$$

This can be proved by carefully examining the definitions of the pairings, but to make it at least plausible note that it is obvious if a and B are represented by manifolds intersecting transversely.

Recall that $L = \ker(\iota)$. Our goal is to prove that $L^\perp = L$, and we start by proving that $L \subset L^\perp$. Consider $x, y \in L$. We must show that $\omega_\partial(x, y) = 0$. Since $y \in \mathrm{H}_n(\partial M^{2n+1})$ satisfies $\iota(y) = 0$, we can find some $Y \in \mathrm{H}_{n+1}(M^{2n+1}, \partial M^{2n+1})$ with $\partial(Y) = y$. We then have

$$\omega_\partial(x, y) = \omega_M(\iota(x), Y) = \omega_M(0, Y) = 0,$$

as desired.

We next prove that $L^\perp \subset L$. Consider some z with $z \notin L$. Our goal is to prove that $z \notin L^\perp$. Since $z \notin L$, we have $\iota(z) \neq 0$, so since $\omega_M(-,-)$ is a perfect pairing, we can find some $W \in \mathrm{H}_{n+1}(M^{2n+1}, \partial M^{2n+1})$ with $\omega_M(\iota(z), W) \neq 0$. We then have

$$\omega_{\partial M}(z, \partial(W)) = \omega_M(\iota(z), W) \neq 0. \tag{0.1}$$

However, since $\partial(W)$ is a boundary in M^{2n+1}, we have $\iota(\partial(W)) = 0$, so $\partial(W) \in L$. The equation (0.1) then implies that $z \notin L^\perp$, as desired. □

0.5. **Signatures of boundaries.** Let V be a finite-dimensional vector space over $\mathbb{R}$ equipped with a symmetric form ω. We can diagonalize the matrix representing ω, and the *signature* of ω is the number of positive eigenvalues minus the number of negative eigenvalues. If M^{4n} is a closed oriented $4n$-dimensional manifold, then the algebraic intersection form on $\mathrm{H}_{2n}(M; \mathbb{R})$ is symmetric, and its signature is called the signature of M. We then have the following fundamental result:

Theorem 0.5. *Let M^{4n} is a closed oriented $4n$-dimensional manifold. Assume that $M^{4n} = \partial W^{4n+1}$ for a compact oriented $(4n+1)$-dimensional manifold W^{4n+1}. Then the signature of M^{4n} is 0.*

Remark 0.6. One reason that this is important is that since the signature is clearly additive under disjoint unions, it implies that the signature is a homomorphism from the oriented $4n$-dimensional bordism group to $\mathbb{Z}$. This is one of the easiest ways of seeing that this bordism group is not trivial. □

Proof of Theorem 0.5. Theorem 0.4 implies that $\mathrm{H}_{2n}(M^{4n}; \mathbb{R})$ has a Lagrangian, so this follows immediate from Lemma 0.7 below. □

Lemma 0.7. *Let V be a finite-dimensional vector space over $\mathbb{R}$ equipped with a nondegenerate symmetric bilinear form $\omega(-,-)$. Assume that there exists a Lagrangian L in V. Then the signature of ω is 0.*

Proof. Lemma 0.2 implies that V is the orthogonal direct sum of 2-dimensional subspaces on which the form ω is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These are often called "hyperbolic planes". Since their signature is 0 and the signature is additive under orthogonal direct sums, the signature of ω is 0. □

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