

The Johnson homomorphism and its kernel

Andrew Putman*

Abstract

We give a new proof of a celebrated theorem of Dennis Johnson that asserts that the kernel of the Johnson homomorphism on the Torelli subgroup of the mapping class group is generated by separating twists. In fact, we prove a more general result that also applies to “subsurface Torelli groups”. Using this, we extend Johnson’s calculation of the rational abelianization of the Torelli group not only to the subsurface Torelli groups, but also to finite-index subgroups of the Torelli group that contain the kernel of the Johnson homomorphism.

Contents

1	Introduction	1
2	Preliminaries	6
2.1	Basic facts about the mapping class group	6
2.2	Basic facts about the Torelli group	7
2.3	Birman exact sequence for Torelli group	8
2.4	The Johnson homomorphism and kernel	12
3	The subsurface Johnson homomorphisms and kernels	13
4	Birman exact sequence for the Johnson kernel	14
4.1	One boundary	14
4.2	Separating boundary	15
4.3	Nonseparating boundary	17
5	Generating the Johnson kernel	22
5.1	Generating the Johnson kernel disc-pushing subgroup	23
5.2	The Johnson kernel is generated by separating twists	24
6	Finite-index subgroups of the Torelli group	28
6.1	Separating twists are torsion, special case	29
6.2	Separating twists are torsion, general case	30

1 Introduction

Let Σ_g^n be a compact oriented genus g surface with n boundary components (we will often omit the n if it vanishes). The *mapping class group* of Σ_g^n , denoted $\text{Mod}(\Sigma_g^n)$, is the group of isotopy classes of orientation preserving homeomorphisms of Σ_g^n that act as the identity on

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$\partial\Sigma_g^n$ (see [9]). The group $\text{Mod}(\Sigma_g^n)$ acts on $H_1(\Sigma_g^n; \mathbb{Z})$ and preserves the algebraic intersection form. If $n \leq 1$, then this is a nondegenerate alternating form, so we obtain a representation $\text{Mod}(\Sigma_g^n) \rightarrow \text{Sp}_{2g}(\mathbb{Z})$. This representation is well-known to be surjective. Its kernel $\mathcal{I}(\Sigma_g^n)$ is known as the *Torelli group*. This is all summarized in the exact sequence

$$1 \longrightarrow \mathcal{I}(\Sigma_g^n) \longrightarrow \text{Mod}(\Sigma_g^n) \longrightarrow \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

Johnson’s work. In the early 1980’s, Dennis Johnson wrote a sequence of remarkable papers about $\mathcal{I}(\Sigma_g^n)$ (see [17] for a survey). Many of his results center on the *Johnson homomorphisms*, which he defined in [16]. Letting $H = H_1(\Sigma_g^n; \mathbb{Z})$, these are homomorphisms

$$\tau_{\Sigma_g^1} : \mathcal{I}(\Sigma_g^1) \longrightarrow \wedge^3 H \quad \text{and} \quad \tau_{\Sigma_g} : \mathcal{I}(\Sigma_g) \longrightarrow (\wedge^3 H)/H$$

which measure the “unipotent” part of the action of $\mathcal{I}(\Sigma_g^n)$ on the second nilpotent truncation of $\pi_1(\Sigma_g^n)$. The Johnson homomorphisms subsequently showed up in many different parts of mathematics; for instance, 3-manifold topology (see, e.g. [11]), the geometry of the moduli space of curves (see, e.g. [12, 28]), and number theory (see, e.g., [22]). The kernel $\mathcal{K}(\Sigma_g^n)$ of $\tau_{\Sigma_g^n}$ is known as the *Johnson kernel*.

Our results. The goal of this paper is to generalize two of Johnson’s theorems.

1. For $g \geq 0$ and $n \leq 1$, Johnson [19] proved that $\mathcal{K}(\Sigma_g^n)$ is generated by *separating twists*, that is, Dehn twists T_x about separating curves x . We will generalize this to the “subsurface Torelli groups” defined by the author in [25]. Our proof is independent of Johnson’s work, so in particular it provides a new proof of his theorem. Our new proof is much less computationally intensive than Johnson’s proof.
2. For $g \geq 3$ and $n \leq 1$, Johnson [20] proved that modulo torsion the Johnson homomorphism gives the abelianization of $\mathcal{I}(\Sigma_g^n)$. We will generalize this not only to subsurface Torelli groups, but also to certain finite-index subgroups of $\mathcal{I}(\Sigma_g^n)$.

Subsurface Torelli groups. At this point, we have only discussed the Torelli group on surfaces with at most one boundary component. Surfaces with multiple boundary components often arise during proofs by induction on the genus of a surface. One argues that some phenomena is “concentrated” on subsurfaces of a closed surface, which are lower-genus surfaces with boundary. For the Torelli group, this leads to the following definition (see [25]). Let S be a compact connected surface with boundary and let $S \hookrightarrow \Sigma_g^n$ be an embedding with $n \leq 1$. We then define $\widehat{\mathcal{I}}(S, \Sigma_g^n)$ to be the subgroup of $\mathcal{I}(\Sigma_g^n)$ consisting of mapping classes which are *supported* on S , that is, which can be realized by homeomorphisms that are the identity on $\Sigma_g^n \setminus S$. The group so obtained depends strongly on the embedding $S \hookrightarrow \Sigma_g^n$; see [25] for a discussion.

Remark. See below for the reason for the “hat” in $\widehat{\mathcal{I}}(S, \Sigma_g^n)$.

The Johnson kernel. For a compact connected subsurface S of Σ_g^n with $n \leq 1$, define $\widehat{\mathcal{K}}(S, \Sigma_g^n)$ to be the kernel of the restriction of the Johnson homomorphism $\tau_{\Sigma_g^n}$ to $\widehat{\mathcal{I}}(S, \Sigma_g^n) \subset \mathcal{I}(\Sigma_g^n)$. Our first theorem is as follows. When $S = \Sigma_g^n$, it reduces to the aforementioned theorem of Johnson giving generators for $\mathcal{K}(\Sigma_g^n)$.

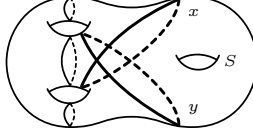


Figure 1: $S \cong \Sigma_{1,3}$ is embedded in Σ_3 . We conjecture that $[T_x, T_y]$ cannot be written as a product of separating twists in $\widehat{\mathcal{I}}(S, \Sigma_3)$

Theorem A (Generators for Johnson kernel). *For $g \geq 2$ and $n \leq 1$, let $S \hookrightarrow \Sigma_g^n$ be a compact connected subsurface of genus at least 2. Then $\widehat{\mathcal{K}}(S, \Sigma_g^n)$ is generated by $\{T_\gamma \mid \gamma \text{ is a separating curve on } \Sigma_g^n \text{ and } \gamma \subset S\}$.*

Remark. We believe that the restriction on the genus of S is necessary. Let $S \hookrightarrow \Sigma_3$ be the subsurface depicted in Figure 1 and let $x, y \subset S$ be the curves shown there. Then it is not hard to see that $[T_x, T_y] \in \widehat{\mathcal{K}}(S, \Sigma_3)$, but we conjecture that $[T_x, T_y]$ cannot be written as a product of separating twists lying in S .

Remark. As indicated in the proof sketch below, the first step of our proof reduces Theorem A to Johnson's result. However, the extra generality of Theorem A is needed for the second step, which reduces Johnson's result to understanding subsurfaces. In other words, from our point of view the more general statement is necessary even if you only care about the classical case.

Abelianizing Torelli and its subgroups. Our second theorem is as follows. When $S = \Sigma_g^n$, it reduces to Johnson's aforementioned theorem asserting that the Johnson homomorphism gives the abelianization of $\mathcal{I}(\Sigma_g^n)$ modulo torsion.

Theorem B (Rational H_1 of finite-index subgroups of Torelli). *For $g \geq 3$ and $n \leq 1$, let $S \hookrightarrow \Sigma_g^n$ be a compact connected subsurface of genus at least 3. Let Γ be a finite-index subgroup of $\widehat{\mathcal{I}}(S, \Sigma_g^n)$ with $\widehat{\mathcal{K}}(S, \Sigma_g^n) < \Gamma$. Then $H_1(\Gamma; \mathbb{Q}) \cong \tau_{\Sigma_g^n}(\widehat{\mathcal{I}}(S, \Sigma_g^n)) \otimes \mathbb{Q}$.*

Remark. After an early version of this paper was circulated, Dimca, Hain, and Papadima [8] calculated $H_1(\mathcal{K}(\Sigma_g^n); \mathbb{Q})$ for $g \geq 6$. Their proof made use of Theorem B.

Remark. Mess [21] showed that $\mathcal{I}(\Sigma_2)$ is an infinitely generated free group. Using this, it is not hard to show that if $S \hookrightarrow \Sigma_g^n$ is a subsurface of genus 2, then $H_1(\widehat{\mathcal{I}}(S, \Sigma_g^n); \mathbb{Q})$ has infinite rank. In particular, the conclusion of Theorem B does not hold.

Remark. After this paper was written, Church [6] showed how to define the Johnson homomorphism on $\widehat{\mathcal{I}}(S, \Sigma_g^n)$ intrinsically, that is, without reference to the Johnson homomorphism on $\mathcal{I}(\Sigma_g^n)$. This allowed him to compute the rank of $\tau_{\Sigma_g^n}(\widehat{\mathcal{I}}(S, \Sigma_g^n)) \otimes \mathbb{Q}$.

Remark. Theorem B is inspired by a well-known conjecture of Ivanov that asserts that if $g \geq 3$ and Γ is a finite-index subgroup of $\text{Mod}(\Sigma_g)$, then $H_1(\Gamma; \mathbb{Q}) \cong H_1(\text{Mod}(\Sigma_g); \mathbb{Q}) = 0$ (see [14] for a recent discussion). In analogy with this, one might guess that the condition $\widehat{\mathcal{K}}(S, \Sigma_g^n) < \Gamma$ is unnecessary. We remark that Ivanov's conjecture is only known in a few special cases – Hain [12] proved it if $\mathcal{I}(\Sigma_g) < \Gamma$, and Boggi [4] and the author [26] later independently proved that it holds if $\mathcal{K}(\Sigma_g) < \Gamma$.

About the hats. Let $S \hookrightarrow \Sigma_g^n$ be a subsurface with $n \leq 1$. For simplicity, assume that S is a *clean* embedding, that is, no components of $\Sigma_g^n \setminus \text{Int}(S)$ are discs. Instead of the subgroups $\widehat{\mathcal{I}}(S, \Sigma_g^n)$ and $\widehat{\mathcal{K}}(S, \Sigma_g^n)$ of $\text{Mod}(\Sigma_g^n)$ discussed above, most of this paper will study the groups $\mathcal{I}(S, \Sigma_g^n), \mathcal{K}(S, \Sigma_g^n) \subset \text{Mod}(S)$ obtained by pulling these groups back under the map $\text{Mod}(S) \rightarrow \text{Mod}(\Sigma_g^n)$ that extends mapping classes by the identity. The natural surjections $\mathcal{I}(S, \Sigma_g^n) \rightarrow \widehat{\mathcal{I}}(S, \Sigma_g^n)$ and $\mathcal{K}(S, \Sigma_g^n) \rightarrow \widehat{\mathcal{K}}(S, \Sigma_g^n)$ are usually (but not always) isomorphisms; see §2.1. Theorem A above will be derived from Theorem 5.1 below, which is an analogous (but slightly more awkward) theorem about $\mathcal{K}(S, \Sigma_g^n)$. See the remark after Theorem 2.5 below for a discussion of why we mostly focus on $\mathcal{I}(S, \Sigma_g^n)$ and $\mathcal{K}(S, \Sigma_g^n)$.

Proof ideas for Theorem A. Let S and Σ_g^n be as in Theorem A. Our proof of Theorem A is by induction on the genus and number of boundary components of S . The base case of the induction is when $S \cong \Sigma_2$, in which case S is the entire surface Σ_g^n (in particular, $g = 2$ and $n = 0$). It turns out that the Johnson homomorphism vanishes on $\mathcal{I}(\Sigma_2)$, so we have $\mathcal{K}(\Sigma_2) = \mathcal{I}(\Sigma_2)$. We can therefore appeal to a theorem of Powell [24] which says that $\mathcal{I}(\Sigma_2)$ is generated by separating twists. The inductive argument then has two steps.

Step 1 (Reduction to the closed case). *Assume that $S \cong \Sigma_h^m$ and that $\mathcal{K}(\Sigma_h)$ is generated by separating twists. Then $\widehat{\mathcal{K}}(S, \Sigma_g^n)$ is generated by separating twists.*

This step is achieved by a version of the classical Birman exact sequence, which we will comment more on below. The second step of our induction is as follows.

Step 2 (Reducing the genus). *Assume that $\widehat{\mathcal{K}}(S, \Sigma_g)$ is generated by separating twists for all subsurfaces S of Σ_g satisfying $S \cong \Sigma_{g-1}^2$. Then $\mathcal{K}(\Sigma_g)$ is generated by separating twists.*

For this step, we use a result of the author [27] which says that $\mathcal{I}(\Sigma_g)$ is generated by elements living on certain subsurfaces of $\mathcal{I}(\Sigma_g)$.

Remark. An earlier version of this paper performed Step 2 by examining the action of $\mathcal{K}(\Sigma_g)$ on a simplicial complex related to the curve complex. Our new approach is both shorter and less technical.

The Birman exact sequence. As we said above, the first step of our proof of Theorem A uses a version of the Birman exact sequence. The classical version of this is a fundamental tool for relating the mapping class groups of surfaces with differing numbers of boundary components. For $g \geq 2$ and $n \leq 1$, it takes the form

$$1 \longrightarrow \pi_1(U\Sigma_g^{n-1}) \longrightarrow \text{Mod}(\Sigma_g^n) \longrightarrow \text{Mod}(\Sigma_g^{n-1}) \longrightarrow 1.$$

Here $U\Sigma_g^{n-1}$ is the unit tangent bundle of Σ_g^{n-1} and the subgroup $\pi_1(U\Sigma_g^{n-1})$ of $\text{Mod}(\Sigma_g^n)$ is the “disc-pushing” subgroup; it “pushes” a boundary component β of Σ_g^n around the surface while allowing it to rotate. The map $\text{Mod}(\Sigma_g^n) \rightarrow \text{Mod}(\Sigma_g^{n-1})$ is the map obtained by gluing a disc to β and extending mapping classes over the disc by the identity. A version of this for the subsurface Torelli groups was proven by the author in [25]; see §2.2 below. In this paper, we give an appropriate version of this for the subsurface Johnson kernels; see §4.1–4.3 for precise statements.

Remark. After an early version of this paper was circulated, our Birman exact sequence was applied by Bestvina, Bux, and Margalit [1] during their calculation of the homological dimension of the Johnson kernel.

Proof ideas for Theorem B. Johnson deduced that the Johnson homomorphism gives the abelianization of the Torelli group modulo torsion by proving that separating twists become torsion when $\mathcal{I}(\Sigma_g^n)$ is abelianized for $g \geq 3$ and $n \leq 1$. This reduces the result to showing that the Johnson kernel is generated by Dehn twists about separating curves. For Theorem B, we do the same thing; namely, we prove that separating twists become torsion in the abelianizations of finite-index subgroups of $\widehat{\mathcal{I}}(S, \Sigma_g^n)$ that contain $\widehat{\mathcal{K}}(S, \Sigma_g^n)$. The proof combines a small generalization of Johnson’s argument with the trick used in the author’s paper [26] to show that powers of Dehn twists go to torsion in the abelianizations of finite-index subgroups of the mapping class group.

Comments on Johnson’s work. To put this paper in perspective, we now give a brief sketch of Johnson’s proof [19] that $\mathcal{K}(\Sigma_g^n)$ is generated by separating twists. We will focus on the case $n = 1$ and ignore some low-genus issues. Set $H = H_1(\Sigma_g^1; \mathbb{Z})$.

Johnson’s proof proceeds by a sequence of ingenious and difficult calculations. Let $\mathcal{S} \triangleleft \mathcal{I}(\Sigma_g^1)$ be the subgroup generated by separating twists. Recall that the Johnson homomorphism takes the form

$$\tau_{\Sigma_g^1} : \mathcal{I}(\Sigma_g^1) \longrightarrow \wedge^3 H.$$

The free abelian group $\wedge^3 H$ has rank $\binom{g}{3}$. Johnson begins by writing down an explicit set X of $\binom{g}{3}$ elements of $\mathcal{I}(\Sigma_g^1)$ such that $\tau(X)$ is a basis for $\wedge^3 H$. He then does the following.

1. By examining the conjugates of X under a standard generating set for $\text{Mod}(\Sigma_g^1)$, he shows that X generates a normal subgroup of $\text{Mod}(\Sigma_g^1)/\mathcal{S}$. The set X contains a normal generating set for $\mathcal{I}(\Sigma_g^1)$ identified by Powell ([24]; see §2.2 for the details of Powell’s generating set). Thus Johnson can conclude that X generates $\mathcal{I}(\Sigma_g^1)/\mathcal{S}$.
2. Via a brute-force calculation, Johnson next shows that the elements of X commute modulo \mathcal{S} , so $\mathcal{I}(\Sigma_g^1)/\mathcal{S}$ is an abelian group generated by $\binom{g}{3}$ elements.
3. The Johnson homomorphism induces a surjection $\mathcal{I}(\Sigma_g^1)/\mathcal{S} \rightarrow \wedge^3 H$. By the previous step, this map has to be an isomorphism, and the result follows.

Outline of paper. In §2, we review some preliminaries about the mapping class group, the Birman exact sequence, the Torelli group, and the Johnson homomorphisms. We construct our version of the Birman exact sequence for the Johnson kernel in §4. We then prove Theorem A in §5 and Theorem B in §6.

Notation and conventions. In this paper, by a *surface* we will mean a compact oriented connected surface with boundary. We will denote the algebraic intersection pairing on $H_1(\Sigma_g; \mathbb{Z})$ by $i_{\text{alg}}(\cdot, \cdot)$. A *symplectic basis* for $H_1(\Sigma_g; \mathbb{Z})$ is a basis $\{a_1, b_1, \dots, a_g, b_g\}$ for $H_1(\Sigma_g; \mathbb{Z})$ such that $i_{\text{alg}}(a_i, b_i) = 1$ and $i_{\text{alg}}(a_i, a_j) = i_{\text{alg}}(a_i, b_j) = i_{\text{alg}}(b_i, b_j) = 0$ for distinct $1 \leq i, j \leq g$. If c is an oriented 1-submanifold of the surface, then $[c]$ will denote the homology class of c . We will confuse a curve on a surface with its isotopy class. Several

times we will have formulas involving $[\gamma]$, where γ is an unoriented simple closed curve. By this we will mean that the formula holds for an appropriate choice of orientation.

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2 Preliminaries

In §2.1 and §2.2, we will review some basic facts about the mapping class group and the Torelli group. Next, in §2.3 we will discuss the various versions of the Birman exact sequence for the Torelli group. Finally, in §2.4, we will discuss the Johnson homomorphism.

2.1 Basic facts about the mapping class group

We start by discussing some basic facts about the mapping class group.

Subsurfaces. If S is a surface and $S' \hookrightarrow S$ is a subsurface, then there is an induced map $i : \text{Mod}(S') \rightarrow \text{Mod}(S)$ (“extend by the identity”). This map is injective except in the following situations.

- If a component A of $S \setminus \text{Int}(S')$ is an annulus with $\partial A \subset S'$, then letting x and y be the components of ∂A we have $i(T_x T_y^{-1}) = 1$.
- If a component of $S \setminus \text{Int}(S')$ is a disc, then $\ker(i)$ contains the “disc-pushing subgroup” which forms the kernel of the Birman exact sequence (see below).

See [23], which proves that $\ker(i)$ is generated by the above elements.

Birman exact sequence. The original Birman exact sequence (see Birman [2]) dealt with the effect on mapping class groups of punctured surfaces of “forgetting the punctures”. Later, Johnson [18] gave an appropriate version for surfaces with boundary. Let S be a surface of genus at least 2 with nonempty boundary. Let β be a boundary component of S and let \hat{S} be the result of gluing a disc to β . The subsurface inclusion $S \hookrightarrow \hat{S}$ induces a map $i : \text{Mod}(S) \rightarrow \text{Mod}(\hat{S})$. Define $\mathcal{P}(S; \beta) = \ker(i)$. The group $\mathcal{P}(S; \beta)$ is called the *disc-pushing subgroup* of $\text{Mod}(S)$. Elements of it “push” the boundary component β around S while allowing it to rotate; keeping track of this path and rotation gives an isomorphism $\mathcal{P}(S; \beta) \xrightarrow{\cong} \pi_1(U\hat{S})$. This is all summarized in the Birman exact sequence

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \mathcal{P}(S; \beta) & \longrightarrow & \text{Mod}(S) & \xrightarrow{i} & \text{Mod}(\hat{S}) \longrightarrow 1 \\
 & & \downarrow \cong & & & & \\
 & & \pi_1(U\hat{S}) & & & &
 \end{array} \tag{1}$$

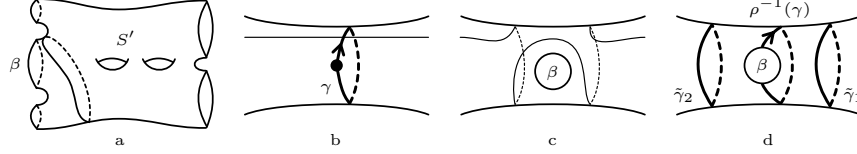


Figure 2: *a.* The subsurface $S' \cong \Sigma_2^4$ of $S \cong \Sigma_2^5$ is a splitting surface for β . *b.* A simple closed curve $\gamma \in \pi_1(\hat{S})$. *c.* The effect of $\text{Push}(\gamma)$. *d.* $\text{Push}(\gamma) = T_{\tilde{\gamma}_1} T_{\tilde{\gamma}_2}^{-1}$

The sequence (1) splits if S has at least 2 boundary components. Say that a subsurface $S' \hookrightarrow S$ is a *splitting surface* for β if $S \setminus \text{Int}(S')$ is a 3-holed sphere two of whose boundary components are components of ∂S and one of whose boundary components is β (see Figure 2.a). The composition of inclusion maps $S' \hookrightarrow S \hookrightarrow \hat{S}$ is then homotopic to a homeomorphism and the induced map $\text{Mod}(S') \rightarrow \text{Mod}(S)$ is injective. Regarding $\text{Mod}(S')$ as a subgroup of $\text{Mod}(S)$, the map i restricts to an isomorphism $\text{Mod}(S') \cong \text{Mod}(\hat{S})$, so (1) splits as

$$\text{Mod}(S) = \mathcal{P}(S; \beta) \rtimes \text{Mod}(S'). \quad (2)$$

Elements of the disc-pushing subgroup. We will need explicit formulas for some elements of $\mathcal{P}(S; \beta) \cong \pi_1(U\hat{S})$. The loop around the fiber of $U\hat{S}$ over the basepoint of \hat{S} (with an appropriate orientation) corresponds to T_β . It is easiest to understand the other elements via the projection $\mathcal{P}(S; \beta) \cong \pi_1(U\hat{S}) \rightarrow \pi_1(\hat{S})$. Consider $\gamma \in \pi_1(\hat{S})$. Assume that γ can be realized by a simple closed curve, which we can assume is smoothly embedded in \hat{S} . Taking the derivative of such an embedding and rescaling to get a unit vector, we get a well-defined lift $\text{Push}(\gamma) \in \pi_1(U\hat{S}) \cong \mathcal{P}(S; \beta) < \text{Mod}(S)$.

Remark. This is well-defined since any two realizations of γ as a smoothly embedded simple closed curve are smoothly isotopic through an isotopy fixing the basepoint.

As shown in Figure 2.b–d, for such a γ we have $\text{Push}(\gamma) = T_{\tilde{\gamma}_1} T_{\tilde{\gamma}_2}^{-1}$ for simple closed curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in S constructed as follows. Let $\rho : S \rightarrow \hat{S}$ be the map which collapses β to a point (the basepoint of \hat{S}). Then $\rho^{-1}(\gamma)$ consists of β together with an oriented arc joining two points of β . The boundary of a regular neighborhood of $\rho^{-1}(\gamma)$ has two components, and the curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are the components lying to the right and left of the arc, respectively.

2.2 Basic facts about the Torelli group

We now turn to the Torelli group. Recall that for $g \geq 0$ and $n \leq 1$ this is the kernel $\mathcal{I}(\Sigma_g^n)$ of the action of $\text{Mod}(\Sigma_g^n)$ on $H_1(\Sigma_g^n; \mathbb{Z})$.

Generators. Following work of Birman [3], Powell [24] proved that $\mathcal{I}(\Sigma_g^n)$ is generated by the following elements for $g \geq 0$ and $n \leq 1$.

- Separating twists, that is, Dehn twists T_x about separating simple closed curves x (see Figure 3.a).
- Bounding pair maps, that is, products $T_y T_z^{-1}$ for disjoint nonseparating homologous simple closed curves y and z , (so called because $y \cup z$ separates Σ_g^n ; see Figure 3.a).

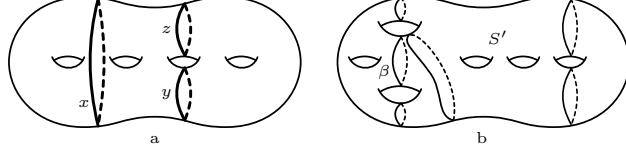


Figure 3: *a.* A separating twist T_x and a bounding pair map $T_y T_z^{-1}$. *b.* The subsurface $S \cong \Sigma_2^5$ of Σ_6 is bounded by the five “vertical” curves. The subsurface $S' \cong \Sigma_2^4$ of S is a Σ_6 -splitting surface for β .

Subsurface Torelli groups. Thus far, we have only defined the Torelli group on a surface Σ_g^n with $n \leq 1$. Now consider an arbitrary surface S . Fix an embedding $S \hookrightarrow \Sigma_g^n$ with $n \leq 1$, let $i : \text{Mod}(S) \rightarrow \text{Mod}(\Sigma_g^n)$ be the induced map, and define

$$\mathcal{I}(S, \Sigma_g^n) = i^{-1}(\mathcal{I}(\Sigma_g^n)).$$

The subgroup $\mathcal{I}(S, \Sigma_g^n)$ of $\text{Mod}(S)$ so obtained depends on the embedding $S \hookrightarrow \Sigma_g^n$. In [25], the author showed that the Torelli groups on S associated to different embeddings into a surface with at most one boundary component are parametrized by a partition of the boundary components of S , but we will not need this.

Remark. Defining $\widehat{\mathcal{I}}(S, \Sigma_g^n)$ as in the introduction, we have $i(\mathcal{I}(S, \Sigma_g^n)) = \widehat{\mathcal{I}}(S, \Sigma_g^n)$. Also, i restricts to an isomorphism from $\mathcal{I}(S, \Sigma_g^n)$ to $\widehat{\mathcal{I}}(S, \Sigma_g^n)$ except for the situations discussed in §2.1 where i is not injective.

Lemma 2.1. *The following hold for all $n \leq 1$.*

- (a) *If $S \cong \Sigma_h^1$ and $S \hookrightarrow \Sigma_g^n$ is any embedding, then $\mathcal{I}(S, \Sigma_g^n) = \mathcal{I}(S) \cong \mathcal{I}(\Sigma_h^1)$.*
- (b) *If $S \hookrightarrow \Sigma_g^1$ and $\Sigma_g^1 \hookrightarrow \Sigma_h^n$ are embeddings, then $\mathcal{I}(S, \Sigma_g^1) = \mathcal{I}(S, \Sigma_h^n)$.*

Proof. Trivial (see [25, Theorem Summary 1.1] for a more general result). □

The paper [25] extends Powell’s computation of generators of $\mathcal{I}(\Sigma_g^n)$ to the groups $\mathcal{I}(S, \Sigma_g^n)$. If x is a nonnullhomotopic simple closed curve on S , then say that $T_x \in \text{Mod}(S)$ is a Σ_g^n -separating twist if x separates Σ_g^n . Similarly, if y and z are disjoint nonnullhomotopic simple closed curves on S , then say that $T_y T_z^{-1} \in \text{Mod}(S)$ is a Σ_g^n -bounding pair map if y and z are nonseparating and homologous in Σ_g^n . Then $\mathcal{I}(S, \Sigma_g^n)$ is generated by the set of Σ_g^n -separating twists and Σ_g^n -bounding pair maps if the genus of S is at least 1 (see [25, Theorem 1.3]).

2.3 Birman exact sequence for Torelli group

Let $S \hookrightarrow \Sigma_g^n$ be a subsurface with $n \leq 1$ and let β be a boundary component of S . Define $\mathcal{P}_{\mathcal{I}}(S, \Sigma_g^n; \beta) = \mathcal{P}(S; \beta) \cap \mathcal{I}(S, \Sigma_g^n)$. The Birman exact sequence for the Torelli group (proved by the author in [25]) relates $\mathcal{I}(S, \Sigma_g^n)$ to a Torelli group defined on the result of gluing a disc to β . There are three versions.

Birman exact sequence for Torelli, one boundary component. The first is as follows, which mildly generalizes a theorem of Johnson [18].

Theorem 2.2 ([25, Theorem 1.2]). *Let $S \hookrightarrow \Sigma_g^n$ be a subsurface with $n \leq 1$ and let β be a boundary component of S . Assume that the genus of S is at least 2 and that β is the only boundary component of S . Let \hat{S} be the result of gluing a disc to β . Then there is an exact sequence*

$$1 \longrightarrow \mathcal{P}_{\mathcal{I}}(S, \Sigma_g^n; \beta) \longrightarrow \mathcal{I}(S, \Sigma_g^n) \longrightarrow \mathcal{I}(\hat{S}) \longrightarrow 1.$$

Also, $\mathcal{P}_{\mathcal{I}}(S, \Sigma_g^n; \beta) = \mathcal{P}(S; \beta) \cong \pi_1(U\hat{S})$.

Remark. Observe that in Theorem 2.2, the surface \hat{S} is closed, so $\mathcal{I}(\hat{S})$ makes sense.

Birman exact sequence for Torelli, separating boundary. The second version of the Birman exact sequence for the Torelli group is as follows. In it (and throughout this paper), we regard a boundary component of a surface Σ_g^n as being a separating curve on Σ_g^n .

Theorem 2.3 ([25, Theorem 1.2]). *Let $S \hookrightarrow \Sigma_g^n$ be a subsurface with $n \leq 1$ and let β be a boundary component of S . Assume that the genus of S is at least 2, that S has more than one boundary component, and that β is a separating curve on Σ_g^n . Let $S' \hookrightarrow S$ be a splitting surface for β and let \hat{S} be the result of gluing a disc to β . Then*

$$\mathcal{I}(S, \Sigma_g^n) = \mathcal{P}_{\mathcal{I}}(S, \Sigma_g^n; \beta) \rtimes \mathcal{I}(S', \Sigma_g^n).$$

Also, $\mathcal{P}_{\mathcal{I}}(S, \Sigma_g^n; \beta) = \mathcal{P}(S; \beta) \cong \pi_1(U\hat{S})$.

Subsurface homology map. To deal with the case where β does not separate Σ_g^n , we need to introduce the *subsurface homology map* of S . We will actually need this map for separating boundary components too, so β is still an arbitrary boundary component of S . Let \hat{S} be the surface obtained by gluing a disc to S along β . The inclusion map $S \hookrightarrow \Sigma_g^n$ induces a map $H_1(S; \mathbb{Z}) \rightarrow H_1(\Sigma_g^n; \mathbb{Z})$. This in turn descends to a map $H_1(\hat{S}; \mathbb{Z}) \rightarrow H_1(\Sigma_g^n; \mathbb{Z}) / \langle [\beta] \rangle$ which we will call the *subsurface homology map* of S and will denote $\sigma_{S, \Sigma_g^n, \beta}$. *Remark.* If β separates Σ_g^n , then $\sigma_{S, \Sigma_g^n, \beta}$ has codomain $H_1(\Sigma_g^n; \mathbb{Z})$.

Relative commutator subgroup. We will denote by $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^n}$ the kernel of the composition

$$\pi_1(\hat{S}) \rightarrow H_1(\hat{S}; \mathbb{Z}) \xrightarrow{\sigma_{S, \Sigma_g^n, \beta}} H_1(\Sigma_g^n; \mathbb{Z}) / \langle [\beta] \rangle.$$

If $K \subset H_1(\hat{S}; \mathbb{Z})$ is the kernel of $\sigma_{S, \Sigma_g^n, \beta}$, then we have a short exact sequence

$$1 \longrightarrow [\pi_1(\hat{S}), \pi_1(\hat{S})] \longrightarrow [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^n} \longrightarrow K \longrightarrow 1. \quad (3)$$

We will need the following observation.

Lemma 2.4. *Let $S \hookrightarrow \Sigma_g^1$ and $\Sigma_g^1 \hookrightarrow \Sigma_h^n$ be subsurfaces with $n \leq 1$. Let β be a boundary component of S and let \hat{S} be the result of gluing a disc to β . Then $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^1} = [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_h^n}$.*

Proof. Since the map $H_1(\Sigma_g^1; \mathbb{Z}) \rightarrow H_1(\Sigma_h^n; \mathbb{Z})$ is injective, the kernels of $\sigma_{S, \Sigma_g^1, \beta}$ and $\sigma_{S, \Sigma_h^n, \beta}$ are the same. The lemma now follows from (3). \square

Σ_g^n -splitting surfaces. Say that a splitting surface $S' \hookrightarrow S$ for β is a Σ_g^n -splitting surface if $\Sigma_g^n \setminus \text{Int}(S')$ has the same number of components as $\Sigma_g^n \setminus \text{Int}(S)$ (see Figure 3.b; in the language of [25], this assumption implies that both boundary components of $S \setminus S'$ lie in the same partition element). Observe that a Σ_g^n -splitting surface exists for β if and only if β is a nonseparating curve on Σ_g^n .

Birman exact sequence for Torelli, nonseparating boundary. The third version of the Birman exact sequence for the Torelli group is as follows.

Theorem 2.5 ([25, Theorem 1.2]). *Let $S \hookrightarrow \Sigma_g^n$ be a subsurface with $n \leq 1$ and let β be a boundary component of S . Assume that the genus of S is at least 2 and that β is a nonseparating curve on Σ_g^n . Let $S' \hookrightarrow S$ be a Σ_g^n -splitting surface for β and let \hat{S} be the result of gluing a disc to β . Then*

$$\mathcal{I}(S, \Sigma_g^n) = \mathcal{P}_{\mathcal{I}}(S, \Sigma_g^n; \beta) \rtimes \mathcal{I}(S', \Sigma_g^n).$$

Also, $\mathcal{P}_{\mathcal{I}}(S, \Sigma_g^n; \beta) \cong [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^n}$.

Remark. Unlike $\mathcal{I}(S, \Sigma_g^n) \subset \text{Mod}(S)$ in Theorem 2.5, the group $\hat{\mathcal{I}}(S, \Sigma_g^n) \subset \text{Mod}(\Sigma_g^n)$ does not always split as a semidirect product (there is a Birman exact sequence, but it need not be split; this issue arises when the component C of $\Sigma_g^n \setminus \text{Int}(S)$ containing β is an annulus). This is the whole reason we went to the trouble of introducing the groups $\mathcal{I}(S, \Sigma_g^n)$ – the above semidirect product decomposition will be very convenient, especially when we introduce the Johnson homomorphism.

The embedding. For the rest of this section, let the notation be as in Theorem 2.5. The assumptions about β in that theorem imply that \hat{S} has a nonempty boundary, so there is an (unnatural) direct sum decomposition $\pi_1(U\hat{S}) \cong \pi_1(\hat{S}) \times \mathbb{Z}$. However, though $\mathcal{P}_{\mathcal{I}}(S, \Sigma_g^n; \beta)$ is isomorphic to a subgroup of $\pi_1(\hat{S})$, it does not lie in the first factor of such a direct sum decomposition. Rather, for every such decomposition there exists a homomorphism $\psi : [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^n} \rightarrow \mathbb{Z}$ such that

$$\mathcal{P}_{\mathcal{I}}(S, \Sigma_g^n; \beta) \cong \{(x, \psi(x)) \mid x \in [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^n}\} \subset \pi_1(\hat{S}) \times \mathbb{Z} \cong \pi_1(U\hat{S}). \quad (4)$$

For a proof of this, see [25, Theorem 4.1].

Generating the relative commutator subgroup. We will need to know generators for the group $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^n}$ that appears in Theorem 2.5. The first type of generator are *torus-bounding curves*, which are $\gamma \in [\pi_1(\hat{S}), \pi_1(\hat{S})]$ that can be realized by simple closed curves that separate S into two subsurfaces, one of which is a one-holed torus (see Figures 4.a,b). The following lemma says that these generate $[\pi_1(\hat{S}), \pi_1(\hat{S})]$.

Lemma 2.6 ([25, Theorem A.1]). *Let X be a surface whose genus is at least 1. Pick $*$ $\in \text{Int}(X)$. Then $[\pi_1(X, *), \pi_1(X, *)]$ is generated by the set of elements that can be realized by torus-bounding curves.*

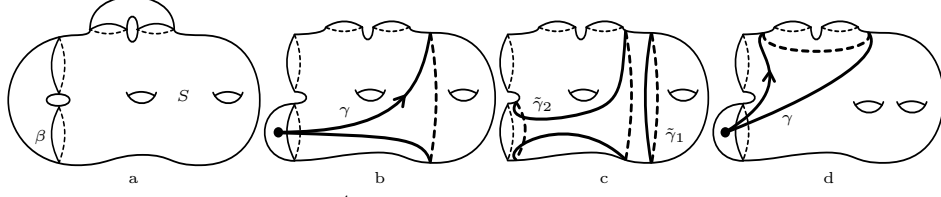


Figure 4: a. A subsurface $S \cong \Sigma_2^4$ of Σ_4 and a boundary component β of S b. A torus-bounding curve $\gamma \in \pi_1(\hat{S})$ c. We have $\text{Push}(\gamma) = T_{\tilde{\gamma}_1} T_{\tilde{\gamma}_2}^{-1}$ and $\widehat{\text{Push}}(\gamma) = T_{\tilde{\gamma}_1}(T_\beta T_{\tilde{\gamma}_2}^{-1})$. d. A Σ_4 -bounding curve $\gamma \in \pi_1(\hat{S})$.

The other type of element needed to generate $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^n}$ are Σ_g^n -boundary curves, which are elements $\gamma \in \pi_1(\hat{S})$ that can be realized by simple closed curve such that there exists a component M of $\Sigma_g^n \setminus \text{Int}(S)$ satisfying the following two conditions (see Figures 4.a,d).

- $M \cap \beta = \emptyset$, so $M \cap S$ lies in \hat{S} , and
- γ separates \hat{S} into two components, one of which is homeomorphic to a sphere whose boundary components consist of γ together with $M \cap S \subset \hat{S}$.

We will say that γ surrounds M . We then have the following lemma.

Lemma 2.7. *Let S be a surface whose genus is at least 1 and let $S \hookrightarrow \Sigma_g^n$ be an embedding with $n \leq 1$. Let β be a boundary component of S and let \hat{S} be the result of gluing a disc to S along β . Then $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^n}$ is generated by the set of elements that can be realized by torus-bounding curves or Σ_g^n -boundary curves.*

Proof. Let Γ be the subgroup of $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^n}$ generated by the set of elements that can be realized by torus-bounding curves or Σ_g^n -boundary curves. By Lemma 2.6, the subgroup Γ contains the entire kernel of the exact sequence (3). Also, the set of elements that can be realized by Σ_g^n -boundary curves project to a generating set for the cokernel of (3). We conclude that $\Gamma = [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^n}$, as desired. \square

Explicit formulas. For $\gamma \in [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^n}$, denote by $\widehat{\text{Push}}(\gamma) \in \mathcal{P}_{\mathcal{I}}(S, \Sigma_g^n; \beta)$ the associated element of Torelli. We will need some explicit formulas for $\widehat{\text{Push}}(\gamma)$. These can be obtained as follows. Assume that $\gamma \in [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^n}$ can be realized by a simple closed curve. We therefore have the element $\text{Push}(\gamma) \in \mathcal{P}(S; \beta)$, which projects to γ under the natural projection $\mathcal{P}(S; \beta) \cong \pi_1(U\hat{S}) \rightarrow \pi_1(\hat{S})$. Examining (4), we see that there is a unique $k \in \mathbb{Z}$ such that $\text{Push}(\gamma) T_\beta^k \in \mathcal{P}_{\mathcal{I}}(S, \Sigma_g^n; \beta)$. We then have $\widehat{\text{Push}}(\gamma) = \text{Push}(\gamma) T_\beta^k$. Using this recipe, one can show that if γ can be realized by a torus-bounding curve or a Σ_g^n -boundary curve, then $\widehat{\text{Push}}(\gamma)$ is the product of a Σ_g^n -separating twist and a Σ_g^n -bounding pair. For example, consider the subsurface S of Σ_4 in Figure 4.a. Let γ be the torus-bounding curve depicted in Figure 4.b. We then have $\text{Push}(\gamma) = T_{\tilde{\gamma}_1} T_{\tilde{\gamma}_2}^{-1}$ for $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ as in Figure 4.c. The mapping class $T_{\tilde{\gamma}_1}$ is a Σ_4 -separating twist, so $T_{\tilde{\gamma}_1} \in \mathcal{I}(S, \Sigma_4)$. The mapping class $T_{\tilde{\gamma}_2}$ is not a Σ_4 -separating twist, but $T_{\tilde{\gamma}_2} T_\beta^{-1}$ is a Σ_4 -bounding pair map, so $T_{\tilde{\gamma}_2} T_\beta^{-1} \in \mathcal{I}(S, \Sigma_4)$. We conclude that $\widehat{\text{Push}}(\gamma) = T_{\tilde{\gamma}_1}(T_\beta T_{\tilde{\gamma}_2}^{-1})$

2.4 The Johnson homomorphism and kernel

The Johnson homomorphism was originally defined in [16] by examining the action of the Torelli group on nilpotent truncations of the surface group. There are now several alternate definitions (see [17]). We will only need its formal properties.

Characteristic element. Let $V \cong \mathbb{Z}^{2g}$ be a free \mathbb{Z} -module equipped with a symplectic form. The *characteristic element* $\omega_V \in \wedge^2 V$ is $a_1 \wedge b_1 + \cdots + a_g \wedge b_g$, where $\{a_1, b_1, \dots, a_g, b_g\}$ is a symplectic basis for V . The element ω_V is independent of the choice of symplectic basis.

Surfaces with one boundary component. Set $H = H_1(\Sigma_g^1; \mathbb{Z})$. The Johnson homomorphism on $\mathcal{I}(\Sigma_g^1)$ is a surjective homomorphism $\tau_{\Sigma_g^1} : \mathcal{I}(\Sigma_g^1) \rightarrow \wedge^3 H$. It has the following behavior on our generators for $\mathcal{I}(\Sigma_g^1)$.

- If $T_x \in \mathcal{I}(\Sigma_g^1)$ is a separating twist, then $\tau_{\Sigma_g^1}(T_x) = 0$.
- If $T_y T_z^{-1} \in \mathcal{I}(\Sigma_g^1)$ is a bounding pair map, then $\tau_{\Sigma_g^1}(T_y T_z^{-1}) = [y] \wedge \omega_V$ where $[y] \in H$ and $\omega_V \in \wedge^2 H$ are as follows. The curves $y \cup z$ divide Σ_g^1 into two subsurfaces Q_1 and Q_2 . Order them so that $\partial \Sigma_g^1 \subset Q_2$. We then have $Q_1 \cong \Sigma_h^2$ and $Q_2 \cong \Sigma_{g-h-1}^3$ for some $0 < h < g$. Orienting y so that Q_1 lies on its right, the element $[y] \in H_1(\Sigma_g^1; \mathbb{Z})$ is the homology class of y . Let $Q'_1 \subset Q_1$ be a genus h subsurface with 1 boundary component. Thus $V := H_1(Q'_1; \mathbb{Z})$ is a symplectic \mathbb{Z} -module with a characteristic element $\omega_V \in \wedge^2 V$, which we will identify with an element of $\wedge^2 H$ via the inclusion $\wedge^2 V \subset \wedge^2 H$.

Remark. It is instructive to verify that $[y] \wedge \omega_V$ is independent of the choice of Q'_1 .

The Johnson kernel $\mathcal{K}(\Sigma_g^1)$ is the kernel of $\tau_{\Sigma_g^1}$.

Closed surfaces. Continue to let $H = H_1(\Sigma_g^1; \mathbb{Z})$. Let β be the boundary component of Σ_g^1 and let $\omega \in \wedge^2 H$ be the characteristic element. Johnson [20] proved the following lemma. In it, we identify $H = H_1(\Sigma_g^1; \mathbb{Z})$ with $H_1(\Sigma_g; \mathbb{Z})$ in the obvious way. Also, we use the inclusion $\mathcal{P}(\Sigma_g^1; \beta) \subset \mathcal{I}(\Sigma_g^1)$ from Theorem 2.2.

Lemma 2.8. *The restriction of $\tau_{\Sigma_g^1}$ to the disc-pushing subgroup $\mathcal{P}(\Sigma_g^1; \beta) \cong \pi_1(U\Sigma_g)$ is the composition*

$$\pi_1(U\Sigma_g) \longrightarrow \pi_1(\Sigma_g) \longrightarrow H \xrightarrow{\kappa} \wedge^3 H,$$

where $\kappa : H \rightarrow \wedge^3 H$ is the inclusion $\kappa(h) = h \wedge \omega$.

There is therefore an induced Johnson homomorphism $\tau_{\Sigma_g} : \mathcal{I}(\Sigma_g) \rightarrow (\wedge^3 H)/H$ which fits into a commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{P}(\Sigma_g^1; \beta) & \longrightarrow & \mathcal{I}(\Sigma_g^1) & \longrightarrow & \mathcal{I}(\Sigma_g) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \tau_{\Sigma_g^1} & & \downarrow \tau_{\Sigma_g} & & \\ 1 & \longrightarrow & H & \xrightarrow{\kappa} & \wedge^3 H & \longrightarrow & (\wedge^3 H)/H & \longrightarrow & 1. \end{array} \quad (5)$$

The Johnson kernel $\mathcal{K}(\Sigma_g)$ is the kernel of τ_{Σ_g} .

3 The subsurface Johnson homomorphisms and kernels

Let S be an arbitrary surface and let $S \hookrightarrow \Sigma_g^n$ be an embedding with $n \leq 1$. We want to define the Johnson homomorphism on $\mathcal{I}(S, \Sigma_g^n)$. To avoid unnecessary technicalities, we will restrict ourselves to *clean embeddings* $S \hookrightarrow \Sigma_g^n$, that is, embeddings such that no components of $\Sigma_g^n \setminus \text{Int}(S)$ are discs. Given a clean embedding $S \hookrightarrow \Sigma_g^n$ with $n \leq 1$, we define the Johnson homomorphism τ_{S, Σ_g^n} on $\mathcal{I}(S, \Sigma_g^n)$ as follows. Let $H = \mathbb{H}_1(\Sigma_g^n; \mathbb{Z})$.

- If $n = 1$, then τ_{S, Σ_g^n} is the composition

$$\mathcal{I}(S, \Sigma_g^n) \longrightarrow \mathcal{I}(\Sigma_g^n) \xrightarrow{\tau_{\Sigma_g^n}} \wedge^3 H.$$

- If $n = 0$, then τ_{S, Σ_g^n} is the composition

$$\mathcal{I}(S, \Sigma_g^n) \longrightarrow \mathcal{I}(\Sigma_g^n) \xrightarrow{\tau_{\Sigma_g^n}} (\wedge^3 H)/H.$$

We emphasize that τ_{S, Σ_g^n} is not defined if $S \hookrightarrow \Sigma_g^n$ is not clean (we do this to avoid complicated and unnecessary special cases in our results). We define the Johnson kernel $\mathcal{K}(S, \Sigma_g^n)$ to be the kernel of τ_{S, Σ_g^n} .

To understand the relationships between different subsurface Johnson homomorphisms, we will need the following lemma. In it, the orthogonal complement is taken with respect to the algebraic intersection pairing.

Lemma 3.1. *For some $g \geq 1$, set $H = \mathbb{H}_1(\Sigma_g)$ and let $\omega \in \wedge^2 H$ be the characteristic element. Define $\kappa : H \rightarrow \wedge^3 H$ by $\kappa(h) = h \wedge \omega$.*

- (a) *If x is a primitive vector in H , then $(\wedge^3(x^\perp)) \cap \kappa(H) = \langle \kappa(x) \rangle \cong \mathbb{Z}$.*
(b) *If x and y are primitive vectors in H with $i_{\text{alg}}(x, y)$ equal to 0 or 1 that span a rank-2 summand of H , then $(\wedge^3(\langle x, y \rangle^\perp)) \cap \kappa(H) = 0$.*

Proof. The proofs of the two conclusions are similar; we will give the details for (a) first and leave (b) to the reader. Choose a symplectic basis $\{a_1, b_1, \dots, a_g, b_g\}$ for H with $a_1 = x$. We have $\omega = a_1 \wedge b_1 + \dots + a_g \wedge b_g$. Since $H = a_1^\perp \oplus \langle b_1 \rangle$, we have

$$\wedge^3 H = (\wedge^2(a_1^\perp)) \oplus (\wedge^3(a_1^\perp)), \tag{6}$$

where the inclusion $\wedge^2(a_1^\perp) \rightarrow \wedge^3 H$ of the first factor takes $\theta \in \wedge^2(a_1^\perp)$ to $\theta \wedge b_1$.

Consider $h \in H$. Write $h = ca_1 + db_1 + h'$ with $c, d \in \mathbb{Z}$ and $h' \in \langle a_2, b_2, \dots, a_g, b_g \rangle$. Letting $\bar{\omega} = a_2 \wedge b_2 + \dots + a_g \wedge b_g$, an easy calculation shows that the expression for $\kappa(h)$ in terms of (6) is $(d\bar{\omega} + h' \wedge a_1, (ca_1 + h') \wedge \bar{\omega})$. The first coordinate of this is 0 if and only if $d = 0$ and $h' = 0$, and the lemma follows. \square

Corollary 3.2. *Let $\Sigma_h^1 \hookrightarrow \Sigma_g$ be a clean embedding. Then the natural map $\wedge^3 \mathbb{H}_1(\Sigma_h^1; \mathbb{Z}) \rightarrow (\wedge^3 \mathbb{H}_1(\Sigma_g; \mathbb{Z}))/\mathbb{H}_1(\Sigma_g; \mathbb{Z})$ is injective.*

Proof. Since the embedding is clean, we have $g > h$. The corollary now follows from the second conclusion of Lemma 3.1. \square

If $\Sigma_h^1 \hookrightarrow \Sigma_g^n$ is a clean embedding with $n \leq 1$, then Lemma 2.1 implies that $\mathcal{I}(\Sigma_h^1, \Sigma_g^n) = \mathcal{I}(\Sigma_h^1)$. We therefore have two different Johnson homomorphisms on $\mathcal{I}(\Sigma_h^1, \Sigma_g^n)$, one whose codomain is $\wedge^3 \mathbf{H}_1(\Sigma_h^1; \mathbb{Z})$ and one whose codomain is either $\wedge^3 \mathbf{H}_1(\Sigma_g^n; \mathbb{Z})$ if $n = 1$ or $(\wedge^3 \mathbf{H}_1(\Sigma_g^n; \mathbb{Z})) / \mathbf{H}_1(\Sigma_g^n; \mathbb{Z})$ if $n = 0$. Thankfully both of these Johnson homomorphisms have the same kernel.

Lemma 3.3. *Let $\Sigma_h^1 \hookrightarrow \Sigma_g^n$ be a clean embedding with $n \leq 1$. Set $H = \mathbf{H}_1(\Sigma_g^n; \mathbb{Z})$.*

- *If $n = 1$, then $\tau_{\Sigma_h^1, \Sigma_g^n} : \mathcal{I}(\Sigma_h^1, \Sigma_g^n) \rightarrow \wedge^3 H$ factors as*

$$\mathcal{I}(\Sigma_h^1, \Sigma_g^n) = \mathcal{I}(\Sigma_h^1) \xrightarrow{\tau_{\Sigma_h^1}} \wedge^3 \mathbf{H}_1(\Sigma_h^1; \mathbb{Z}) \longrightarrow \wedge^3 H. \quad (7)$$

- *If $n = 0$, then $\tau_{\Sigma_h^1, \Sigma_g^n} : \mathcal{I}(\Sigma_h^1, \Sigma_g^n) \rightarrow (\wedge^3 H) / H$ factors as*

$$\mathcal{I}(\Sigma_h^1, \Sigma_g^n) = \mathcal{I}(\Sigma_h^1) \xrightarrow{\tau_{\Sigma_h^1}} \wedge^3 \mathbf{H}_1(\Sigma_h^1; \mathbb{Z}) \longrightarrow \wedge^3 H \longrightarrow (\wedge^3 H) / H.$$

Also, in both cases we have $\mathcal{K}(\Sigma_h^1) = \mathcal{K}(\Sigma_h^1, \Sigma_g^n)$.

Proof. The function with the factorization (7) has the same effect on separating twists and bounding pair maps as $\tau_{\Sigma_h^1, \Sigma_g^n}$. These generate $\mathcal{I}(\Sigma_h^1, \Sigma_g^n)$, so it must equal $\tau_{\Sigma_h^1, \Sigma_g^n}$. A similar proof works for the case $n = 0$. As for the final claim, for $n = 1$ it follows from the fact that the map $\wedge^3 \mathbf{H}_1(\Sigma_h^1; \mathbb{Z}) \rightarrow \wedge^3 \mathbf{H}_1(\Sigma_g^n; \mathbb{Z})$ is injective, and for $n = 0$ it follows from Corollary 3.2. \square

4 Birman exact sequence for the Johnson kernel

Let S be a surface, let $S \hookrightarrow \Sigma_g^n$ be a clean embedding with $n \leq 1$, and let β be a boundary component of S . Define $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta) = \mathcal{P}(S; \beta) \cap \mathcal{K}(S, \Sigma_g^n)$. In this section, we construct an analogue of the Birman exact sequence for the Johnson kernel; the ‘‘differences’’ between the Johnson kernels on surfaces with different numbers of boundary components will be encoded by the disc-pushing subgroups $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta)$. Just like for the Torelli group, there are three cases (corresponding to Theorems 2.2, 2.3, and 2.5).

4.1 One boundary

The following theorem is the analogue for the Johnson kernel of Theorem 2.2. It is very close to results of Johnson from [19]; see in particular [19, §4].

Theorem 4.1. *Let $S \hookrightarrow \Sigma_g^n$ be a clean embedding with $n \leq 1$ and let β be a boundary component of S . Assume that the genus of S is at least 2 and that β is the only boundary component of S . Let \hat{S} be the result of gluing a disc to β . Then there is an exact sequence*

$$1 \longrightarrow \mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta) \longrightarrow \mathcal{K}(S, \Sigma_g^n) \longrightarrow \mathcal{K}(\hat{S}) \longrightarrow 1.$$

Also, $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta) \subset \mathcal{P}(S; \beta)$ is the pullback of $[\pi_1(\hat{S}), \pi_1(\hat{S})]$ under the map

$$\mathcal{P}(S; \beta) \cong \pi_1(U\hat{S}) \longrightarrow \pi_1(\hat{S}).$$

Proof. By Lemma 3.3, we have $\mathcal{K}(S, \Sigma_g^n) = \mathcal{K}(S)$. The Johnson homomorphism on a closed surface was defined precisely so that the surjection $\mathcal{I}(S) \rightarrow \mathcal{I}(\hat{S})$ from Theorem 2.2 restricts to a surjection $\mathcal{K}(S) \rightarrow \mathcal{K}(\hat{S})$ (see the commutative diagram (5)). The indicated exact sequence follows. As for the identification of $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta)$, Theorem 2.2 says that $\mathcal{P}_{\mathcal{I}}(S, \Sigma_g^n; \beta) = \mathcal{P}(S; \beta)$. It then follows from the aforementioned identity $\mathcal{K}(S, \Sigma_g^n) = \mathcal{K}(S)$ that $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta) \subset \mathcal{P}(S; \beta)$ is the kernel of the restriction of τ_S to $\mathcal{P}(S; \beta)$. Letting $H = H_1(\hat{S}; \mathbb{Z})$ and letting $\omega \in \wedge^2 H$ be the characteristic element, Lemma 2.8 says that this restriction equals the composition

$$\mathcal{P}(S; \beta) \cong \pi_1(U\hat{S}) \longrightarrow \pi_1(\hat{S}) \longrightarrow H \xrightarrow{\kappa} \wedge^3 H,$$

where $\kappa : H \rightarrow \wedge^3 H$ is the map $\kappa(h) = h \wedge \omega$. The map κ is injective, so we conclude that $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta) \subset \mathcal{P}(S; \beta)$ is the kernel of the composition

$$\mathcal{P}(S; \beta) \cong \pi_1(U\hat{S}) \longrightarrow \pi_1(\hat{S}) \longrightarrow H,$$

as desired. □

4.2 Separating boundary

The following theorem is the analogue for the Johnson kernel of Theorem 2.3.

Theorem 4.2. *Let $S \hookrightarrow \Sigma_g^n$ be a clean embedding with $n \leq 1$ and let β be a boundary component of S . Assume that the genus of S is at least 2, that S has more than one boundary component, and that β separates Σ_g^n . Let $S' \hookrightarrow S$ be a splitting surface for β and let \hat{S} be the result of gluing a disc to β . Then if all the boundary components of S separate Σ_g^n , we have*

$$\mathcal{K}(S, \Sigma_g^n) = \mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta) \rtimes \mathcal{K}(S', \Sigma_g^n).$$

Also, even if the boundary components of S other than β do not separate Σ_g^n , the group $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta)$ is the pullback of $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^n}$ under the map

$$\mathcal{P}_{\mathcal{I}}(S, \Sigma_g^n; \beta) \cong \pi_1(U\hat{S}) \rightarrow \pi_1(\hat{S}).$$

Remark. The condition that all the boundary components of S separate Σ_g^n might seem quite restrictive. However, when we use the Birman exact sequence for the Johnson kernel, the first step will be to cap off nonseparating boundary components one at a time (using Theorem 4.4 below) until only separating boundary components remain, so Theorem 4.2 suffices for our purposes. There are subtleties involved in formulating the correct statement without this condition. For example, if S and S' and x and y and z are as in Figure 5.a, then $T_x T_y^{-1} \in \mathcal{P}_{\mathcal{I}}(S, \Sigma_g; \beta) \setminus \mathcal{K}(S, \Sigma_g)$ and $T_y T_z^{-1} \in \mathcal{I}(S', \Sigma_g) \setminus \mathcal{K}(S', \Sigma_g)$, but $(T_x T_y^{-1})(T_y T_z^{-1}) = T_x T_z^{-1} \in \mathcal{K}(S, \Sigma_g)$.

The proof of Theorem 4.2 requires the following lemma.

Lemma 4.3. *Let $f : G_1 \rightarrow G_2$ be a group homomorphism. Assume that $G_1 = K \rtimes Q$. Then $\ker(f) = \ker(f|_K) \rtimes \ker(f|_Q)$ if and only if $f(K) \cap f(Q) = 1$.*

Proof. Trivial. □

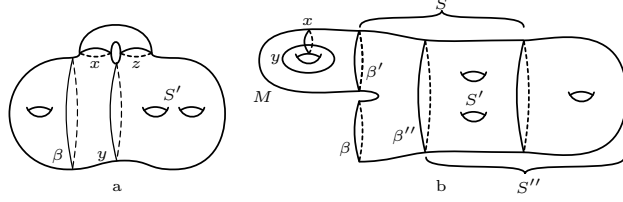


Figure 5: *a.* S is the subsurface to the right of β and below x and z . Considered as curves in S , the curves x and y become isotopic when a disc is glued to β , so the Σ_4 -bounding pair map $T_x T_y^{-1}$ lies in $\mathcal{P}_{\mathcal{I}}(S, \Sigma_4; \beta)$. *b.* The subsurfaces from the last part of the proof of Theorem 4.2.

Proof of Theorem 4.2. If $n = 1$, then by Lemmas 2.1(b) and 2.4, we can replace Σ_g^n with Σ_{g+1} without changing the truth of the theorem, so without loss of generality we can assume that $n = 0$. Next, another application of Lemmas 2.1(b) and 2.4 shows that we can replace Σ_g with the component of Σ_g cut along β that contains S without changing the truth of the theorem, so without loss of generality we can assume that $n = 1$ and that $\beta = \partial\Sigma_g^n$.

Set $H = H_1(\Sigma_g^1; \mathbb{Z})$ and let $\omega \in \wedge^2 H$ be the characteristic element. Define $\kappa : H \rightarrow \wedge^3 H$ via the formula $\kappa(h) = h \wedge \omega$. Theorem 2.3 says that $\mathcal{P}_{\mathcal{I}}(S, \Sigma_g^1; \beta) = \mathcal{P}(S; \beta) \cong \pi_1(U\hat{S})$. By Lemma 2.8, the restriction of τ_{S, Σ_g^1} to $\mathcal{P}(S; \beta)$ equals the composition

$$\mathcal{P}(S; \beta) \hookrightarrow \mathcal{P}(\Sigma_g^1; \beta) \cong \pi_1(U\Sigma_g) \rightarrow \pi_1(\Sigma_g) \rightarrow H \xrightarrow{\kappa} \wedge^3 H,$$

where we are identifying H with $H_1(\Sigma_g; \mathbb{Z})$ in the obvious way. This composition equals

$$\mathcal{P}(S; \beta) \cong \pi_1(U\hat{S}) \rightarrow \pi_1(\hat{S}) \rightarrow H_1(\hat{S}; \mathbb{Z}) \xrightarrow{\sigma_{S, \Sigma_g^1, \beta}} H \xrightarrow{\kappa} \wedge^3 H. \quad (8)$$

Since κ is injective, we see that the kernel of the restriction of τ_{S, Σ_g^1} to $\mathcal{P}(S; \beta)$ equals the kernel of the composition

$$\mathcal{P}(S; \beta) \cong \pi_1(U\hat{S}) \rightarrow \pi_1(\hat{S}) \rightarrow H_1(\hat{S}; \mathbb{Z}) \xrightarrow{\sigma_{S, \Sigma_g^1, \beta}} H.$$

By definition, this is the pullback to $\mathcal{P}(S; \beta)$ of $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\Sigma_g^1}$, as claimed.

It remains to prove that if all the boundary components of S separate Σ_g^1 , then

$$\mathcal{K}(S, \Sigma_g^1) = \mathcal{P}_{\mathcal{K}}(S, \Sigma_g^1; \beta) \times \mathcal{K}(S', \Sigma_g^1).$$

By Theorem 2.3, we have

$$\mathcal{I}(S, \Sigma_g^1) = \mathcal{P}_{\mathcal{I}}(S, \Sigma_g^1; \beta) \times \mathcal{I}(S', \Sigma_g^1).$$

Lemma 4.3 thus implies that it is enough to prove that

$$\tau_{S, \Sigma_g^1}(\mathcal{P}_{\mathcal{I}}(S, \Sigma_g^1; \beta)) \cap \tau_{S, \Sigma_g^1}(\mathcal{I}(S', \Sigma_g^1)) = 0. \quad (9)$$

Let $\{\beta, \beta', \beta''\}$ be the boundary components of $S \setminus \text{Int}(S')$, ordered such that $\beta'' = S \cap S'$. Let M be the component of Σ_g^1 cut along β' that does not contain S . Also, let S'' be the component of Σ_g^1 cut along β'' that contains S' (see Figure 5(b)). Using our assumption that all the boundary components of S separate Σ_g^1 , we see that M and S'' are disjoint surfaces with one boundary component whose genera are positive. We can thus find oriented simple

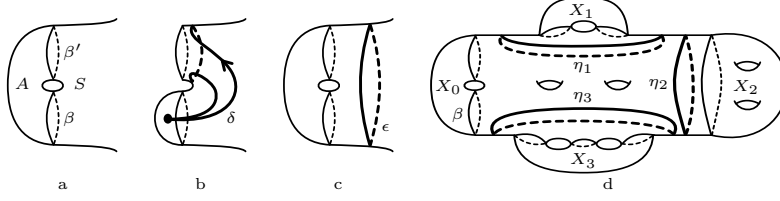


Figure 6: *a–c.* Why an extra generator is needed in Theorem 4.4 if C is an annulus $d.$ $S \cong \Sigma_2^8$ is the subsurface in the “middle” of Σ_8 .

closed curves x and y in M that intersect once, and $H_1(S''; \mathbb{Z}) \subset \langle [x], [y] \rangle^\perp$. Lemma 3.1(b) therefore implies that

$$(H \wedge \omega) \cap (\wedge^3 H_1(S''; \mathbb{Z})) = 0.$$

Lemma 3.3 implies that $\tau_{S, \Sigma_g^1}(\mathcal{I}(S', \Sigma_g^1)) \subset \wedge^3 H_1(S''; \mathbb{Z})$, and it is immediate from (8) that $\tau_{S, \Sigma_g^1}(\mathcal{P}_{\mathcal{I}}(S, \Sigma_g^1; \beta)) \subset H \wedge \omega$. The desired equation (9) follows. \square

4.3 Nonseparating boundary

The following theorem is the analogue for the Johnson kernel of Theorem 2.5.

Theorem 4.4. *Let $S \hookrightarrow \Sigma_g^n$ be a clean embedding with $n \leq 1$ and let β be a boundary component of S . Assume that the genus of S is at least 2 and that β is a nonseparating curve on Σ_g^n . Let $S' \hookrightarrow S$ be a Σ_g^n -splitting surface for β and let \hat{S} be the result of gluing a disc to β . Then*

$$\mathcal{K}(S, \Sigma_g^n) = \mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta) \rtimes \mathcal{K}(S', \Sigma_g^n).$$

Also, the following hold. Let C be the component of $\Sigma_g^n \setminus \text{Int}(S)$ adjacent to β .

- *If C is not an annulus, then $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta) = [\pi, [\pi, \pi]_{\Sigma_g^n}] \subset [\pi, \pi]_{\Sigma_g^n}$.*
- *If C is an annulus and $\beta' \subset S$ is its other boundary component, then $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta)$ is generated by $[\pi, [\pi, \pi]_{\Sigma_g^n}]$ together with a simple closed curve $\delta \in [\pi, \pi]_{\Sigma_g^n}$ which is freely homotopic to the component of $\partial \hat{S}$ corresponding to β' (see Figures 6.a–b).*

Remark. The reason for the additional generator of $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta)$ when C is an annulus with boundary components β and β' is as follows. As shown in Figure 6.a–c, if $\delta \in [\pi, \pi]_{\Sigma_g^n}$ is a simple closed curve freely homotopic to β' , then using the explicit formula from §2.2 we have (up to signs) $\widehat{\text{Push}}(\delta) = T_\beta T_{\beta'}^{-1} T_\epsilon$, where T_ϵ is a Σ_g^n -separating twist. Thus

$$\tau_{S, \Sigma_g^n}(\widehat{\text{Push}}(\delta)) = \tau_{S, \Sigma_g^n}(T_\beta T_{\beta'}^{-1}) + \tau_{S, \Sigma_g^n}(T_\epsilon) = \tau_{S, \Sigma_g^n}(T_\beta T_{\beta'}^{-1}).$$

However, β and β' are isotopic in Σ_g^n , so $T_\beta T_{\beta'}^{-1}$ is in the kernel of the map $\mathcal{I}(S, \Sigma_g^n) \rightarrow \mathcal{I}(\Sigma_g^n)$ and thus certainly in the kernel of τ_{S, Σ_g^n} .

Proof of Theorem 4.4. If $n = 1$, then by Lemmas 2.1(b) and 2.4, we can replace Σ_g^n with Σ_{g+1} without changing the truth of the theorem, so without loss of generality we can assume that $n = 0$.

The first thing we must do is to construct an analogue for our situation of the factorization in Lemma 2.8. To simplify our notation, we make the following definitions.

- $H = H_1(\Sigma_g; \mathbb{Z})$.
- $\pi = \pi_1(\hat{S})$.
- $\sigma = \sigma_{S, \Sigma_g, \beta}$ (this is the subsurface homology map defined in §2.3).

Let $V \subset H/\langle[\beta]\rangle$ be the image of $\sigma : H_1(\hat{S}; \mathbb{Z}) \rightarrow H/\langle[\beta]\rangle$ and let $K = \ker(\sigma)$, so we have a short exact sequence

$$0 \longrightarrow K \longrightarrow H_1(\hat{S}; \mathbb{Z}) \xrightarrow{\sigma} V \longrightarrow 0.$$

Let the connected components of $\Sigma_g \setminus \text{Int}(S)$ be X_0, \dots, X_r , where X_0 is the component containing β . Orienting the connected components of $\partial X_i \subset S$ in the usual way, the Σ_g -homology class of ∂X_i is 0, so we obtain an element $[\partial X_i]' \in \ker(H_1(S; \mathbb{Z}) \rightarrow H_1(\Sigma_g; \mathbb{Z}))$. Let $[\partial X_i] \in K$ be the image of $[\partial X_i]' \in H_1(S; \mathbb{Z})$. Then K is generated by $[\partial X_0], \dots, [\partial X_r]$, and the only relation between the $[\partial X_i]$ is that $[\partial X_0] + \dots + [\partial X_r] = 0$ (see Figure 6.d). The upshot is that $K \cong \mathbb{Z}^r$ with basis $\{[\partial X_1], \dots, [\partial X_r]\}$. We now choose $\gamma_1, \dots, \gamma_r \in \pi$ with the following properties (see Figure 6.d).

- For $1 \leq i \leq r$, the curve γ_i is a Σ_g -bounding curve surrounding X_i (we recall that Σ_g -bounding curves and torus-bounding curves were defined towards the end of §2.2), oriented such that $[\gamma_i] = [\partial X_i]$.
- For $1 \leq i \leq r$, the curve γ_i is freely homotopic to an unbased simple closed curve η_i such that the curves $\{\eta_1, \dots, \eta_r\}$ are pairwise disjoint.

Observe that $\gamma_i \in [\pi, \pi]_{\Sigma_g}$. We then have the following.

Claim 1. *There is a surjective homomorphism $\phi : [\pi, \pi]_{\Sigma_g} \rightarrow K \oplus \wedge^2 V$ with the following properties.*

- For $1 \leq i \leq r$, we have $\phi(\gamma_i) = [\partial X_i] \in K$.
- For $x, y \in \pi$, we have $\phi([x, y]) = \sigma([x]) \wedge \sigma([y])$, where $[x], [y] \in H_1(\hat{S}; \mathbb{Z})$ are the associated homology classes.
- $\ker(\phi) = [\pi, [\pi, \pi]_{\Sigma_g}]$.

Proof of claim. We have a short exact sequence

$$1 \longrightarrow [\pi, \pi]_{\Sigma_g} \longrightarrow \pi \longrightarrow V \longrightarrow 1.$$

Associated to this is a 5-term exact sequence in group homology (see [5, Corollary VII.6.4]). The group π is free, so $H_2(\pi; \mathbb{Z}) = 0$. Also, $H_2(V; \mathbb{Z}) \cong \wedge^2 V$. The 5-term sequence is thus of the form

$$0 \longrightarrow \wedge^2 V \longrightarrow (H_1([\pi, \pi]_{\Sigma_g}; \mathbb{Z}))_{\pi} \longrightarrow H_1(\pi; \mathbb{Z}) \longrightarrow V \longrightarrow 0.$$

Here $(H_1([\pi, \pi]_{\Sigma_g}; \mathbb{Z}))_{\pi}$ are the coinvariants of the action of π on $H_1([\pi, \pi]_{\Sigma_g}; \mathbb{Z})$, i.e. the largest quotient of $H_1([\pi, \pi]_{\Sigma_g}; \mathbb{Z})$ on which the action of π induced by the conjugation action of π on $[\pi, \pi]_{\Sigma_g}$ is trivial. Chasing through the definitions, we have $(H_1([\pi, \pi]_{\Sigma_g}; \mathbb{Z}))_{\pi} = [\pi, \pi]_{\Sigma_g} / [\pi, [\pi, \pi]_{\Sigma_g}]$. Also, the kernel of the map $H_1(\pi; \mathbb{Z}) \rightarrow V$ is K . We therefore obtain a short exact sequence

$$0 \longrightarrow \wedge^2 V \longrightarrow [\pi, \pi]_{\Sigma_g} / [\pi, [\pi, \pi]_{\Sigma_g}] \longrightarrow K \longrightarrow 0.$$

Since K is a free abelian group, this exact sequence splits; in fact, there is a splitting that takes $[\partial X_i] \in K$ to the image of $\gamma_i \in [\pi, \pi]_{\Sigma_g}$ in $[\pi, \pi]_{\Sigma_g} / [\pi, [\pi, \pi]_{\Sigma_g}]$. From this splitting,

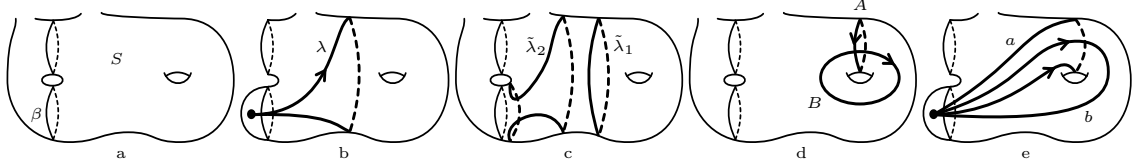


Figure 7: *a.* S is a subsurface of Σ_g and β is a boundary component of S that does not separate Σ_g *b.* $\lambda \in \pi$ is a torus-bounding curve *c.* $\widehat{\text{Push}}(\lambda) = T_{\tilde{\lambda}_1}(T_{\beta}T_{\tilde{\lambda}_2}^{-1})$ *d.* $\tau_{S,\Sigma_g}(T_{\tilde{\lambda}_1}(T_{\beta}T_{\tilde{\lambda}_2}^{-1})) = \tau_{S,\Sigma_g}(T_{\tilde{\lambda}_1}) + \tau_{S,\Sigma_g}(T_{\beta}T_{\tilde{\lambda}_2}^{-1}) = 0 + [\beta] \wedge [A] \wedge [B] = [\beta] \wedge [A] \wedge [B]$ *e.* $\lambda = [a, b]$

we get an isomorphism

$$\phi' : [\pi, \pi]_{\Sigma_g} / [\pi, [\pi, \pi]_{\Sigma_g}] \xrightarrow{\cong} K \oplus \wedge^2 V.$$

The desired map $\phi : [\pi, \pi]_{\Sigma_g} \rightarrow K \oplus \wedge^2 V$ is then the composition of ϕ' with the projection $[\pi, \pi]_{\Sigma_g} \rightarrow [\pi, \pi]_{\Sigma_g} / [\pi, [\pi, \pi]_{\Sigma_g}]$. \square

For $1 \leq i \leq r$, let Y_i be the component of Σ_g cut along η_i that contains X_i . Thus Y_i is a surface with one boundary component and we have an injection $H_1(Y_i; \mathbb{Z}) \hookrightarrow H_1(\Sigma_g; \mathbb{Z})$; let $\omega_i \in \wedge^2 H$ be the image of the characteristic element of $\wedge^2 H_1(Y_i; \mathbb{Z})$. Define a homomorphism $\psi_1 : K \rightarrow \wedge^3 H$ via the formula $\psi_1([\partial X_i]) = [\beta] \wedge \omega_i$. Also, define a homomorphism $\psi_2 : \wedge^2 V \rightarrow \wedge^3 H$ via the formula $\psi_2(v_1 \wedge v_2) = [\beta] \wedge \tilde{v}_1 \wedge \tilde{v}_2$, where $\tilde{v}_i \in H$ is any lift of $v_i \in V \subset H / \langle [\beta] \rangle$ (observe that this is well-defined). Finally, define a homomorphism $\psi : K \oplus \wedge^2 V \rightarrow \wedge^3 H$ via the formula $\psi(k, v_1 \wedge v_2) = \psi_1(k) + \psi_2(v_1 \wedge v_2)$. We then have the following, which is the analogue for our situation of Lemma 2.8. The first isomorphism in this claim follows from Theorem 2.5.

Claim 2. *The restriction of $\tau_{S,\Sigma_g} : \mathcal{I}(S, \Sigma_g) \rightarrow (\wedge^3 H) / H$ to $\mathcal{P}_{\mathcal{I}}(S, \Sigma_g; \beta)$ factors as*

$$\mathcal{P}_{\mathcal{I}}(S, \Sigma_g; \beta) \cong [\pi, \pi]_{\Sigma_g} \xrightarrow{\phi} K \oplus \wedge^2 V \xrightarrow{\psi} \wedge^3 H \longrightarrow (\wedge^3 H) / H.$$

Proof of claim. Let $\nu : [\pi, \pi]_{\Sigma_g} \rightarrow (\wedge^3 H) / H$ be the indicated composition. The group $[\pi, \pi]_{\Sigma_g}$ is generated by $[\pi, \pi]$ together with $\gamma_1, \dots, \gamma_r$, and Lemma 2.6 says that $[\pi, \pi]$ is generated by torus-bounding curves. It is enough therefore to prove that $\nu(\lambda) = \tau_{S,\Sigma_g}(\widehat{\text{Push}}(\lambda))$ for λ either one of the γ_i or a torus-bounding curve. We will give the details for the latter case; the former can be handled similarly.

Let $\lambda \in [\pi, \pi]$ be torus-bounding curve. Replacing λ by its inverse if necessary, we can assume that the one-holed torus lies on its right; see Figures 7.a–b. Letting $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ be the curves depicted in Figure 7.c, the explicit formulas in §2.2 show that $\widehat{\text{Push}}(\lambda) = T_{\tilde{\lambda}_1}(T_{\beta}T_{\tilde{\lambda}_2}^{-1})$. Letting A and B be the curves shown in Figure 7.d, the formulas in §2.4 show that $\tau_{S,\Sigma_g}(\widehat{\text{Push}}(\lambda))$ equals

$$\tau_{S,\Sigma_g}(T_{\tilde{\lambda}_1}) + \tau_{S,\Sigma_g}(T_{\beta}T_{\tilde{\lambda}_2}^{-1}) = 0 + [\beta] \wedge [A] \wedge [B] = [\beta] \wedge [A] \wedge [B] \in (\wedge^3 H) / H$$

Letting $a, b \in \pi$ be the curves in Figure 7.e, we have $\lambda = [a, b]$. Regarding A and B as curves in \hat{S} via the inclusion $S \hookrightarrow \hat{S}$, the curve a is freely homotopic to A and the curve b is freely homotopic to B . Tracing through the definitions, we see that $\nu(\lambda) = [\beta] \wedge [A] \wedge [B] \in (\wedge^3 H) / H$, as desired. \square

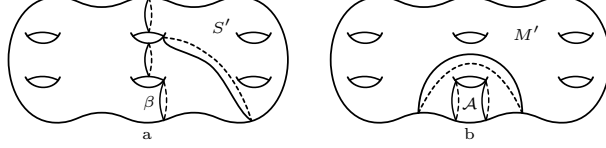


Figure 8: *a.* S is the subsurface to the right of the three “vertical” curves. S' is a Σ_6 -splitting surface for S . *b.* $\mathcal{A} \subset \Sigma_6 \setminus \text{Int}(S)$ is an annulus one of whose boundary components is β . We have $M = \Sigma_6 \setminus \text{Int}(\mathcal{A})$, and $M' \subset M$ is a Σ_6 -splitting surface for β satisfying $S' \subset M'$.

We are now in a position to prove the first assertion of the theorem, namely the following claim.

Claim 3. *We have $\mathcal{K}(S, \Sigma_g) = \mathcal{P}_{\mathcal{K}}(S, \Sigma_g; \beta) \rtimes \mathcal{K}(S', \Sigma_g)$.*

Proof of claim. Theorem 2.5 implies that

$$\mathcal{I}(S, \Sigma_g) = \mathcal{P}_{\mathcal{I}}(S, \Sigma_g; \beta) \rtimes \mathcal{I}(S', \Sigma_g).$$

By Lemma 4.3, the claim is equivalent to

$$\tau_{S, \Sigma_g}(\mathcal{P}_{\mathcal{I}}(S, \Sigma_g; \beta)) \cap \tau_{S, \Sigma_g}(\mathcal{I}(S', \Sigma_g)) = 0,$$

which we now prove. Let \mathcal{A} be an annulus in $\Sigma_g \setminus \text{Int}(S)$ one of whose boundary components is β and let $M = \Sigma_g \setminus \text{Int}(\mathcal{A})$. We have $S \subset M$, and we can choose a Σ_g -splitting surface M' for β in M such that $S' \subset M'$ (see Figure 8). We have

$$\tau_{S, \Sigma_g}(\mathcal{P}_{\mathcal{I}}(S, \Sigma_g; \beta)) \subset \tau_{M, \Sigma_g}(\mathcal{P}_{\mathcal{I}}(M, \Sigma_g; \beta)) \quad \text{and} \quad \tau_{S, \Sigma_g}(\mathcal{I}(S', \Sigma_g)) \subset \tau_{M, \Sigma_g}(\mathcal{I}(M', \Sigma_g)),$$

so it suffices to prove that

$$\tau_{M, \Sigma_g}(\mathcal{P}_{\mathcal{I}}(M, \Sigma_g; \beta)) \cap \tau_{M, \Sigma_g}(\mathcal{I}(M', \Sigma_g)) = 0. \quad (10)$$

Set $H_{M'} = H_1(M'; \mathbb{Z}) \subset H$. Let $\rho : \wedge^3 H \rightarrow (\wedge^3 H)/H$ be the projection. Claim 2 implies that

$$\tau_{M, \Sigma_g}(\mathcal{P}_{\mathcal{I}}(M, \Sigma_g; \beta)) = \rho([\beta] \wedge (\wedge^2 H_{M'})) \subset (\wedge^3 H)/H.$$

Also, Lemma 3.3 implies that

$$\tau_{M, \Sigma_g}(\mathcal{I}(M', \Sigma_g)) = \rho(\wedge^3 H_{M'}) \subset (\wedge^3 H)/H.$$

Equation (10) therefore asserts that

$$\rho([\beta] \wedge (\wedge^2 H_{M'})) \cap \rho(\wedge^3 H_{M'}) = 0. \quad (11)$$

We have that $([\beta] \wedge (\wedge^2 H_{M'})) \cap \wedge^3 H_{M'} = 0$ (as subsets of $\wedge^3 H$). In fact, we have

$$\wedge^3 [\beta]^\perp = ([\beta] \wedge (\wedge^2 H_{M'})) \oplus \wedge^3 H_{M'}; \quad (12)$$

here the orthogonal complement is taken with respect to the intersection form. The first conclusion of Lemma 3.1 says that the kernel of $\rho|_{\wedge^3 [\beta]^\perp}$ lies entirely in $[\beta] \wedge (\wedge^2 H_{M'})$. Therefore, the decomposition (12) projects to

$$\rho(\wedge^3 [\beta]^\perp) = \rho([\beta] \wedge (\wedge^2 H_{M'})) \oplus \rho(\wedge^3 H_{M'}). \quad (13)$$

In particular, (11) holds. \square

For the second assertion of the theorem (concerning $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g; \beta)$), we need the following.

Claim 4. *The homomorphism ψ is injective.*

Proof of claim. The map $\psi|_{\wedge^2 V}$ is injective with image $[\beta] \wedge (\wedge^2 V)$. To prove that ψ itself is injective, it is enough to prove that the elements $[\beta] \wedge \omega_1, \dots, [\beta] \wedge \omega_r \in \wedge^3 H$ are linearly independent modulo $[\beta] \wedge (\wedge^2 V)$.

Recall that ω_i is the characteristic element of $\wedge^2 H_1(Y_i; \mathbb{Z})$, where Y_i is the subsurface of Σ_g to the right of η_i . Choose a symplectic basis B for H with the following properties.

- For $1 \leq i \leq r$, there is some subset of B forming a symplectic basis for $H_1(Y_i; \mathbb{Z})$.
- $[\beta] \in B$.
- There is a subset of B that projects to a basis for $V \subset H/\langle[\beta]\rangle$.

Choose a total ordering \prec on B such that $[\beta]$ is a minimal element. The set $B_{\wedge} = \{x \wedge y \wedge z \mid x, y \in B, x \prec y \prec z\}$ is a basis for $\wedge^3 H$. Define

$$B'_{\wedge} = \{[\beta] \wedge y \wedge z \mid y, z \in B, [\beta] \prec y \prec z\} \subset B_{\wedge}.$$

By assumption, there is a subset B''_{\wedge} of B'_{\wedge} which forms a basis for $[\beta] \wedge (\wedge^2 V)$. Also, for all $1 \leq i \leq r$, the element $[\beta] \wedge \omega_i$ is in the span of B'_{\wedge} . Finally, for all $1 \leq i \leq r$, there exists some $v_i \in B'_{\wedge}$ with the following properties.

- In the expansion of $[\beta] \wedge \omega_i$ in the basis B'_{\wedge} , the coordinate of v_i is nonzero.
- $v_i \notin B''_{\wedge}$.
- $v_i \neq v_j$ for $i \neq j$.

It follows that the elements $[\beta] \wedge \omega_i$ are linearly independent modulo $[\beta] \wedge (\wedge^2 V)$, and we are done. \square

The following claim is the remaining part of the theorem

Claim 5. *Let C be the component of $\Sigma_g^n \setminus \text{Int}(S)$ adjacent to β .*

- *If C is not an annulus, then $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta) = [\pi, [\pi, \pi]_{\Sigma_g^n}] \subset [\pi, \pi]_{\Sigma_g^n}$.*
- *If C is an annulus and $\beta' \subset S$ is its other boundary component, then $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g^n; \beta)$ is generated by $[\pi, [\pi, \pi]_{\Sigma_g^n}]$ together with a simple closed curve $\delta \in [\pi, \pi]_{\Sigma_g^n}$ which is freely homotopic to the component of $\partial \hat{S}$ corresponding to β' (see Figures 6.a–b).*

Proof of claim. Let $\tau_1 : \mathcal{P}_{\mathcal{I}}(S, \Sigma_g; \beta) \rightarrow \wedge^3 H$ be the composition

$$\mathcal{P}_{\mathcal{I}}(S, \Sigma_g; \beta) \cong [\pi, \pi]_{\Sigma_g} \xrightarrow{\phi} K \oplus \wedge^2 V \xrightarrow{\psi} \wedge^3 H$$

and let $\tau_2 : \wedge^3 H \rightarrow (\wedge^3 H)/H$ be the projection. Claim 2 says that $\tau_{S, \Sigma_g}|_{\mathcal{P}_{\mathcal{I}}(S, \Sigma_g; \beta)} = \tau_2 \circ \tau_1$. Claims 1 and 4 show that $\ker(\tau_1) = [\pi, [\pi, \pi]_{\Sigma_g}]$.

If the component C of $\Sigma_g \setminus \text{Int}(S)$ adjacent to β is not an annulus, then we can find a simple closed curve ζ in C such that ζ is not homologous to β . The image of τ_1 lies in $\wedge^3 \langle [\beta], [\zeta] \rangle^{\perp}$, and the second conclusion of Lemma 3.1 implies that $\tau_2|_{\wedge^3 \langle [\beta], [\zeta] \rangle^{\perp}}$ is injective. We conclude that the kernel of $\tau_2 \circ \tau_1$ is $[\pi, [\pi, \pi]_{\Sigma_g}]$, as desired.

If C is an annulus, then we cannot find such a ζ . In this case, the image of τ_1 lies in $\wedge^3\langle[\beta]\rangle^\perp$, and the first conclusion of Lemma 3.1 says that $\ker(\tau_2|_{\wedge^3\langle[\beta]\rangle^\perp})$ is spanned by $[\beta] \wedge \omega$, where $\omega \in \wedge^2 H$ is the characteristic element. Letting $\delta \in [\pi, \pi]_{\Sigma_g}$ be the element described in the theorem, an easy calculation shows that $\tau_1(\delta) = \pm[\beta] \wedge \omega$. We conclude that the kernel of $\tau_2 \circ \tau_1$ is generated by $[\pi, [\pi, \pi]_{\Sigma_g}]$ and δ , as desired. \square

This concludes the proof of Theorem 4.4. \square

In the course of the above proof, we proved the following lemma (see in particular equation (13)), which we will need later.

Lemma 4.5. *For some $g \geq 3$, let β be a nonseparating simple closed curve on Σ_g . Let $A \subset \Sigma_g$ be an annulus one of whose boundary components is β , let $M = \Sigma_g \setminus \text{Int}(A)$, and let $M' \subset M$ be a Σ_g -splitting surface for β . Define $H = H_1(\Sigma_g; \mathbb{Z})$ and $H_{M'} = H_1(M'; \mathbb{Z}) \subset H$. Then*

$$\tau_{M, \Sigma_g}(\mathcal{I}(M, \Sigma_g)) \cong ((\wedge^2 H_{M'})/\mathbb{Z}) \oplus (\wedge^3 H_{M'}) \subset (\wedge^3 H)/H,$$

where $((\wedge^2 H_{M'})/\mathbb{Z})$ is the image of $\mathcal{P}_{\mathcal{I}}(M, \Sigma_g; \beta) \subset \mathcal{I}(M, \Sigma_g)$ and $\wedge^3 H_{M'}$ is the image of $\mathcal{I}(M', \Sigma_g) \subset \mathcal{I}(M, \Sigma_g)$.

5 Generating the Johnson kernel

The main result of this section is as follows.

Theorem 5.1. *Let S be a surface whose genus is at least 2 and let $S \hookrightarrow \Sigma_g^n$ be a clean embedding with $n \leq 1$. Then the group $\mathcal{K}(S, \Sigma_g^n)$ is generated by the following elements.*

- Σ_g^n -separating twists.
- Σ_g^n -bounding pair maps $T_\beta T_{\beta'}^{-1}$, where β and β' are the boundary components of a component A of $\Sigma_g^n \setminus \text{Int}(S)$ such that A is homeomorphic to an annulus and $\partial A \subset S$.

The proof of Theorem 5.1 is in §5.2 following a preliminary lemma in §5.1. Before doing this, we derive Theorem A from Theorem 5.1.

Proof of Theorem A. Let us recall the statement. Let S be a surface whose genus is at least 2 and let $S \hookrightarrow \Sigma_g^n$ be an embedding with $n \leq 1$. We want to show that $\widehat{\mathcal{K}}(S, \Sigma_g^n)$ is generated by $\{T_\gamma \mid \gamma \text{ is a separating curve on } \Sigma_g^n \text{ and } \gamma \subset S\}$. First, without changing $\widehat{\mathcal{K}}(S, \Sigma_g^n)$ we can add to S any components of $\Sigma_g^n \setminus \text{Int}(S)$ that are discs. Thus we can assume without loss of generality that $S \hookrightarrow \Sigma_g^n$ is clean. The natural map $\rho : \mathcal{K}(S, \Sigma_g^n) \rightarrow \widehat{\mathcal{K}}(S, \Sigma_g^n)$ is a surjection which takes Σ_g^n -separating twists to the claimed generators. It also satisfies $\rho(T_\beta T_{\beta'}^{-1}) = 0$ whenever β and β' are the boundary components of a component A of $\Sigma_g^n \setminus \text{Int}(S)$ such that A is homeomorphic to an annulus and $\partial A \subset S$. The desired result thus follows from Theorem 5.1. \square

5.1 Generating the Johnson kernel disc-pushing subgroup

In preparation for proving Theorem 5.1, we prove the following.

Lemma 5.2. *Let S be a surface whose genus is at least 2, let $S \hookrightarrow \Sigma_g$ be a clean embedding, and let β be a boundary component of S . Let Γ be the subgroup of $\mathcal{K}(S, \Sigma_g)$ generated by the purported generating set from Theorem 5.1. Then $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g; \beta) \subset \Gamma$.*

Proof. Let \hat{S} be the result of gluing a disc to S along β and let $\pi = \pi_1(\hat{S})$.

Case 1. β separates Σ_g .

Theorem 4.2 says that $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g; \beta)$ is the pullback of $[\pi, \pi]_{\Sigma_g}$ under the map

$$\mathcal{P}_{\mathcal{I}}(S, \Sigma_g; \beta) \cong \pi_1(U\hat{S}) \rightarrow \pi_1(\hat{S}).$$

Using Lemma 2.7, we deduce that $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g; \beta)$ is generated by two kinds of elements.

- The Σ_g -separating twist T_β corresponding to the loop around the fiber in $\pi_1(U\hat{S})$.
- $\text{Push}(\gamma)$ for $\gamma \in [\pi, \pi]_{\Sigma_g}$ that can be realized by either a torus-bounding curve or a Σ_g -boundary curve. Using the explicit formula in §2.1, we have $\text{Push}(\gamma) = T_{\tilde{\gamma}_1} T_{\tilde{\gamma}_2}^{-1}$ where $T_{\tilde{\gamma}_1}$ and $T_{\tilde{\gamma}_2}$ are Σ_g -separating twists.

This implies that $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g; \beta) \subset \Gamma$, as desired.

Case 2. β does not separate Σ_g and the component of $\Sigma_g \setminus \text{Int}(S)$ adjacent to β is not an annulus.

Theorem 4.4 says that $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g; \beta)$ is the subgroup $[\pi, [\pi, \pi]_{\Sigma_g}]$ of $\mathcal{P}_{\mathcal{I}}(S, \Sigma_g; \beta) \cong [\pi, \pi]_{\Sigma_g}$. Define

$$A = \{\gamma \in [\pi, \pi]_{\Sigma_g} \mid \gamma \text{ can be realized by a torus-bounding or } \Sigma_g\text{-boundary curve}\}.$$

Lemma 2.7 says that A is a generating set for $[\pi, \pi]_{\Sigma_g}$. For $\gamma \in A$, define

$$B_\gamma = \{\delta \in \pi \mid \delta \text{ can be realized by a nonseparating curve that only intersects } \gamma \text{ at the basepoint}\}.$$

For all $\gamma \in A$, the set B_γ generates π . It follows that the set

$$C = \{[\delta, \gamma] \mid \gamma \in A \text{ and } \delta \in B_\gamma\}$$

normally generates $[\pi, [\pi, \pi]_{\Sigma_g}]$ as a subgroup of π .

We now prove that for $\zeta \in \pi$ and $[\delta, \gamma] \in C$, we also have $[\zeta\delta\zeta^{-1}, \zeta\gamma\zeta^{-1}] \in C$. By the Dehn-Nielsen-Baer theorem, inner automorphisms of π are induced by homeomorphisms of \hat{S} that fix the basepoint. It follows that there exists some homeomorphism $f : \hat{S} \rightarrow \hat{S}$ that fixes the basepoint and satisfies $f_*(x) = \zeta x \zeta^{-1}$ for all $x \in \pi$. We conclude that $[\zeta\delta\zeta^{-1}, \zeta\gamma\zeta^{-1}] = [f_*(\delta), f_*(\gamma)]$, which lies in C , as claimed. This implies that C generates $[\pi, [\pi, \pi]_{\Sigma_g}]$.

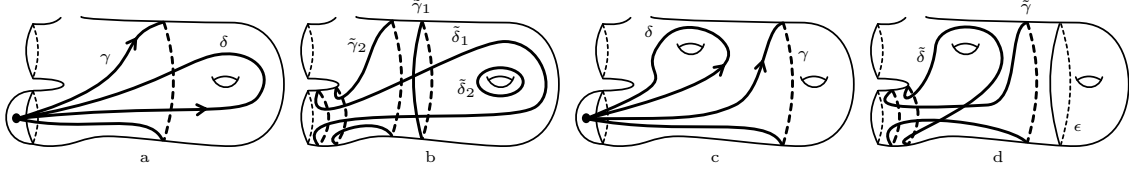


Figure 9: a. γ is a torus-bounding curve and δ is inside the torus. b. $\text{Push}(\delta) = T_{\tilde{\delta}_1} T_{\tilde{\delta}_2}^{-1}$ and $\widehat{\text{Push}}(\gamma) = T_{\tilde{\gamma}_1} (T_{\beta} T_{\tilde{\gamma}_2}^{-1})$. c. γ is a torus-bounding curve and δ is outside the torus. d. $[\text{Push}(\delta), \widehat{\text{Push}}(\gamma)] = T_{T_{\tilde{\delta}}(\tilde{\gamma})} T_{\tilde{\gamma}}^{-1}$.

Consider $[\delta, \gamma] \in C$. The associated element of $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g; \beta)$ is $[\text{Push}(\delta), \widehat{\text{Push}}(\gamma)]$. We must show that this lies in Γ . We will give the details for the case where γ is a torus-bounding curve; the case where γ is a Σ_g -boundary curve is similar. Let X be the component of S cut along γ that is a one-holed torus. There are two cases. In the first, δ can be realized by a simple closed nonseparating curve lying inside X . For definiteness, assume that the orientations on γ and δ are the same as those in Figure 9.a; the other cases are similar. Letting $\tilde{\gamma}_i$ and $\tilde{\delta}_i$ be the curves in Figure 9.b, the formulas in §2.1 and §2.2 imply that

$$[\text{Push}(\delta), \widehat{\text{Push}}(\gamma)] = [T_{\tilde{\delta}_1} T_{\tilde{\delta}_2}^{-1}, T_{\tilde{\gamma}_1} (T_{\beta} T_{\tilde{\gamma}_2}^{-1})] = [T_{\tilde{\delta}_1}, T_{\tilde{\gamma}_1}] = T_{\tilde{\delta}_1} T_{\tilde{\gamma}_1} T_{\tilde{\delta}_1}^{-1} T_{\tilde{\gamma}_1}^{-1} = T_{T_{\tilde{\delta}_1}(\tilde{\gamma}_1)} T_{\tilde{\gamma}_1}^{-1}.$$

Both $T_{T_{\tilde{\delta}_1}(\tilde{\gamma}_1)}$ and $T_{\tilde{\gamma}_1}$ are Σ_g -separating twists, so this lies in Γ .

The other case is when δ can be realized by a simple closed nonseparating curve which aside from the basepoint lies entirely outside X . For definiteness, assume that the orientations on γ and δ are the same as those in Figure 9.c; the other cases are similar. Letting $\tilde{\gamma}$ and $\tilde{\delta}$ be the curves in Figure 9.d, a similar calculation to the one above shows that

$$[\text{Push}(\delta), \widehat{\text{Push}}(\gamma)] = T_{T_{\tilde{\delta}}(\tilde{\gamma})} T_{\tilde{\gamma}}^{-1}.$$

Unfortunately, $T_{\tilde{\gamma}}$ is not a Σ_g -separating twist. However, let ϵ be the curve shown in Figure 9.d and let S' be the subsurface of S to the left of ϵ . Then we can regard $\phi := T_{T_{\tilde{\delta}}(\tilde{\gamma})} T_{\tilde{\gamma}}^{-1}$ as an element of $\mathcal{K}(S', \Sigma_g)$. Moreover, ϕ becomes homotopic to the identity when a disc is glued to S' along ϵ , so $\phi \in \mathcal{P}_{\mathcal{K}}(S', \Sigma_g; \epsilon)$. The previous case therefore applies to show that ϕ can be written as a product of Σ_g -separating twists, as desired.

Case 3. β does not separate Σ_g and the component of $\Sigma_g \setminus \text{Int}(S)$ adjacent to β is an annulus.

This is similar to case 2, the only difference being the extra generator for $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g; \beta)$ given by Theorem 4.4. As indicated in the remark following Theorem 4.4, this extra generator lies in Γ . \square

5.2 The Johnson kernel is generated by separating twists

We now prove Theorem 5.1. Our proof will use the following theorem of the author.

Theorem 5.3. *Fix some $g \geq 3$. Let α and β be simple closed curves on Σ_g that intersect once. Then $\mathcal{I}(\Sigma_g)$ is generated by the stabilizer subgroups $(\mathcal{I}(\Sigma_g))_{\alpha}$ and $(\mathcal{I}(\Sigma_g))_{\beta}$.*

Proof. An immediate consequence of Lemmas 3.1, 5.1, and 5.2 of [25]. \square

Proof of Theorem 5.1. Let us recall the statement. Let S be a surface whose genus is at least 2 and let $S \hookrightarrow \Sigma_g^n$ be a clean embedding with $n \leq 1$. Then we want to prove that the group $\mathcal{K}(S, \Sigma_g^n)$ is generated by the following elements.

- Σ_g^n -separating twists.
- Σ_g^n -bounding pair maps $T_\beta T_{\beta'}^{-1}$, where β and β' are the boundary components of a component A of $\Sigma_g^n \setminus \text{Int}(S)$ such that A is homeomorphic to an annulus and $\partial A \subset S$.

If $n = 1$, then there is a clean embedding $\Sigma_g^n \hookrightarrow \Sigma_{g+1}$ and Lemma 3.3 says that $\mathcal{K}(S, \Sigma_g^n) = \mathcal{K}(S, \Sigma_{g+1})$. We can therefore assume without loss of generality that $n = 0$.

Let $\Gamma(S, \Sigma_g)$ be the subgroup of $\mathcal{K}(S, \Sigma_g)$ generated by the above elements. If $S = \Sigma_g$, then we will write $\Gamma(\Sigma_g)$ instead of $\Gamma(S, \Sigma_g)$. We want to prove that $\Gamma(S, \Sigma_g) = \mathcal{K}(S, \Sigma_g)$. We break this into several steps.

Step 1. $\Gamma(\Sigma_2) = \mathcal{K}(\Sigma_2)$.

The Johnson homomorphism τ_{Σ_2} vanishes, so $\mathcal{K}(\Sigma_2) = \mathcal{I}(\Sigma_2)$ and the result follows from the fact that $\mathcal{I}(\Sigma_2)$ is generated by separating twists (this theorem should probably be attributed to Powell [24], though he does not state it; see [25] and [13] for modern proofs).

Step 2. *For some $h \geq 2$, assume that $\Gamma(\Sigma_h) = \mathcal{K}(\Sigma_h)$. Then $\Gamma(S, \Sigma_g) = \mathcal{K}(S, \Sigma_g)$ whenever S has genus h .*

The proof is by induction on the number of boundary components of S . For the base case when S is a closed surface (and thus $g = h$ and $S = \Sigma_g$), the result holds by assumption. Now assume that S has nonempty boundary. There are three cases. The first is that S has one boundary component. Letting β be the boundary component of S and \hat{S} be the result of gluing a disc to β , Theorem 4.1 says that there is a short exact sequence

$$1 \longrightarrow \mathcal{P}_{\mathcal{K}}(S, \Sigma_g; \beta) \longrightarrow \mathcal{K}(S, \Sigma_g) \longrightarrow \mathcal{K}(\hat{S}) \longrightarrow 1.$$

Lemma 5.2 says that $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g; \beta)$ is contained in $\Gamma(S, \Sigma_g)$. Also, examining our purported generating set we see that $\Gamma(S, \Sigma_g)$ projects to $\Gamma(\hat{S})$. We have assumed that $\Gamma(\Sigma_h) = \mathcal{K}(\Sigma_h)$, so we conclude that $\Gamma(S, \Sigma_g) = \mathcal{K}(S, \Sigma_g)$.

The second case is when S has multiple boundary components, but all of them separate Σ_g . Let β be one of the boundary components of S and $S' \hookrightarrow S$ be a splitting surface for β . Theorem 4.2 says that

$$\mathcal{K}(S, \Sigma_g) = \mathcal{P}_{\mathcal{K}}(S, \Sigma_g; \beta) \rtimes \mathcal{K}(S', \Sigma_g).$$

Lemma 5.2 says that we have $\mathcal{P}_{\mathcal{K}}(S, \Sigma_g; \beta) \subset \Gamma(S, \Sigma_g)$. Our inductive hypothesis says that $\mathcal{K}(S', \Sigma_g) = \Gamma(S', \Sigma_g)$; examining our purported generating set, we see that every purported generator for $\mathcal{K}(S', \Sigma_g)$ is also one for the larger group $\mathcal{K}(S, \Sigma_g)$, so we deduce that $\mathcal{K}(S', \Sigma_g) \subset \Gamma(S, \Sigma_g)$. We conclude that $\mathcal{K}(S, \Sigma_g) = \Gamma(S, \Sigma_g)$, as desired.

Finally, if S has multiple boundary components and one of them does not separate Σ_g , then we can substitute Theorem 4.4 for Theorem 4.2 (and use a Σ_g -splitting surface) in the above argument.

Step 3. *For some $g \geq 3$, assume that $\Gamma(S, \Sigma_g) = \mathcal{K}(S, \Sigma_g)$ for all cleanly embedded subsurfaces S of Σ_g whose genus is less than g . Then $\Gamma(\Sigma_g) = \mathcal{K}(\Sigma_g)$.*

Set $H = H_1(\Sigma_g; \mathbb{Z})$ and $\mathcal{Q} = \mathcal{I}(\Sigma_g)/\Gamma(\Sigma_g)$. The surjection $\tau_{\Sigma_g} : \mathcal{I}(\Sigma_g) \rightarrow (\wedge^3 H)/H$ induces a surjection $\psi : \mathcal{Q} \rightarrow (\wedge^3 H)/H$. Our goal is to show that ψ is an isomorphism. We divide this into four substeps. Let α and β be two curves on Σ_g that intersect once.

Substep 3.1. *Let x be either α or β . We construct groups \mathcal{Q}_x and maps $\phi_x : \mathcal{Q}_x \rightarrow \mathcal{Q}$.*

Let \mathcal{A}_x be an annulus in Σ_g one of whose boundary components is x and let $M_x = \Sigma_g \setminus \text{Int}(\mathcal{A}_x)$. Thus $M_x \cong \Sigma_{g-1}^2$ and we have a surjection $\mathcal{I}(M_x, \Sigma_g) \rightarrow (\mathcal{I}(\Sigma_g))_x$. Our assumptions say that $\Gamma(M_x, \Sigma_g) = \mathcal{K}(M_x, \Sigma_g)$, and image of $\Gamma(M_x, \Sigma_g)$ under the map $\mathcal{I}(M_x, \Sigma_g) \rightarrow \mathcal{I}(\Sigma_g)$ is contained in $\Gamma(\Sigma_g)$. Defining $\mathcal{Q}_x = \mathcal{I}(M_x, \Sigma_g)/\mathcal{K}(M_x, \Sigma_g)$, we get an induced map $\phi_x : \mathcal{Q}_x \rightarrow \mathcal{Q}$.

By definition,

$$\mathcal{Q}_x \cong \tau_{M_x, \Sigma_g}(\mathcal{I}(M_x, \Sigma_g)) \subset (\wedge^3 H)/H.$$

Let $M' \subset \Sigma_g$ be the complement of an open regular neighborhood of $\alpha \cup \beta$. Thus $M' \subset M_x$ and $M' \cong \Sigma_{g-1}^1$. Moreover, M' is a Σ_g -splitting surface for the boundary component x of M_x . Letting $H_{M'} = H_1(M'; \mathbb{Z}) \subset H$, Lemma 4.5 implies that

$$\mathcal{Q}_x \cong ((\wedge^2 H_{M'})/\mathbb{Z}) \oplus (\wedge^3 H_{M'}) \subset (\wedge^3 H)/H.$$

We will identify \mathcal{Q}_x with $((\wedge^2 H_{M'})/\mathbb{Z}) \oplus (\wedge^3 H_{M'})$. The next two substeps focus on the effect of ϕ_x on these two summands.

Substep 3.2. *For all $v \in \wedge^3 H_{M'}$, we have $\phi_\alpha(v) = \phi_\beta(v)$.*

Since $\wedge^3 H_{M'} \subset \mathcal{Q}_\alpha$ and $\wedge^3 H_{M'} \subset \mathcal{Q}_\beta$ are the images of $\mathcal{I}(M', \Sigma_g)$, this follows from the commutative diagram

$$\begin{array}{ccc} & \mathcal{I}(M_\alpha, \Sigma_g) & \\ & \nearrow & \searrow \\ \mathcal{I}(M', \Sigma_g) & & \mathcal{I}(\Sigma_g) \\ & \searrow & \nearrow \\ & \mathcal{I}(M_\beta, \Sigma_g) & \end{array}$$

Substep 3.3. *The images $\phi_\alpha((\wedge^2 H_{M'})/\mathbb{Z}) \subset \mathcal{Q}$ and $\phi_\beta((\wedge^2 H_{M'})/\mathbb{Z}) \subset \mathcal{Q}$ commute.*

For x equal to either α or β , let \hat{M}_x be the result of gluing a disc to M_x along x (see Figures 10a–c). Recall from §2.2 that $\mathcal{P}_{\mathcal{I}}(M_x, \Sigma_g; x) \cong [\pi_1(\hat{M}_x), \pi_1(\hat{M}_x)]$ and that for $\gamma \in [\pi_1(\hat{M}_x), \pi_1(\hat{M}_x)]$ the associated element of $\mathcal{P}_{\mathcal{I}}(M_x, \Sigma_g; x)$ is denoted $\widehat{\text{Push}}(\gamma)$. Let $\rho_x : \mathcal{I}(M_x, \Sigma_g) \rightarrow \mathcal{Q}_x$ be the projection.

Lemma 2.6 says that $[\pi_1(\hat{M}_\alpha), \pi_1(\hat{M}_\alpha)]$ is generated by torus-bounding curves a . Fixing such an $a \in [\pi_1(\hat{M}_\alpha), \pi_1(\hat{M}_\alpha)]$, it is enough to find a set $B \subset [\pi_1(\hat{M}_\beta), \pi_1(\hat{M}_\beta)]$ with the following two properties.

1. The set $\{\rho_\beta(\widehat{\text{Push}}(b)) \mid b \in B\}$ generates $(\wedge^2 H_{M'})/\mathbb{Z} \subset \mathcal{Q}_\beta$.
2. For all $b \in B$, we have $[\phi_\alpha(\rho_\alpha(\widehat{\text{Push}}(a))), \phi_\beta(\rho_\beta(\widehat{\text{Push}}(b)))] = 1 \in \mathcal{Q}$.

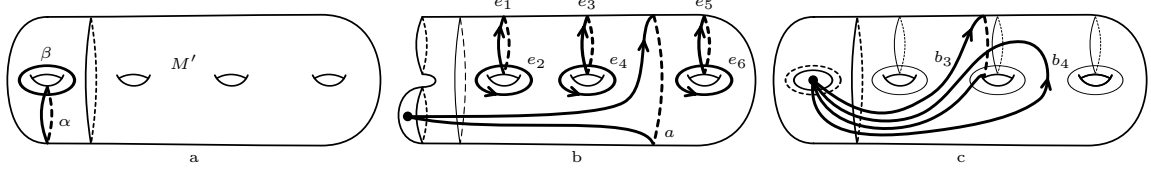


Figure 10: *a.* Two curves α and β that intersect once and the complement M' of an open regular neighborhood of $\alpha \cup \beta$. *b.* The surface \hat{M}_α , the torus-bounding curve $a \in \pi_1(\hat{M}_\alpha)$, and the curves e_1, \dots, e_{2g-2} . *c.* The surface \hat{M}_β ; the dotted loop on the left hand side is a boundary component on the back of the surface. As an example of how to find the curves b_i , the curves b_3 and b_4 are shown. The curves e_1, \dots, e_{2g-2} are drawn faintly.

As shown in Figure 10.b, the curve a divides M' into two components X and Y with Y a one-holed torus. Choose unbased oriented simple closed curves e_1, \dots, e_{2g-2} in M' with the following properties (see Figure 10.b).

- $e_1, \dots, e_{2g-4} \subset X$ and $e_{2g-3}, e_{2g-2} \subset Y$.
- For $1 \leq i < j \leq 2g - 2$, the curves e_i and e_j are disjoint unless $i = 2k$ and $j = 2k + 1$ for some $1 \leq k \leq g - 1$, in which case they intersect once with positive sign.

This implies that $[e_1], \dots, [e_{2g-2}]$ is a symplectic basis for $H_{M'}$, and thus that $\wedge^2 H_{M'}$ is generated by $\{[e_i] \wedge [e_j] \mid 1 \leq i < j \leq 2g - 2\}$.

As is shown in Figure 10.c, we can find elements $b_1, \dots, b_{2g-2} \in \pi_1(\hat{M}_\beta)$ with the following properties.

- For $1 \leq i \leq 2g - 2$, the curve b_i is freely homotopic to and disjoint from e_i .
- For $1 \leq i, j \leq 2g - 2$ such that $i \neq j$ and such that e_i is disjoint from e_j , the curve b_i is also disjoint from e_j .

The first of these shows that for $1 \leq i < j \leq 2g - 2$, the element $\rho_\beta(\widehat{\text{Push}}([b_i, b_j])) \in (\wedge^2 H_{M'})/\mathbb{Z} \subset \mathcal{Q}_\beta$ equals the image of $[e_i] \wedge [e_j] \in \wedge^2 H_{M'}$ in $(\wedge^2 H_{M'})/\mathbb{Z}$. It follows that the set

$$B = \{[b_i, b_j] \mid 1 \leq i < j \leq 2g - 2\}$$

satisfies the first property of B listed above.

It remains to prove the second claimed property of B . Fixing $1 \leq i < j \leq 2g - 2$, this asserts that

$$[\phi_\alpha(\rho_\alpha(\widehat{\text{Push}}(a))), \phi_\beta(\rho_\beta(\widehat{\text{Push}}([b_i, b_j])))] = 1. \quad (14)$$

Let $i_\alpha : \mathcal{I}(M_\alpha, \Sigma_g) \rightarrow \mathcal{I}(\Sigma_g)$ and $i_\beta : \mathcal{I}(M_\beta, \Sigma_g) \rightarrow \mathcal{I}(\Sigma_g)$ be the maps induced by the inclusions $M_\alpha \hookrightarrow \Sigma_g$ and $M_\beta \hookrightarrow \Sigma_g$. The equation (14) is equivalent to the assertion that the element $\theta := [i_\alpha(\widehat{\text{Push}}(a)), i_\beta(\widehat{\text{Push}}([b_i, b_j]))] \in \mathcal{K}(\Sigma_g)$ lies in $\Gamma(\Sigma_g)$.

By construction (and the fact that $g \geq 3$), there is some $1 \leq k \leq 2g - 2$ such that e_k is disjoint from both b_i and b_j . Since e_k is also disjoint from a , we see that e_k is fixed by $\widehat{\text{Push}}([b_i, b_j])$ and $\widehat{\text{Push}}(a)$, and thus also fixed by θ . Letting R be the complement in Σ_g of an open regular neighborhood of e_k , we deduce that θ lies in the image of $\mathcal{K}(R, \Sigma_g)$. By assumption, we have that $\Gamma(R, \Sigma_g) = \mathcal{K}(R, \Sigma_g)$, so we conclude that $\theta \in \Gamma(\Sigma_g)$, as desired.

Substep 3.4. *We use ϕ_α and ϕ_β to prove that ψ is an isomorphism.*

The universal property of the free product says that ϕ_α and ϕ_β can be combined into a map

$$\phi_\alpha * \phi_\beta : \mathcal{Q}_\alpha * \mathcal{Q}_\beta \cong [((\wedge^2 H_{M'})/\mathbb{Z}) \oplus (\wedge^3 H_{M'})] * [((\wedge^2 H_{M'})/\mathbb{Z}) \oplus (\wedge^3 H_{M'})] \longrightarrow \mathcal{Q}.$$

Theorem 5.3 implies that $\phi_\alpha * \phi_\beta$ is surjective, and Substeps 3.2 and 3.3 show that $\phi_\alpha * \phi_\beta$ factors through a surjective homomorphism

$$\phi : [(\wedge^2 H_{M'})/\mathbb{Z}] \oplus [(\wedge^2 H_{M'})/\mathbb{Z}] \oplus [\wedge^3 H_{M'}] \longrightarrow \mathcal{Q}.$$

Summing up, we have surjective maps of the form

$$[(\wedge^2 H_{M'})/\mathbb{Z}] \oplus [(\wedge^2 H_{M'})/\mathbb{Z}] \oplus [\wedge^3 H_{M'}] \xrightarrow{\phi} \mathcal{Q} \xrightarrow{\psi} (\wedge^3 H)/H.$$

Since $g \geq 3$ (and, in particular, $g > 1$), the group $(\wedge^3 H)/H$ is a free abelian group of rank $\binom{2g}{3} - 2g$. Also, $[(\wedge^2 H_{M'})/\mathbb{Z}] \oplus [(\wedge^2 H_{M'})/\mathbb{Z}] \oplus [\wedge^3 H_{M'}]$ is a free abelian group of rank

$$\left[\binom{2g-2}{2} - 1 \right] + \left[\binom{2g-2}{2} - 1 \right] + \binom{2g-2}{3}.$$

A little elementary algebra shows that this also equals $\binom{2g}{3} - 2g$. In other words, $\psi \circ \phi$ is a surjective map between free abelian groups of the same rank, so it must be an isomorphism. We conclude that ψ is also an isomorphism, as desired. \square

6 Finite-index subgroups of the Torelli group

In this section, we prove Theorem B. We first recall its statement. Let S be a surface whose genus is at least 3 and $S \hookrightarrow \Sigma_g^n$ an embedding with $n \leq 1$. Also, let Γ be a finite-index subgroup of $\widehat{\mathcal{I}}(S, \Sigma_g^n)$ with $\widehat{\mathcal{K}}(S, \Sigma_g^n) < \Gamma$. Then Theorem B asserts that $H_1(\Gamma; \mathbb{Q}) \cong \tau_{\Sigma_g^n}(\widehat{\mathcal{I}}(S, \Sigma_g^n)) \otimes \mathbb{Q}$.

The target of $\tau_{\Sigma_g^n}$ is a free abelian group and $[\widehat{\mathcal{I}}(S, \Sigma_g^n) : \Gamma] < \infty$, so

$$\tau_{\Sigma_g^n}(\widehat{\mathcal{I}}(S, \Sigma_g^n)) \otimes \mathbb{Q} \cong \tau_{\Sigma_g^n}(\Gamma) \otimes \mathbb{Q}.$$

Since $\widehat{\mathcal{K}}(S, \Sigma_g^n) < \Gamma$, we have $\ker(\tau_{\Sigma_g^n}|_\Gamma) = \widehat{\mathcal{K}}(S, \Sigma_g^n)$, so Theorem A implies that $\ker(\tau_{\Sigma_g^n}|_\Gamma)$ is generated by separating twists. We conclude that Theorem B is equivalent to the following lemma. If G is a group and $g \in G$, then denote by $[g]_G$ the associated element of $H_1(G; \mathbb{Q})$.

Lemma 6.1 (Separating twists vanish). *For $g \geq 3$ and $n \leq 1$, let $S \hookrightarrow \Sigma_g^n$ be a compact connected subsurface of genus at least 3. Let Γ be a finite-index subgroup of $\widehat{\mathcal{I}}(S, \Sigma_g^n)$ with $\widehat{\mathcal{K}}(S, \Sigma_g^n) < \Gamma$ and let δ be a simple closed curve on S which separates Σ_g^n . Then $[T_\delta]_\Gamma = 0$.*

Remark. In the special case that $S = \Sigma_g^n$ and $\Gamma = \widehat{\mathcal{I}}(S, \Sigma_g^n)$, this was proven by Johnson in [20]. Our proof builds on Johnson's proof.

The proof of Lemma 6.1 is contained in §6.2. This is preceded by §6.1.

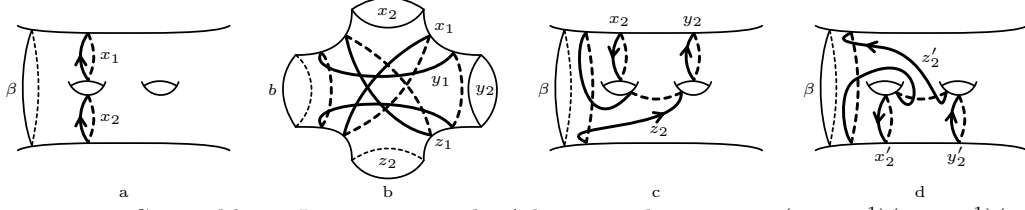


Figure 11: *a.* Curves like in Lemma 6.3 *b.* A lantern relation $T_\beta = (T_{x_1} T_{x_2}^{-1})(T_{y_1} T_{y_2}^{-1})(T_{z_1} T_{z_2}^{-1})$ *c,d.* Two lantern relations as indicated in the proof of Lemma 6.2. We only draw the four “exterior” curves, namely $\beta \cup x_2 \cup y_2 \cup z_2$ and $\beta \cup x_2' \cup y_2' \cup z_2'$.

6.1 Separating twists are torsion, special case

In this section, we prove the following lemma.

Lemma 6.2. *For $g \geq 2$ and $n \leq 1$, let $S \hookrightarrow \Sigma_g^n$ be a compact connected subsurface such that $S \cong \Sigma_h^2$ with $h \geq 2$ and such that both boundary components of S separate Σ_g^n . Let β be a boundary component of S . Then $[T_\beta]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)} = 0$.*

The proof requires the following lemma. Recall that we defined bounding pair maps in §2.2. Also, if γ is an oriented simple closed curve on S , write $[\gamma]$ for its homology class in $H_1(\Sigma_g^n; \mathbb{Z})$.

Lemma 6.3. *For $g \geq 2$ and $n \leq 1$, let $S \hookrightarrow \Sigma_g^n$ be a compact connected subsurface such that $S \cong \Sigma_h^2$ and such that both boundary components of S separate Σ_g^n . Let β be a boundary component of S and let $T_{x_1} T_{x_2}^{-1}$ and $T_{x_1'} T_{x_2'}^{-1}$ be Σ_g^n -bounding pair maps in $\mathcal{I}(S, \Sigma_g^n)$ that satisfy the following conditions.*

1. *For $1 \leq i \leq 2$, the 1-submanifolds $\beta \cup x_1 \cup x_2$ and $\beta \cup x_1' \cup x_2'$ of S each bound subsurfaces homeomorphic to $\Sigma_{0,3}$ (see Figure 11.a).*
2. *Orient x_1 and x_2 (resp. x_1' and x_2') so that β is in the component of S cut along $x_1 \cup x_2$ (resp. $x_1' \cup x_2'$) to the left of x_1 and x_2 (resp. x_1' and x_2'). Then $[x_1] = [x_1']$.*

Then $T_{x_1} T_{x_2}^{-1}$ and $T_{x_1'} T_{x_2'}^{-1}$ are conjugate elements of $\widehat{\mathcal{I}}(S, \Sigma_g^n)$.

This is similar to a result of Johnson [15, Theorem 1B] and can be proved in exactly the same way.

Remark. The statement of [15, Theorem 1B] contains an extra condition about a certain splitting of homology, but that condition is trivially satisfied in our situation.

Proof of Lemma 6.2. We will use the *lantern relation*, which is the relation

$$T_\beta = (T_{x_1} T_{x_2}^{-1})(T_{y_1} T_{y_2}^{-1})(T_{z_1} T_{z_2}^{-1}),$$

where the curves x_i and y_i and β are the curves on the 4-holed sphere Σ_0^4 shown in Figure 11.b (see [7, §7g]). Let $x_2, y_2, z_2 \subset S$ be the curves shown in Figure 11.c. The multicurve $\beta \cup x_2 \cup y_2 \cup z_2$ bounds a subsurface \mathcal{L} of S with $\mathcal{L} \cong \Sigma_0^4$. Inside this subsurface \mathcal{L} , we can find curves $x_1 \cup y_1 \cup z_1$ such that we have a lantern relation

$$T_\beta = (T_{x_1} T_{x_2}^{-1})(T_{y_1} T_{y_2}^{-1})(T_{z_1} T_{z_2}^{-1}). \quad (15)$$

We remark that the curves $x_1 \cup y_1 \cup z_1$ are not unique (they depend on a choice of homeomorphism $\mathcal{L} \cong \Sigma_0^4$), but any choice will work in our proof. Similarly, if $x'_2, y'_2, z'_2 \subset S$ are the curves in Figure 11.d, then we can find curves $x'_1 \cup y'_2 \cup z'_2$ such that we have a lantern relation

$$T_\beta = (T_{x'_1} T_{x'_2}^{-1})(T_{y'_1} T_{y'_2}^{-1})(T_{z'_1} T_{z'_2}^{-1}). \quad (16)$$

Orient x_1 and x_2 (resp. x'_1 and x'_2) so that β is in the component of S cut along $x_1 \cup x_2$ (resp. $x'_1 \cup x'_2$) to the left of x_1 and x_2 (resp. x'_1 and x'_2). This choice of orientation yields $[x_1] = -[x_2]$ and $[x'_1] = -[x'_2]$. As shown in Figure 11.c,d, we have $[x_2] = -[x'_2]$, so

$$[x_1] = -[x_2] = [x'_2].$$

Lemma 6.3 thus implies that $T_{x_1} T_{x_2}^{-1}$ is conjugate in $\widehat{\mathcal{I}}(S, \Sigma_g^n)$ to $T_{x'_2} T_{x'_1}^{-1}$ (notice that we are using $T_{x_2} T_{x_1}^{-1}$ here and not $T_{x'_1} T_{x'_2}^{-1}$), and hence $[T_{x_1} T_{x_2}^{-1}]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)} = -[T_{x'_1} T_{x'_2}^{-1}]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)}$. Similarly, we have $[T_{y_1} T_{y_2}^{-1}]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)} = -[T_{y'_1} T_{y'_2}^{-1}]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)}$ and $[T_{z_1} T_{z_2}^{-1}]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)} = -[T_{z'_1} T_{z'_2}^{-1}]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)}$. Combining these facts with (15) and (16), we deduce that

$$\begin{aligned} 2[T_\beta]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)} &= ([T_{x_1} T_{x_2}^{-1}]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)} + [T_{y_1} T_{y_2}^{-1}]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)} + [T_{z_1} T_{z_2}^{-1}]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)}) \\ &\quad + ([T_{x'_1} T_{x'_2}^{-1}]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)} + [T_{y'_1} T_{y'_2}^{-1}]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)} + [T_{z'_1} T_{z'_2}^{-1}]_{\widehat{\mathcal{I}}(S, \Sigma_g^n)}) = 0, \end{aligned}$$

as desired. \square

6.2 Separating twists are torsion, general case

In this section, we derive Lemma 6.1 from Lemma 6.2. This requires the following lemma, which is an abstraction of the main idea of the proof of [26, Theorem 1.1].

Lemma 6.4. *Let G be a group, let $G' < G$ be a finite index subgroup, and let $g \in G$ be a central element. Assume that $[g]_G = 0$ and that $g \in G'$. Then $[g]_{G'} = 0$.*

Proof. We can assume that g has infinite order, so $\langle g \rangle \cong \mathbb{Z}$. Since g is central, we have $\langle g \rangle \triangleleft G$. Define $\overline{G} = G/\langle g \rangle$ and $\overline{G}' = G'/\langle g \rangle$. We have a commutative diagram of central extensions

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & G' & \longrightarrow & \overline{G}' & \longrightarrow & 1 \\ & & \downarrow \cong & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & G & \longrightarrow & \overline{G} & \longrightarrow & 1 \end{array}$$

There is an associated map of 5-term exact sequences in rational group homology (see [5, Corollary VII.6.4]), which takes the form

$$\begin{array}{ccccccccc} \mathrm{H}_2(G'; \mathbb{Q}) & \longrightarrow & \mathrm{H}_2(\overline{G}'; \mathbb{Q}) & \xrightarrow{j'} & \mathbb{Q} & \xrightarrow{i'} & \mathrm{H}_1(G'; \mathbb{Q}) & \longrightarrow & \mathrm{H}_1(\overline{G}'; \mathbb{Q}) & \longrightarrow & 0 \\ \downarrow & & \downarrow \phi & & \downarrow \cong & & \downarrow & & \downarrow & & \\ \mathrm{H}_2(G; \mathbb{Q}) & \longrightarrow & \mathrm{H}_2(\overline{G}; \mathbb{Q}) & \xrightarrow{j} & \mathbb{Q} & \xrightarrow{i} & \mathrm{H}_1(G; \mathbb{Q}) & \longrightarrow & \mathrm{H}_1(\overline{G}; \mathbb{Q}) & \longrightarrow & 0 \end{array}$$

By assumption, the map i in this diagram is the 0 map, so j is surjective. Since \overline{G}' is a finite index subgroup of \overline{G} , we can make use of the transfer map in group homology (see [5, Chapter III.9]) to deduce that ϕ is surjective. We conclude that j' is surjective, so $i' = 0$, and the lemma follows. \square

Proof of Lemma 6.1. Since S has genus at least 3, we can find a subsurface S' of S such that $S' \cong \Sigma_2^2$ and such that δ is a boundary component of S' . Since δ separates Σ_g^n , the other boundary component of S' also separates Σ_g^n . Setting $\Gamma' = \Gamma \cap \widehat{\mathcal{I}}(S', \Sigma_g^n)$, it is enough to show that $[T_\delta]_{\Gamma'} = 0$. Lemma 6.2 implies that $[T_\delta]_{\widehat{\mathcal{I}}(S', \Sigma_g^n)} = 0$. Since T_δ is central in $\widehat{\mathcal{I}}(S', \Sigma_g^n)$ and Γ' is a finite-index subgroup of $\widehat{\mathcal{I}}(S', \Sigma_g^n)$, Lemma 6.4 implies that $[T_\delta]_{\mathcal{I}(\Sigma, S')} = 0$, as desired. \square

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Andrew Putman
Department of Mathematics
Rice University, MS 136
6100 Main St.
Houston, TX 77005
E-mail: andyp@rice.edu