The generalized Schoenflies theorem

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Abstract

The generalized Schoenflies theorem asserts that if $\phi : S^{n-1} \to S^n$ is a topological embedding and A is the closure of a component of $S^n \smallsetminus \phi(S^{n-1})$, then $A \cong \mathbb{D}^n$ as long as A is a manifold. This was originally proved by Barry Mazur and Morton Brown using rather different techniques. We give both of these proofs.

1 Introduction

Let $\phi : S^{n-1} \to S^n$ be a topological embedding with $n \geq 2$. It follows from Alexander duality (see [6, Theorem 3.44]) that $S^n \smallsetminus \phi(S^{n-1})$ has two connected components. Let A and B their closures, so $S^n = A \cup B$ and $A \cap B = \phi(S^{n-1})$. If $n = 2$, then the classical Jordan-Schoenflies theorem says that $A \cong \mathbb{D}^2$ and $B \cong \mathbb{D}^2$. However, this need not hold for $n \geq 3$. Indeed, the Alexander horned sphere is an embedding $\alpha : S^2 \to S^3$ such that one of the two components of $S^3 \smallsetminus \alpha(S^2)$ is not simply connected. In fact, something even worse is true: the closure of the non-simply-connected component of $S^3 \smallsetminus \alpha(S^2)$ is not even a manifold!

It turns out that this is the only thing that can go wrong.

Generalized Schoenflies Theorem. *Let $\phi : S^{n-1} \to S^n$ be a topological embedding with $n \geq 2$ and let A be the closure of a component of $S^n \smallsetminus \phi(S^{n-1})$. Assume that A is a manifold with boundary. Then $A \cong \mathbb{D}^n$.*

The generalized Schoenflies theorem was originally proved by Barry Mazur [7] and Morton Brown [3] in rather different ways, though both approaches are striking and completely elementary. These notes contain an exposition of both of these proofs.

Remark. In fact, Mazur proved a seemingly weaker theorem earlier than Brown which Morse [9] proved implied the general result. The paper [9] of Morse appears in the same volume as Brown's paper [3].

Remark. The closures of the components of $S^n \smallsetminus \phi(S^{n-1})$ are known as *crumpled n-cubes*. When they are not manifolds, they have complicated fractal singularities. However, Bing [2, Theorem 4] proved that a crumpled n-cube is a retract of $\mathbb{R}^n$, and in particular is contractible.

Remark. It is possible for the closures of both components of $S^n \smallsetminus \phi(S^{n-1})$ to be non-manifolds; indeed, Bing [1] proved that the space obtained by "doubling" the closure of the "bad" component of the complement of the Alexander horned sphere along the horned sphere "boundary" is homeomorphic to S^3.

Both proofs of the generalized Schoenflies theorem start by using the assumption that A is a manifold to find a *collar neighborhood* of $\partial A \cong S^{n-1}$, i.e. an embedding $\partial A \times [0,1] \to A$ that takes $(a, 0) \in \partial A \times [0,1]$ to a. The existence of collar neighborhoods is a theorem of

Morton Brown [4]; we give a very short proof due to Connelly [5] in §2. Next, one considers a small round disc $D' \subset S^n$ lying in $\partial A \times [0,1] \subset A$. Setting $D = S^n \setminus D'$, we have $D \cong \mathbb{D}^n$. The strategy of both proofs is to parlay the homeomorphism $D \cong \mathbb{D}^n$ into a homeomorphism $A \cong \mathbb{D}^n$. They do this in different ways. Mazur's proof, which we discuss in §3, uses a clever infinite boundary connect sum to deduce the desired result. This argument resembles the Eilenberg swindle in algebra; at a formal level, is is based on the ersatz "proof"

$$0 = (1-1) + (1-1) + \cdots = 1 + (-1+1) + (-1+1) + \cdots = 1.$$

Brown's proof, which we discuss in §4, instead uses a technique called Bing shrinking to understand the complement $S^n \setminus (\partial A \times [0,1])$.

2 Collar neighborhoods

In this section, we give a short proof due to Connelly [5] of the following theorem of Morton Brown [4]. Recall that if M is a manifold with boundary, then a *collar neighborhood* of ∂M is a closed neighborhood C of ∂M such that $C \cong \partial M \times [0,1]$.

Theorem 2.1. *Let M be a compact manifold with boundary. Then ∂M has a collar neighborhood.*

Proof. Define N to be the result of gluing $\partial M \times (-\infty, 0]$ to M by identifying $(m,0) \in \partial M \times [-\infty, 0]$ with $m \in \partial M$. For $s \in (-\infty, 0]$, let $N_s \subset N$ be the subset consisting of M and $\partial M \times [s, 0]$. The theorem is equivalent to the assertion that $M \cong N_{-1}$, which we will prove by "dragging" ∂M over the collar a little at a time using a sequence of homeomorphisms $\eta_i : N \to N$.

Let $\{U_1, \ldots, U_k\}$ be an open cover of ∂M such that each U_i is equipped with an embedding $\phi_i : U_i \times [0,1] \to M$. Extend ϕ_i to an embedding $\psi_i : U_i \times (-\infty, 1] \to N$ in the obvious way. Let $\{\rho_i : U_i \to [0,1]\}_{i=1}^k$ be a partition of unity subordinate to the U_i. For $0 \le a \le 1$, define a function $\zeta_a : (\infty, 1] \to (-\infty, 1]$ via the formula

$$\zeta_a(t) = \begin{cases} t - a & \text{if } -\infty < t \le 0, \\ (1+2a)t - a & \text{if } 0 \le t \le 1/2, \\ t & \text{if } 1/2 \le t \le 1. \end{cases}$$

In particular, $\zeta_0 = \mathrm{id}$. Each function ζ_a is a homeomorphism satisfying $\zeta_a(0) = -a$ and $\zeta_a|_{[1/2,1]} = \mathrm{id}$. For $1 \le i \le k$, let $\widehat{\eta_i} : U_i \times (-\infty, 1] \to U_i \times (-\infty, 1]$ be the homeomorphism given by the formula

$$\widehat{\eta_i}(u,t) = (u, \zeta_{\rho_i(u)}(t)).$$

The homeomorphism $\widehat{\eta_i}$ is the identity outside the set $\mathrm{supp}(\rho_i) \times (-\infty, 1/2] \subset U_i \times (-\infty, 1]$. We can therefore extend it by identity to a homeomorphism $\eta_i : N \to N$. The homeomorphism $\eta_1 \circ \eta_2 \circ \cdots \circ \eta_k : N \to N$ then restricts to a homeomorphism between M and N_{-1}, as desired. $\square$

3 Schoenflies via infinite repetition

In this section, we give Barry Mazur's proof of the generalized Schoenflies theorem, which originally appeared in [7]. In fact, the paper [7] proves the following seemingly weaker theorem; we will deduce the general case using an argument of Morse [9]. We say that a subspace $D' \subset \mathrm{Int}(S^{n-1} \times [0,1])$ is a round n-disc if it is such when $S^{n-1} \times [0,1]$ is regarded as the usual tubular neighborhood of the equator in S^n.

Lemma 3.1. *Let $\widehat{\phi}: S^{n-1} \times [0,1] \to S^n$ be an embedding with $n \geq 2$ and let A be the closure of the component of $S^n \smallsetminus \widehat{\phi}(S^{n-1} \times 0)$ that contains $\widehat{\phi}(S^{n-1} \times (0,1])$. Assume that there exists a round n-disc $D' \subset \mathrm{Int}(S^{n-1} \times [0,1])$ such that $D := S^n \smallsetminus \widehat{\phi}(\mathrm{Int}(D'))$ satisfies $D \cong \mathbb{D}^n$. Then $A \cong \mathbb{D}^n$.*

Proof. We begin by introducing some notation. We will identify S^{n-1} with

$$\{(t_1, \ldots, t_n) \in [0,1]^n \mid \text{there exists some } 1 \leq i \leq n \text{ with } t_i \in \{0,1\}\}.$$

If C and C' are n-manifolds whose boundaries are identified with S^{n-1} in a fixed way, then define $C + C'$ to be the result of identifying $(1, t_2, \ldots, t_n) \in \partial C \cong S^{n-1}$ with $(0, t_2, \ldots, t_n) \in \partial C' \cong S^{n-1}$ for all $(t_2, \ldots, t_n) \in [0,1]^{n-1}$. It is easy to see that $C + C' \cong C' + C$. If $C_1, C_2, \ldots$ are n-manifolds whose boundaries are identified with S^{n-1} in a fixed way, then we have

$$C_1 \subset C_1 + C_2 \subset C_1 + C_2 + C_3 \subset \cdots.$$

We will write $C_1 + C_2 + \cdots$ for the union of this increasing sequence of spaces.

We now turn to the proof of Lemma 3.1. Let B be the closure of the component of $S^n \smallsetminus \widehat{\phi}(S^{n-1} \times 1)$ that is not contained in A. Both A and B are n-manifolds whose boundaries are homeomorphic to S^{n-1}, and we will fix homeomorphisms between S^{n-1} and these boundaries. The first observation is that $A + B \cong D$, and hence $A + B \cong \mathbb{D}^n$. To see this, observe that the fact that D' is a round n-disc in $\mathrm{Int}(S^{n-1} \times [0,1])$ implies that $S := (S^{n-1} \times [0,1]) \smallsetminus \mathrm{Int}(D')$ is homeomorphic to an n-disc with the interiors of two disjoint n-discs in its interior removed. Letting X and Y be the components of $S^n \smallsetminus \widehat{\phi}(S^{n-1} \times (0,1))$ ordered so that $X \subset A$ and $Y \subset B$, the disc D is formed by gluing X and Y to two of the boundary components of S. As is shown in Figure 1, the result is homeomorphic to $A + B$.

Since $A + B \cong \mathbb{D}^n$, we have

$$A + B + A + B + \cdots \cong \mathbb{D}^n + \mathbb{D}^n + \mathbb{D}^n + \cdots.$$

As is shown in Figure 1, this implies that $A + B + A + B + \cdots$ is homeomorphic to the upper half space

$$\{(s_1, \ldots, s_n) \in \mathbb{R}^n \mid s_1 \geq 0\}.$$

For a space M, let $\mathcal{P}(M)$ be the one-point compactification of M. The above identification of $A + B + A + B + \cdots$ implies that $\mathcal{P}(A + B + A + B + \cdots) \cong \mathbb{D}^n$. In a similar way, the fact that $B + A \cong \mathbb{D}^n$ implies that $\mathcal{P}(B + A + B + A + \cdots) \cong \mathbb{D}^n$. We therefore deduce that

$$\mathbb{D}^n \cong \mathcal{P}(A + B + A + B + \cdots) \cong A + \mathcal{P}(B + A + B + A + \cdots) \cong A + \mathbb{D}^n \cong A,$$

as desired. □

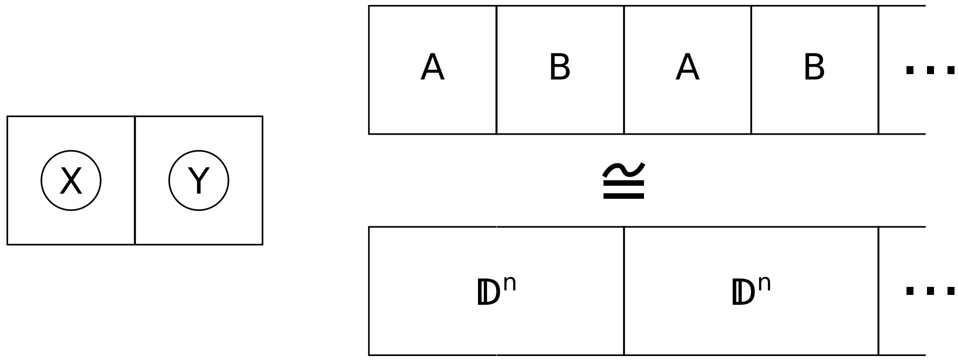

Figure 1: *LHS: A drawing of D. The outer boundary component is $\partial D = \partial D'$. The inner two boundary components are the places to which X and Y are glued. The left square is homeomorphic to A and the right square is homeomorphic to B. RHS: The space $A+B+A+B+\cdots$ is homeomorphic to $\mathbb{D}^n + \mathbb{D}^n + \mathbb{D}^n + \cdots$, which is homeomorphic to the upper half space $\{(s_1,\ldots,s_n) \in \mathbb{R}^n \mid s_1 \geq 0\}$.*

Proof of the generalized Schoenflies theorem. We recall the setup. Let $\phi : S^{n-1} \to S^n$ be a topological embedding and let A be the closure of a component of $S^n \smallsetminus \phi(S^{n-1})$. Assume that A is a manifold. Our goal is to prove that $A \cong \mathbb{D}^n$. Using Theorem 2.1, we can extend ϕ to an embedding $\widehat{\phi} : S^{n-1} \times [0,1] \to S^n$ whose image lies in A. Choose some point $p_0 \in S^{n-1} \times (0,1)$. We will regard $S^{n-1} \times [0,1]$ as lying in S^n as the standard tubular neighborhood of the equator. Using this convention, we can compose everything in sight with a rotation and assume that $\widehat{\phi}(p_0) = p_0$. Let $D' \subset S^{n-1} \times (0,1)$ be a small round disc around p_0. Choosing a second point $q_0 \in S^n \smallsetminus A$, we will construct a continuous map $f : S^n \smallsetminus \{q_0\} \to S^n$ with the following properties.

- The map f is a homeomorphism onto its image, which is an open subset of S^n.
- The embedding $f \circ \widehat{\phi} : S^{n-1} \times [0,1] \to S^n$ restricts to the identity on D'.

The embedding $f \circ \widehat{\phi}$ will thus satisfy the conditions of Lemma 3.1 and we will be able to conclude that $f(A) \cong \mathbb{D}^n$, and hence that $A \cong \mathbb{D}^n$.

It remains to construct f. Let B be a small open round ball around p_0 in S^n such that B lies in $\widehat{\phi}(S^{n-1} \times (0,1))$ and such that $D' \subset B$. Let $g : S^n \smallsetminus \{q_0\} \to B$ be a homeomorphism such that $g|_{D'} = \mathrm{id}$. Also, let $C = \widehat{\phi}^{-1}(B)$. Define $f : S^n \smallsetminus \{q_0\} \to S^n$ to be the composition

$$S^n \smallsetminus \{q_0\} \xrightarrow{g} B \xrightarrow{(\widehat{\phi}|_C)^{-1}} C \hookrightarrow S^n.$$

The map f clearly satisfies the above conditions, and the theorem follows. □

4 Schoenflies via Bing shrinking

In this section, we give Morton Brown's proof of the generalized Schoenflies theorem, which originally appeared in [3]. Before we launch into the details, we discuss the strategy of the proof. Let $\phi : S^{n-1} \to S^n$ be a topological embedding and let A be the closure of a component of $S^n \smallsetminus \phi(S^{n-1})$. Assume that A is a manifold with boundary. Using Theorem 2.1, we can extend ϕ to an embedding $\widehat{\phi} : S^{n-1} \times [0,1] \to S^n$ whose image lies in A. Our goal is to prove that $A \cong \mathbb{D}^n$. Let X and Y be the two components of $S^n \smallsetminus \widehat{\phi}(S^{n-1} \times (0,1))$, ordered so that $X \subset A$. The key observation is that there exists a surjective map $f : S^n \to S^n$ that collapses X and Y to points x and y, respectively, and is otherwise injective. Clearly f restricts to a surjection from A to a disc $\mathbb{D}^n \subset S^n$. What Brown showed was that X has a certain topological property that ensures that $A \cong A/X$.

This topological property enjoyed by X is that X is cellular, which we now define. A subset X of an n-manifold is *cellular* if for all open sets U containing X, we can write $X = \bigcap_{i=1}^{\infty} C_i$, where for all $i \geq 1$ the set C_i satisfies

$$C_i \subset U \quad \text{and} \quad C_i \cong \mathbb{D}^n \quad \text{and} \quad C_{i+1} \subset \operatorname{Int}(C_i).$$

Since each C_i is closed, this implies that X is closed. Before we state the main consequence of being cellular, we must introduce some terminology for collapsing subsets of manifolds. Let M be a compact manifold with boundary and let $X_1, \ldots, X_s$ be pairwise disjoint closed subsets of M. The result of *collapsing* the sets $X_1, \ldots, X_s$ is the quotient space $M/\sim$, where for distinct $z, z' \in M$ we have $z \sim z'$ if and only if there exists some $1 \leq i \leq s$ such that $z, z' \in X_i$. The projection $M \to M/\sim$ is the *collapse map* of $X_1, \ldots, X_s$.

Lemma 4.1. *Let M be a compact n-manifold with boundary and let $X_1, \ldots, X_s$ be pairwise disjoint cellular subsets of* $\operatorname{Int}(M)$. *Define M' to be the result of collapsing $X_1, \ldots, X_s$. Then M is homeomorphic to M'.*

Proof. Using induction, it is enough to deal with the case $s = 1$, so let $X \subset \operatorname{Int}(M)$ be a cellular subset. We will construct a surjective map $f : M \to M$ such that $f|_{M \setminus X}$ is injective and such that there exists some $x_0 \in M$ with $f^{-1}(x_0) = X$. These conditions ensure that f is the collapse map of X, so M will be homeomorphic to the result of collapsing X.

Write $X = \bigcap_{i=1}^{\infty} C_i$, where for all $i \geq 1$ we have

$$C_i \subset \operatorname{Int}(M) \quad \text{and} \quad C_i \cong \mathbb{D}^n \quad \text{and} \quad C_{i+1} \subset \operatorname{Int}(C_i).$$

The surjective map f will be the limit of a sequence of homeomorphisms $f_i : M \to M$ that are constructed inductively. First, $f_1 = \mathrm{id}$. Next, assume that $f_i : M \to M$ has been constructed for some $i \geq 1$. We have

$$f_i(C_i) \cong \mathbb{D}^n \quad \text{and} \quad f_i(C_{i+1}) \cong \mathbb{D}^n \quad \text{and} \quad f_i(C_{i+1}) \subset \operatorname{Int}(f_i(C_i)).$$

We can therefore choose a homeomorphism $\widehat{g}_{i+1} : f_i(C_i) \to f_i(C_i)$ that restricts to the identity on $\partial(f_i(C_i))$ and satisfies $\operatorname{diam}(\widehat{g}_{i+1}(f_i(C_{i+1}))) \leq \frac{1}{i+1}$. Extend $\widehat{g}_{i+1}$ by the identity to a homeomorphism $g_{i+1} : M \to M$ and define $f_{i+1} = g_{i+1} \circ f_i$.

We now prove that for all $p \in M$, the sequence of points $f_j(p)$ approaches a limit. There are two cases. If $p \in X$, then $f_j(p) \in f_j(C_j)$ for all j. By construction, we have

$$f_1(C_1) \supset f_2(C_2) \supset f_3(C_3) \supset \cdots \quad \text{and} \quad \lim_{j \to \infty} \operatorname{diam}(f_j(C_j)) = 0.$$

The set $\cap_{j=1}^{\infty} f_j(C_j)$ therefore reduces to a single point x_0 and $\lim_{j \to \infty} f_j(p) = x_0$. If instead $p \notin X$, then something even stronger happens: the sequence of points

$$f_1(p), f_2(p), f_3(p), \ldots$$

is eventually constant. Indeed, if $k \geq 1$ is such that $p \notin C_k$, then $f_j(p) = f_{j-1}(p)$ for $j \geq k$. Thus $\lim_{j\to\infty} f_j(p)$ equals $f_j(p)$ for $j \gg 0$.

We can therefore define a map $f : M \to M$ via the formula

$$f(p) = \lim_{j\to\infty} f_j(p) \qquad (p \in \mathbb{D}^n).$$

It is clear that f is a continuous map and that $f^{-1}(x_0) = X$. To deduce the lemma, we must show that f is surjective and that $f|_{M\smallsetminus X}$ is injective.

We begin with surjectivity. Clearly the image of f contains x_0, so it is enough to show that it contains an arbitrary point $q \in M \smallsetminus \{x_0\}$. For $\ell \gg 0$, we have $q \notin f_\ell(C_\ell)$, and hence $f_\ell^{-1}(q) \notin C_\ell$ and $f(f_\ell^{-1}(q)) = f_\ell(f_\ell^{-1}(q)) = q$, so q is in the image of f.

We next prove that $f|_{M\smallsetminus X}$ is injective. Consider distinct point $r, r' \in M \smallsetminus X$. We can find $m \gg 0$ such that $f(r) = f_m(r)$ and $f(r') = f_m(r')$. Since f_m is a homeomorphism, we therefore have $f(r) \neq f(r')$. The lemma follows □

Remark. The technique used to prove Lemma 4.1 is called *Bing shrinking*; it was introduced by Bing in [1] to prove that the double of the Alexander horned ball is homeomorphic to the 3-sphere and plays a basic role in many delicate results in geometric topology.

To make use of Lemma 4.1, we need a way of recognizing when a set is cellular. This is subtle in general, but for closed subsets X of the interior of a disc $\mathbb{D}^n$ it turns out that X is cellular if the conclusion of Lemma 4.1 holds, namely if the result of collapsing X is homeomorphic to $\mathbb{D}^n$. We will actually need the following slight strengthening of this fact.

Lemma 4.2. *Let $X_1, \ldots, X_s$ be pairwise disjoint closed subsets of $\mathrm{Int}(\mathbb{D}^n)$. Define M' to be the result of collapsing $X_1, \ldots, X_s$ and let $\pi : \mathbb{D}^n \to M'$ be the collapse map. Assume that there exists an embedding $M' \hookrightarrow S^n$ that takes $\pi(\mathrm{Int}(\mathbb{D}^n)) \subset M'$ to an open subset of S^n. Then each X_i is cellular.*

Proof. The proof will be by induction on s. The base case will be $s = 0$, in which case the lemma has no content. Assume now that $s > 0$ and that the lemma is true for all smaller collections of sets. Let $f : \mathbb{D}^n \to S^n$ be the composition of π and the embedding given by the assumptions and let $x_i = f(X_i)$ for $1 \leq i \leq s$. Let U be an open set in $\mathbb{D}^n$ with $X_s \subset U$, so $x_s \in f(U)$. Fix a metric on S^n, and for all $\delta > 0$ let $B_\delta \subset S^n$ be the ball around x_s of radius δ. Choose $\epsilon > 0$ such that $B_\epsilon \subset f(U)$ and such that $x_i \notin B_\epsilon$ for $1 \leq i \leq s-1$. For $j \geq 1$, let $h_j : S^n \to S^n$ be an injective continuous map such that $h_j(f(\mathbb{D}^n)) \subset B_{\epsilon/j}$ and such that $h_j|_{B_{\epsilon/(j+1)}} = \mathrm{id}$. Next, define $g_j : \mathbb{D}^n \to \mathbb{D}^n$ via the formula

$$g_j(z) = \begin{cases} z & \text{if } z \in X_s, \\ f^{-1} \circ h_j \circ f(z) & \text{if } z \notin X_s. \end{cases}$$

This expression makes sense since $h_j \circ f(z) \neq x_s$ if $z \notin X$, so $f^{-1} \circ h_j \circ f(z)$ is a single well-defined point. Since $h_j|_{B_{\epsilon/(j+1)}} = \mathrm{id}$, the function g_j restricts to the identity on $f^{-1}(B_{\epsilon/(j+1)})$, and hence g_j is a continuous map. Set $C_j = g_j(\mathbb{D}^n) \subset \mathbb{D}^n$. By construction, C_j is the result of collapsing $X_1, \ldots, X_{s-1}$. We can therefore apply our inductive hypothesis to deduce that

X_i is cellular for $1 \le i \le s-1$; here we are using the fact that $\mathbb{D}^n$ can be embedded in S^n. Applying Lemma 4.1, we get that $C_j \cong \mathbb{D}^n$. We also also have

$$X_s \subset f^{-1}(B_{\epsilon/(j+1)}) \subset C_j \subset f^{-1}(B_{\epsilon/j}) \subset U.$$

The sets C_j thus satisfy the conditions in the definition of a cellular set, so X_s is also cellular, as desired. □

Proof of the generalized Schoenflies theorem. The setup is just as in the beginning of this section. Let $\phi : S^{n-1} \to S^n$ be a topological embedding and let A be the closure of a component of $S^n \smallsetminus \phi(S^{n-1})$. Assume that A is a manifold. Using Theorem 2.1, we can extend ϕ to an embedding $\widehat{\phi} : S^{n-1} \times [0,1] \to S^n$ whose image lies in A. Let X and Y be the two components of $S^n \smallsetminus \widehat{\phi}(S^{n-1} \times (0,1))$, ordered so that $X \subset A$. As in the beginning of this section, let $f : S^n \to S^n$ be the collapse map of X and Y. Let $D' \subset S^n \smallsetminus (X \cup Y)$ be a small round disc. Letting $D = S^n \smallsetminus D'$, we have $D \cong \mathbb{D}^n$. The restriction of f to D is the composition of the collapse map of X and Y (considered as subsets of D) with an inclusion into S^n. Applying Lemma 4.2, we deduce that X and Y are cellular (in fact, we only need this for X). Finally, applying Lemma 4.1 we see that $A \cong A/X \cong \mathbb{D}^n$, as desired. □

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