

Representation stability and finite linear groups

Andrew Putman* Steven V Sam†

January 13, 2017

Abstract

We study analogues of FI-modules where the role of the symmetric group is played by the general linear groups and the symplectic groups over finite rings and prove basic structural properties such as local Noetherianity. Applications include a proof of the Lannes–Schwartz Artinian conjecture in the generic representation theory of finite fields, very general homological stability theorems with twisted coefficients for the general linear and symplectic groups over finite rings, and representation-theoretic versions of homological stability for congruence subgroups of the general linear group, the automorphism group of a free group, the symplectic group, and the mapping class group.

Contents

1 Introduction	1
2 Noetherian results	14
3 Partial resolutions and representation stability	28
4 Twisted homological stability	35
5 A machine for finite generation	36
6 Finite generation theorems	41

1 Introduction

The language of **representation stability** was introduced by Church–Farb [ChFa] to describe representation-theoretic patterns they observed in many parts of mathematics. In later work with Ellenberg [ChEFa], they introduced **FI-modules**, which gave a clearer framework for these patterns for the representation theory of the symmetric group. The key technical innovation of their work was a certain **Noetherian** property of FI-modules, which they later established in great generality with Nagpal [ChEFaNag]. This Noetherian property allowed them to prove an **asymptotic structure theorem** for finitely generated FI-modules, giving an elegant mechanism behind the patterns observed earlier in [ChFa]. The theory of FI-modules fits into the larger picture of functor categories, which play an important role in algebraic topology and algebraic K-theory; see [FraFriPiSch, FraTo] for surveys.

In this paper, we study functor categories that can be viewed as analogues of FI-modules where the role of the symmetric group is played by the classical finite groups of matrices, namely the general linear groups and the symplectic groups over finite rings. We establish analogues of the aforementioned Noetherian and asymptotic structure results. A byproduct of our work here is a proof of the Lannes–Schwartz Artinian conjecture (see Theorem B). We

*Supported in part by NSF grant DMS-1255350 and the Alfred P. Sloan Foundation

†Supported by a Miller research fellowship

have two families of applications. The first are very general homological stability theorems with twisted coefficients for the general linear groups and symplectic groups over finite rings. The second are representation-theoretic analogues of homological stability for congruence subgroups of the general linear group, the automorphism group of a free group, the symplectic group, and the mapping class group.

1.1 Review of FI-modules

Since our theorems parallel results about FI-modules, we begin by reviewing the theory of FI-modules. Unless otherwise specified, *all rings in this paper are commutative and contain 1*. Fix a Noetherian ring \mathbf{k} .

Configuration spaces. We start with an example. Let X be a compact oriented manifold with $\partial X \neq \emptyset$. Let

$$\text{PConf}_n(X) = \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

be the configuration space of n ordered points in X . The symmetric group \mathfrak{S}_n acts freely on $\text{PConf}_n(X)$ by permuting points. Let

$$\text{Conf}_n(X) = \text{PConf}_n(X)/\mathfrak{S}_n$$

be the configuration space of n unordered points in X . McDuff [Mc] proved that $\text{Conf}_n(X)$ satisfies **cohomological stability**: for $k \geq 1$, we have $H^k(\text{Conf}_n(X); \mathbf{k}) \cong H^k(\text{Conf}_{n+1}(X); \mathbf{k})$ for $n \gg 0$. For $\text{PConf}_n(X)$, easy examples show that cohomological stability does not hold. Observe that $H^k(\text{PConf}_n(X); \mathbf{k})$ is a representation of \mathfrak{S}_n . Building on work of Church [Ch1], Church–Ellenberg–Farb–Nagpal [ChEFaNag] proved that $H^k(\text{PConf}_n(X); \mathbf{k})$ stabilizes as a representation of \mathfrak{S}_n in a natural sense. The key to this is the theory of FI-modules.

Basic definitions. Let **FI** be the category of finite sets and injections. An **FI-module** over a ring \mathbf{k} is a functor $M: \mathbf{FI} \rightarrow \text{Mod}_{\mathbf{k}}$. More concretely, M consists of the following.

- For each finite set I , a \mathbf{k} -module M_I .
 - For each injection $f: I \hookrightarrow J$ between finite sets I, J , a homomorphism $M_f: M_I \rightarrow M_J$.
- These homomorphisms must satisfy the obvious compatibility conditions.

For $I \in \mathbf{FI}$, the FI-endomorphisms of I are exactly the symmetric group \mathfrak{S}_I , so \mathfrak{S}_I acts on M_I . In particular, setting $[n] = \{1, \dots, n\}$ and $M_n = M_{[n]}$, the group \mathfrak{S}_n acts on M_n .

Example 1.1. Given a finite set I (with the discrete topology), let $\text{Emb}(I, X)$ be the space of injective functions $I \rightarrow X$. There is an FI-module M with $M_I = H^k(\text{Emb}(I, X); \mathbf{k})$ for $I \in \mathbf{FI}$. Observe that $M_n = H^k(\text{PConf}_n(X); \mathbf{k})$. \square

General situation. More generally, let \mathbf{C} be any category. A **C-module** over a ring \mathbf{k} is a functor M from \mathbf{C} to $\text{Mod}_{\mathbf{k}}$ (see [FraFriPiSch, FraTo] for surveys). For each $C \in \mathbf{C}$, we thus have a \mathbf{k} -module M_C , and for each \mathbf{C} -morphism $f: C \rightarrow C'$ we have a \mathbf{k} -module morphism $M_f: M_C \rightarrow M_{C'}$. These morphisms satisfy compatibility conditions coming from the composition law in \mathbf{C} . The class of \mathbf{C} -modules over \mathbf{k} forms a category whose morphisms are natural transformations. A morphism $\phi: M \rightarrow N$ of \mathbf{C} -modules thus consists of \mathbf{k} -module morphisms $\phi_C: M_C \rightarrow N_C$ for each $C \in \mathbf{C}$ such that the diagram

$$\begin{array}{ccc} M_C & \xrightarrow{\phi_C} & N_C \\ M_f \downarrow & & \downarrow N_f \\ M_{C'} & \xrightarrow{\phi_{C'}} & N_{C'} \end{array}$$

commutes for all \mathcal{C} -morphisms $f: C \rightarrow C'$. We will often omit the \mathbf{k} and simply speak of \mathcal{C} -modules.

Notions such as \mathcal{C} -submodules, surjective/injective morphisms, etc. are defined pointwise. For example, a \mathcal{C} -module morphism $\phi: M \rightarrow N$ is surjective/injective if for all objects C , the linear map $\phi_C: M_C \rightarrow N_C$ is surjective/injective. A \mathcal{C} -submodule is the image of an injective morphism of \mathcal{C} -modules. A \mathcal{C} -module M is **finitely generated** if there exist $C_1, \dots, C_n \in \mathcal{C}$ and elements $x_i \in M_{C_i}$ such that the smallest \mathcal{C} -submodule of M containing the x_i is M itself. We will say that the category of \mathcal{C} -modules over \mathbf{k} is **locally Noetherian** if all submodules of finitely generated \mathcal{C} -modules over \mathbf{k} are finitely generated. We will say that the category of \mathcal{C} -modules is **locally Noetherian** if the category of \mathcal{C} -modules over \mathbf{k} is locally Noetherian for all Noetherian rings \mathbf{k} .

Asymptotic structure. Let M be a finitely generated FI-module over a Noetherian ring. Church–Ellenberg–Farb–Nagpal [ChEFaNag] proved that the category of FI-modules is locally Noetherian, and used this to prove three things about M . For $N \geq 0$, let FI^N be the full subcategory of FI spanned by $I \in \text{FI}$ with $|I| \leq N$.

- (Injective representation stability) If $f: I \rightarrow J$ is an FI-morphism, then the homomorphism $M_f: M_I \rightarrow M_J$ is injective when $|I| \gg 0$.
- (Surjective representation stability) If $f: I \rightarrow J$ is an FI-morphism, then the \mathfrak{S}_J -orbit of the image of $M_f: M_I \rightarrow M_J$ spans M_J when $|I| \gg 0$.
- (Central stability) For $N \gg 0$, the functor M is the left Kan extension to FI of the restriction of M to FI^N .

Informally, central stability says that for $I \in \text{FI}$ with $|I| > N$, the \mathbf{k} -module M_I can be constructed from $M_{I'}$ for $|I'| \leq N$ using the “obvious” generators and the “obvious” relations. It should be viewed as a sort of finite presentability condition.

Remark 1.2. The statement of central stability in [ChEFaNag] does not refer to Kan extensions, but is easily seen to be equivalent to the above. The original definition of central stability in the first author’s paper [Pu2] appears to be very different: it gives a presentation for M_I in terms of those $M_{I'}$ with $|I'| \in \{|I| - 1, |I| - 2\}$. One can show that this is equivalent to the above statement. We will prove a more general result during the proof of Theorem F below. \square

Conclusion. The upshot is that the locally Noetherian property implies that to describe how $H^k(\text{PConf}_n(X); \mathbf{k})$ changes as $n \rightarrow \infty$, it was enough for Church–Ellenberg–Farb–Nagpal to prove that the FI-module in Example 1.1 is finitely generated. Our goal is to generalize this kind of result to other settings.

Remark 1.3. In a different direction, Wilson [Wi] has generalized the theory of FI-modules to the other classical families of Weyl groups. \square

1.2 Introduction to the rest of the introduction

It will take some time to state our results, so to help orient the reader we give a brief outline of the rest of the introduction and provide a summary of our four main families of results. We begin in §1.3 by introducing the four main categories we study in this paper. They take as input a commutative ring R and are called $\text{VI}(R)$, $\text{V}(R)$, $\text{V}'(R)$, $\text{VIC}(R, \mathfrak{U})$, and $\text{SI}(R)$. This section states our first main family of theorems.

- **Theorems A/B/C/D/E:** When R is a finite commutative ring, the categories of $\text{VI}(R)$ -, $\text{V}(R)$ -, $\text{V}'(R)$ -, $\text{VIC}(R, \mathfrak{U})$ -, and $\text{SI}(R)$ -modules are locally Noetherian.

Remark 1.4. Our proofs of Theorems A–E use analogues of the theory of Gröbner bases. Such techniques are developed systematically by Sam–Snowden in [SamSn4]; however, the results of this paper can **not** be proved using the main result of [SamSn4]. See Remark 2.16 for more on this. \square

Remark 1.5. The condition that R is finite is necessary; indeed, in Theorem N we prove that none of the above categories of modules are locally Noetherian when R is infinite. \square

In §1.4, we introduce the language of complemented categories, which provides a common framework for stating and proving our results. Examples include FI and $\text{VIC}(R, \mathfrak{U})$ and $\text{SI}(R)$ (though not $\text{VI}(R)$ or $\text{V}(R)$ or $\text{V}'(R)$; we will explain how to handle these in Remark 1.18 below). This section states our second main theorem:

- **Theorem F:** If \mathcal{C} is a complemented category such that the category of \mathcal{C} -modules is locally Noetherian, then finitely generated \mathcal{C} -modules have a simple asymptotic structure analogous to Church–Farb’s theorem from the previous section.

The next section (§1.5) provides large numbers of examples of finitely generated \mathcal{C} -modules to which we can apply the previous result and obtain interesting asymptotic structure theorems. Letting \mathcal{O}_K be the ring of integers in an algebraic number field, the main theorems of this section are as follows:

- **Theorems G/H/I/J/K:** The homology groups of congruence subgroups in $\text{GL}_n(\mathcal{O}_K)$, $\text{SL}_n(\mathcal{O}_K)$, the automorphism group of a free group, $\text{Sp}_{2n}(\mathcal{O}_K)$, and the mapping class group form finitely generated modules over appropriate categories.

Combining these theorems with Theorem F, we obtain a version of representation stability for these homology groups.

Remark 1.6. The proofs of Theorems G–K all use a machine we develop later in this paper. The key input to this machine is an appropriate local Noetherianity theorem (like Theorems A–E above). See Theorem 5.13 for a statement of how this machine works. \square

The final section (§1.6) of our introduction concerns twisted homological stability. Its main theorems can be summarized as follows:

- **Theorems L/M:** If R is a finite commutative ring, then the twisted homology groups of $\text{GL}_n(R)$, $\text{SL}_n(R)$, and $\text{Sp}_{2n}(R)$ with coefficients in a finitely generated module over an appropriate category stabilize.

Remark 1.7. The proofs of Theorems L and M both use another machine we develop later in this paper to prove twisted homological stability theorems from appropriate local Noetherianity theorems. See Theorem 4.2 for a statement of how this machine works. \square

Remark 1.8. Some readers may notice a parallel between Theorems 5.13 and 4.2 and a program outlined independently by Church (in lectures and personal communication, but never published). Church’s program had similar inputs and would have obtained these same conclusions (finite generation for homology of congruence subgroups, and stability for twisted homology), but with one key difference: if successful, it would have yielded effective quantitative bounds (on the degree of generators in the former case, and on the stable range in the latter). However, Church informs us that technical difficulties arose that mean his proposed program cannot be carried out, and so it will not appear. He points out that, although the results in this paper were obtained independently, they can be viewed after the fact as a demonstration that the framework of his approach can be salvaged by using our local Noetherianity theorems to avoid these technical difficulties, at the cost of giving up effective bounds. \square

The introduction closes with some comments and questions together with an outline of the remainder of the paper. This last section also contains Theorem N, which is the non-Noetherian result discussed in Remark 1.5.

1.3 Categories of free modules: VI, V, V', VIC, and SI

We now introduce the categories we will consider in this paper. Fix a ring R (assumed as always to be commutative).

The category VI. A linear map between free R -modules is **splittable** if it has a left inverse. Let $\mathbf{VI}(R)$ be the category whose objects are finite-rank free R -modules and whose morphisms are splittable injections. For $V \in \mathbf{VI}(R)$, the monoid of $\mathbf{VI}(R)$ -endomorphisms of V is $\mathrm{GL}(V)$. Thus if M is a $\mathbf{VI}(R)$ -module, then M_V is a representation of $\mathrm{GL}(V)$ for all $V \in \mathbf{VI}(R)$. The category $\mathbf{VI}(R)$ was introduced by Scorichenko in his thesis ([Sco]; see [FraFriPiSch] for a published account). Our main theorem about $\mathbf{VI}(R)$ is as follows.

Theorem A. *Let R be a finite ring. Then the category of $\mathbf{VI}(R)$ -modules is locally Noetherian.*

Remark 1.9. In contrast, $\mathbf{VI}(\mathbb{Z})$ is not locally Noetherian, and in fact $\mathbf{VI}(R)$ is not locally Noetherian whenever R is an infinite ring; see Theorem N below. \square

Remark 1.10. While preparing this article, we learned that Gan and Li [GanLi] have recently independently shown that the category of functors $\mathbf{VI}(R) \rightarrow \mathrm{Mod}_{\mathbf{k}}$ is locally Noetherian when R is a finite field and \mathbf{k} is a field of characteristic 0. For our applications, it is important that both R and especially \mathbf{k} be more general. \square

A variant: the Artinian conjecture. Define $\mathbf{V}(R)$ to be the category whose objects are finite-rank free R -modules and whose morphisms are all splittable linear maps (not necessarily injective). We then have the following.

Theorem B (Artinian conjecture). *Let R be a finite ring. Then the category of $\mathbf{V}(R)$ -modules is locally Noetherian.*

Theorem B generalizes an old conjecture of Jean Lannes and Lionel Schwartz which asserts that the category of $\mathbf{V}(\mathbb{F}_q)$ -modules over \mathbb{F}_q is locally Noetherian. This is often stated in a dual form and is thus known as the Artinian conjecture. It first appeared in print in [Ku1, Conjecture 3.12] and now has a large literature containing many partial results (mostly focused on the case $q = 2$, though even there nothing close to the full conjecture was previously known). See, e.g., [Ku2, Po1, Po2, Po3, Dj1, Dj2]. See also [Sch, Chapter 5] for connections to the Steenrod algebra. A proof of this conjecture also appears in [SamSn4], see Remark 2.17.

A variant of $\mathbf{V}(R)$ is the category $\mathbf{V}'(R)$ whose objects are finite-rank free R -modules and whose morphisms are all linear maps (not just the splittable ones). Of course, $\mathbf{V}(R) = \mathbf{V}'(R)$ if R is a field, but they differ in general. For $\mathbf{V}'(R)$, we have the following theorem.

Theorem C. *Let R be a finite ring. Then the category of $\mathbf{V}'(R)$ -modules is locally Noetherian.*

Problems with VI. The category $\mathbf{VI}(R)$ is not rich enough to allow many natural constructions. For instance, the map $V \mapsto \mathrm{GL}(V)$ does not extend to a functor from $\mathbf{VI}(R)$ to the category of groups. The problem is that if V is a submodule of W , then there is no canonical way to extend an automorphism of V to an automorphism of W . We overcome this problem by introducing a richer category.

The category VIC, take I. Define $\text{VIC}(R)$ to be the following category. The objects of $\text{VIC}(R)$ are finite-rank free R -modules V . If $V, W \in \text{VIC}(R)$, then a $\text{VIC}(R)$ -morphism from V to W consists of a pair (f, C) as follows.

- $f: V \rightarrow W$ is an R -linear injection.
- $C \subset W$ is a free R -submodule such that $W = f(V) \oplus C$.

If $(f, C): V \rightarrow W$ and $(g, D): W \rightarrow X$ are two $\text{VIC}(R)$ -morphisms, then the composition $(g, D) \circ (f, C)$ equals $(g \circ f, g(C) \oplus D)$.

The monoid of $\text{VIC}(R)$ -endomorphisms of $V \in \text{VIC}(R)$ is $\text{GL}(V)$. Observe that in contrast to $\text{VI}(R)$, the assignment $V \mapsto \text{GL}(V)$ extends to a functor from $\text{VIC}(R)$ to the category of groups: if $(f, C): V \rightarrow W$ is a morphism, then there is an induced map $\text{GL}(V) \rightarrow \text{GL}(W)$ which extends automorphisms over C by the identity, and this assignment is functorial. The category $\text{VIC}(R)$ was first introduced by Djament [Dj3].

Remark 1.11. We will sometimes use an alternative description of the morphisms in $\text{VIC}(R)$ which works whenever all stably free R -modules are free (for instance, when R is a finite commutative ring). Namely, the data (f, C) which defines a morphism $V \rightarrow W$ is equivalent to a pair of R -linear maps $V \rightarrow W \rightarrow V$ whose composition is the identity. Here the first map is f and the second map is the projection $W \rightarrow V$ away from C . \square

The category VIC, take II (cheap). Let $\mathfrak{U} \subset R^\times$ be a subgroup of the group of multiplicative units of R and let $\text{SL}^\mathfrak{U}(V) = \{f \in \text{GL}(V) \mid \det(f) \in \mathfrak{U}\}$. For our applications, we will need a variant of $\text{VIC}(R)$ whose automorphism groups are $\text{SL}^\mathfrak{U}(V)$ rather than $\text{GL}(V)$. The category $\text{VIC}(R)$ is equivalent to the full subcategory generated by $\{R^n \mid n \geq 0\}$. Define $\text{VIC}(R, \mathfrak{U})$ to be the category whose objects are $\{R^n \mid n \geq 0\}$ and whose morphisms from R^n to $R^{n'}$ are $\text{VIC}(R)$ -morphisms $(f, C): R^n \rightarrow R^{n'}$ such that $\det(f) \in \mathfrak{U}$ if $n = n'$. Thus the monoid of $\text{VIC}(R, \mathfrak{U})$ -endomorphisms of R^n is $\text{SL}_n^\mathfrak{U}(R) := \text{SL}^\mathfrak{U}(R^n)$.

Orientations. For many of our constructions, it will be useful to extend $\text{VIC}(R, \mathfrak{U})$ to a category whose objects are all finite-rank free R -modules rather than just the R^n . An **orientation** of a rank n free R -module V is a generator for the rank 1 free R -module $\wedge^n V$. The group of units of R acts transitively on the set of orientations on V , and a **\mathfrak{U} -orientation** on V consists of an orbit under the action of \mathfrak{U} . A **\mathfrak{U} -oriented free R -module** is a finite-rank free R module V equipped with a \mathfrak{U} -orientation. If W_1 and W_2 are \mathfrak{U} -oriented free R -modules, then $W_1 \oplus W_2$ is in a natural way a \mathfrak{U} -oriented free R -module. A **\mathfrak{U} -oriented free R -submodule** of V is an R -submodule V' of V which is a free R -module equipped with a \mathfrak{U} -orientation; if $V' = V$, then the \mathfrak{U} -orientations on V' and V must be the same, and if $V' \subsetneq V$ then there is no condition on the \mathfrak{U} -orientation of V' .

The category VIC, take II (better). The category $\text{VIC}(R, \mathfrak{U})$ can now be defined as follows (this yields a category equivalent to the previous one). The objects of $\text{VIC}(R, \mathfrak{U})$ are \mathfrak{U} -oriented free R -modules V . If $V, W \in \text{VIC}(R, \mathfrak{U})$, then a morphism of $\text{VIC}(R, \mathfrak{U})$ from V to W consists of a pair (f, C) as follows.

- $f: V \rightarrow W$ is an R -linear injection.
- $C \subset W$ is a \mathfrak{U} -oriented free R -submodule such that $W = f(V) \oplus C$.

In the second bullet point, $f(V)$ is equipped with the \mathfrak{U} -orientation $f_*(\omega)$, where ω is the \mathfrak{U} -orientation on V , and the direct sum decomposition $f(V) \oplus C$ is as \mathfrak{U} -oriented free R -modules. Note that when C is nonzero, there is a unique \mathfrak{U} -orientation on C such that the \mathfrak{U} -orientations on $f(V) \oplus C$ and W are the same, so the morphisms in $\text{VIC}(R, \mathfrak{U})$ and $\text{VIC}(R)$ are only different when $\dim(V) = \dim(W)$. The monoid of $\text{VIC}(R, \mathfrak{U})$ -endomorphisms of $V \in \text{VIC}(R, \mathfrak{U})$ is $\text{SL}^\mathfrak{U}(V)$. Observe that $\text{VIC}(R, \mathfrak{U}) \subseteq \text{VIC}(R)$ with equality when \mathfrak{U} is the

entire group of units. Our main theorem about $\text{VIC}(R, \mathfrak{U})$ is as follows. For $\text{VIC}(\mathbb{F})$ with \mathbb{F} a finite field, this theorem was conjectured by Church [Ch2].

Theorem D. *Let R be a finite ring and let $\mathfrak{U} \subset R$ be a subgroup of the group of units. Then the category of $\text{VIC}(R, \mathfrak{U})$ -modules is locally Noetherian.*

Remark 1.12. Just like for $\text{VI}(R)$, it is necessary for R to be finite; see Theorem N. \square

Symplectic forms. We now turn to the symplectic analogue of $\text{VI}(R)$ -modules. A **symplectic form** on a finite-rank free R -module V is a bilinear form $\hat{i}: V \times V \rightarrow R$ which is alternating (i.e., $\hat{i}(v, v) = 0$ for all $v \in V$) and which induces an isomorphism from V to its dual $V^* = \text{Hom}_R(V, R)$. A **symplectic R -module** is a finite-rank free R -module equipped with a symplectic form. A **symplectic R -submodule** of a symplectic R -module V is an R -submodule W such that the symplectic form on V restricts to a symplectic form on W . A **symplectic map** between symplectic R -modules is an R -linear map that preserves the symplectic form. Observe that symplectic maps must be injective.

The category SI . Define $\text{SI}(R)$ to be the category whose objects are symplectic R -modules and whose morphisms are symplectic maps. The monoid of $\text{SI}(R)$ -endomorphisms of $V \in \text{SI}(R)$ is $\text{Sp}(V)$, i.e., the group of R -linear isomorphisms of V that preserve the symplectic form. Just like for $\text{VIC}(R)$, the map $V \mapsto \text{Sp}(V)$ extends to a functor from $\text{SI}(R)$ to the category of groups: a symplectic map $f: V \rightarrow W$ induces a map $\text{Sp}(V) \rightarrow \text{Sp}(W)$ which extends automorphisms over $f(V)^\perp$ by the identity. We then have the following.

Theorem E. *Let R be a finite ring. Then the category of $\text{SI}(R)$ -modules is locally Noetherian.*

Remark 1.13. Just like for $\text{VI}(R)$, it is necessary for R to be finite; see Theorem N. \square

1.4 Complemented categories and their asymptotic structure

Our next result says that finitely generated modules over the above categories have a simple asymptotic structure (like the one for FI -modules we discussed earlier). To avoid having to give separate proofs for all our categories, we introduce an abstract framework.

Complemented categories. A **weak complemented category** is a monoidal category (\mathbf{A}, \otimes) satisfying:

- Every morphism in \mathbf{A} is a monomorphism. Thus for all morphisms $f: V \rightarrow V'$, it makes sense to talk about the subobject $f(V)$ of V' .
- The identity object $\mathbb{1}$ of (\mathbf{A}, \otimes) is initial. Thus for $V, V' \in \mathbf{A}$ there exist natural morphisms $V \rightarrow V \otimes V'$ and $V' \rightarrow V \otimes V'$, namely the compositions

$$V \xrightarrow{\cong} V \otimes \mathbb{1} \longrightarrow V \otimes V' \quad \text{and} \quad V' \xrightarrow{\cong} \mathbb{1} \otimes V' \longrightarrow V \otimes V'.$$

We will call these the **canonical morphisms**.

- For $V, V', W \in \mathbf{A}$, the map $\text{Hom}_{\mathbf{A}}(V \otimes V', W) \rightarrow \text{Hom}_{\mathbf{A}}(V, W) \times \text{Hom}_{\mathbf{A}}(V', W)$ obtained by composing morphisms with the canonical morphisms is an injection.
- Every subobject C of an object V has a unique **complement**, that is, a unique subobject D of V such that there is an isomorphism $C \otimes D \xrightarrow{\cong} V$ where the compositions $C \rightarrow C \otimes D \xrightarrow{\cong} V$ and $D \rightarrow C \otimes D \xrightarrow{\cong} V$ of the isomorphism with the canonical morphisms are the inclusion morphisms.
- The canonical map $\mathfrak{S}_p \wr \text{Aut}(X) \rightarrow \text{Aut}(X^p)$ is injective. (Here \wr denotes wreath product.)

A **complemented category** is a weak complemented category (\mathbf{A}, \otimes) whose monoidal structure is symmetric (more precisely, which is equipped with a symmetry).

Remark 1.14. Things like complemented categories go back to work of Quillen; see [Gr, p. 3]. After a version of this paper was distributed, Nathalie Wahl (and later with Oscar Randal-Williams) posted her paper [WahRW], which uses an axiomatization similar to complemented categories to prove homological stability results of a more classical flavor than ours. A related axiomatization also appears in work of Djament–Vespa [DjVe1]. \square

Generators for a category. If (\mathbf{A}, \otimes) is a monoidal category, then a **generator** for \mathbf{A} is an object X of \mathbf{A} such that all objects V of \mathbf{A} are isomorphic to X^i for some unique $i \geq 0$. We will call i the **X -rank** of V .

Example 1.15. The category \mathbf{FI} of finite sets and injections is a complemented category whose monoidal structure is the disjoint union. A generator is the set $\{1\}$. \square

Example 1.16. The category $\mathbf{VIC}(R)$ is a complemented category whose monoidal structure is the direct sum. The subobject attached to a morphism $V \rightarrow V'$ can be interpreted as the pair (W, W') where W is the image of the linear injection $V \rightarrow V'$ and W' is the chosen complement. The complement to this subobject is the pair (W', W) . A generator is R^1 . \square

Example 1.17. The category $\mathbf{SI}(R)$ is a complemented category whose monoidal structure is the orthogonal direct sum. The complement of a symplectic submodule W of a symplectic R -module (V, \hat{i}) is $W^\perp = \{v \in V \mid \hat{i}(v, w) = 0 \text{ for all } w \in W\}$. A generator is R^2 equipped with the standard symplectic form. \square

Remark 1.18. The category $\mathbf{VI}(R)$ is not a complemented category. However, there is a forgetful functor $\Psi: \mathbf{VIC}(R) \rightarrow \mathbf{VI}(R)$. If R is a finite ring and M is a finitely generated $\mathbf{VI}(R)$ -module, then the pullback $\Psi^*(M)$ is a finitely generated $\mathbf{VIC}(R)$ -module (see the proof of Theorem A in §2.4) which can be analyzed using the technology we will discuss below. A similar remark applies to $\mathbf{V}(R)$ and $\mathbf{V}'(R)$. \square

Asymptotic structure theorem. The following theorem generalizes the aforementioned asymptotic structure theorem for \mathbf{FI} -modules.

Theorem F. *Let (\mathbf{A}, \otimes) be a complemented category with generator X . Assume that the category of \mathbf{A} -modules is locally Noetherian, and let M be a finitely generated \mathbf{A} -module. Then the following hold. For $N \geq 0$, let \mathbf{A}^N denote the full subcategory of \mathbf{A} spanned by elements whose X -ranks are at most N .*

- *(Injective representation stability) If $f: V \rightarrow W$ is an \mathbf{A} -morphism, then the homomorphism $M_f: M_V \rightarrow M_W$ is injective when the X -rank of V is sufficiently large.*
- *(Surjective representation stability) If $f: V \rightarrow W$ is an \mathbf{A} -morphism, then the orbit under $\text{Aut}_{\mathbf{A}}(W)$ of the image of $M_f: M_V \rightarrow M_W$ spans M_W when the X -rank of V is sufficiently large.*
- *(Central stability) For $N \gg 0$, the functor M is the left Kan extension to \mathbf{A} of the restriction of M to \mathbf{A}^N .*

Remark 1.19. See Theorem 3.7 below for a concrete description of the presentation for M implied by central stability; the proof of Theorem F immediately follows the proof of Theorem 3.7. \square

1.5 Representation stability for congruence subgroups

Our next theorems apply the above technology to prove representation-theoretic analogues of homological stability for certain kinds of congruence subgroups.

GL over number rings. Let \mathcal{O}_K be the ring of integers in an algebraic number field K . Charney [Cha1] proved that $\mathrm{GL}_n(\mathcal{O}_K)$ satisfies homological stability. Rationally, these stable values were computed by Borel [Bo]. Let $\alpha \subset \mathcal{O}_K$ be a proper nonzero ideal. The **level α congruence subgroup** of $\mathrm{GL}_n(\mathcal{O}_K)$, denoted $\mathrm{GL}_n(\mathcal{O}_K, \alpha)$, is the kernel of the map $\mathrm{GL}_n(\mathcal{O}_K) \rightarrow \mathrm{GL}_n(\mathcal{O}_K/\alpha)$. It follows from Borel’s work [Bo] that $\mathrm{H}_k(\mathrm{GL}_n(\mathcal{O}_K, \alpha); \mathbf{k})$ stabilizes when $\mathbf{k} = \mathbb{Q}$; however, Lee–Szczarba [LeSz] proved that it does not stabilize integrally, even for $k = 1$. Few of these homology groups are known, and the existing computations do not seem to fit into any pattern. However, they have an important additional piece of structure: for each $n \geq 1$, the conjugation action of $\mathrm{GL}_n(\mathcal{O}_K)$ on $\mathrm{GL}_n(\mathcal{O}_K, \alpha)$ descends to an action of $\mathrm{SL}_n^{\mathfrak{U}}(\mathcal{O}_K/\alpha)$ on $\mathrm{H}_k(\mathrm{GL}_n(\mathcal{O}_K, \alpha); \mathbf{k})$, where $\mathfrak{U} \subset \mathcal{O}_K/\alpha$ is the image of the group of units of \mathcal{O}_K . Our goal is to describe how this representation changes as n increases.

Congruence subgroups of GL as VIC-modules. As we discussed above, there is a functor from $\mathrm{VIC}(\mathcal{O}_K)$ to the category of groups that takes V to $\mathrm{GL}(V)$. Similarly, there is a functor $\mathrm{GL}(\mathcal{O}_K, \alpha)$ from $\mathrm{VIC}(\mathcal{O}_K)$ to the category of groups defined by the formula

$$\mathrm{GL}(\mathcal{O}_K, \alpha)_V = \ker(\mathrm{GL}(V) \rightarrow \mathrm{GL}(V \otimes_{\mathcal{O}_K} (\mathcal{O}_K/\alpha))).$$

Passing to homology, we get a $\mathrm{VIC}(\mathcal{O}_K)$ -module $\mathcal{H}_k(\mathrm{GL}(\mathcal{O}_K, \alpha); \mathbf{k})$ with

$$\mathcal{H}_k(\mathrm{GL}(\mathcal{O}_K, \alpha); \mathbf{k})_V = \mathrm{H}_k(\mathrm{GL}(\mathcal{O}_K, \alpha)_V; \mathbf{k}) \quad (V \in \mathrm{VIC}(\mathcal{O}_K)).$$

Since \mathcal{O}_K is infinite, the category of $\mathrm{VIC}(\mathcal{O}_K)$ -modules is poorly behaved. However, we will prove in §6.1 that $\mathcal{H}_k(\mathrm{GL}(\mathcal{O}_K, \alpha); \mathbf{k})$ actually descends to a $\mathrm{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$ -module, where $\mathfrak{U} \subset \mathcal{O}_K/\alpha$ is the image of the group of units of \mathcal{O}_K . Since \mathcal{O}_K/α is finite, our theorems apply to $\mathrm{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$. All that is necessary is to prove that the $\mathrm{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$ -module $\mathcal{H}_k(\mathrm{GL}(\mathcal{O}_K, \alpha); \mathbf{k})$ is finitely generated.

Relation to FI-modules. It turns out that the needed finite generation follows from known results. The conjugation action of the set of permutation matrices on $\mathrm{GL}_n(\mathcal{O}_K, \alpha)$ turns $\mathrm{H}_k(\mathrm{GL}_n(\mathcal{O}_K, \alpha); \mathbf{k})$ into a representation of \mathfrak{S}_n . In [Pu2], the first author proved that if \mathbf{k} is a field whose characteristic is sufficiently large, then the \mathfrak{S}_n -representations $\mathrm{H}_k(\mathrm{GL}_n(\mathcal{O}_K, \alpha); \mathbf{k})$ satisfy a version of central stability. This was generalized to arbitrary Noetherian rings \mathbf{k} by Church–Ellenberg–Farb–Nagpal [ChEFaNag], who actually proved that the groups $\mathrm{H}_k(\mathrm{GL}_n(\mathcal{O}_K, \alpha); \mathbf{k})$ form a finitely generated FI-module. More precisely, the following defines an embedding of categories $\Phi: \mathrm{FI} \rightarrow \mathrm{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$.

- For $I \in \mathrm{FI}$, set $\Phi(I) = (\mathcal{O}_K/\alpha)^I$.
- For an FI-morphism $\sigma: I \hookrightarrow J$, set $\Phi(\sigma) = (f_\sigma, C_\sigma)$, where $f_\sigma: (\mathcal{O}_K/\alpha)^I \rightarrow (\mathcal{O}_K/\alpha)^J$ is defined by $f_\sigma(\mathbf{b}_i) = \mathbf{b}_{\sigma(i)}$ for $i \in I$ and $C_\sigma = \langle \mathbf{b}_j \mid j \notin \sigma(I) \rangle$. Here $\{\mathbf{b}_i \mid i \in I\}$ and $\{\mathbf{b}_j \mid j \in J\}$ are the standard bases for $(\mathcal{O}_K/\alpha)^I$ and $(\mathcal{O}_K/\alpha)^J$, respectively.

Then [ChEFaNag] proves that the restriction of $\mathrm{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$ to $\Phi(\mathrm{FI}) \subset \mathrm{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$ is finitely generated, so the $\mathrm{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$ -module $\mathcal{H}_k(\mathrm{GL}(\mathcal{O}_K, \alpha); \mathbf{k})$ is finitely generated:

Theorem G. *Let \mathcal{O}_K be the ring of integers in an algebraic number field K , let $\alpha \subset \mathcal{O}_K$ be a proper nonzero ideal, let $\mathfrak{U} \subset \mathcal{O}_K/\alpha$ be the image of the group of units of \mathcal{O}_K , and let \mathbf{k} be a Noetherian ring. Then the $\mathrm{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$ -module $\mathcal{H}_k(\mathrm{GL}(\mathcal{O}_K, \alpha); \mathbf{k})$ is finitely generated for all $k \geq 0$.*

We can therefore apply Theorem F to $\mathcal{H}_k(\mathrm{GL}(\mathcal{O}_K, \alpha); \mathbf{k})$. The result is an asymptotic structure theorem for $\mathrm{H}_k(\mathrm{GL}_n(\mathcal{O}_K, \alpha); \mathbf{k})$. This partially verifies a conjecture of Church–Farb [ChFa, Conjecture 8.2.1]. As a simple illustration of our machinery, we will give a direct proof of Theorem G in this paper.

Remark 1.20. Church–Ellenberg–Farb–Nagpal’s theorem gives a central stability description of $\mathrm{H}_k(\mathrm{GL}_n(\mathcal{O}_K, \alpha); \mathbf{k})$ as an FI-module; however, the central stability conclusion from Theorem F gives more information than this since it deals with $\mathrm{H}_k(\mathrm{GL}_n(\mathcal{O}_K, \alpha); \mathbf{k})$ as a representation of $\mathrm{SL}_n^{\mathfrak{U}}(\mathcal{O}_K/\alpha)$ and not merely as a representation of \mathfrak{S}_n . \square

Remark 1.21. The other examples that we discuss also form FI-modules, but it seems very hard to prove that they are finitely generated as FI-modules. \square

Remark 1.22. One could also consider congruence subgroups of $\mathrm{SL}_n(R)$ for other rings R . In fact, the first author’s results in [Pu2] work in this level of generality; however, we have to restrict ourselves to rings of integers so we can cite a theorem of Borel–Serre [BoSe] which asserts that the homology groups in question are all finitely generated \mathbf{k} -modules. This is the only property of rings of integers we use, and our methods give a similar result whenever this holds (another example would be R a finite commutative ring). \square

Variante: SL. We can define $\mathrm{SL}_n(\mathcal{O}_K, \alpha) \subset \mathrm{SL}_n(\mathcal{O}_K)$ in the obvious way. The homology groups $\mathrm{H}_k(\mathrm{SL}_n(\mathcal{O}_K, \alpha); \mathbf{k})$ are now representations of $\mathrm{SL}_n(\mathcal{O}_K/\alpha) = \mathrm{SL}_n^1(\mathcal{O}_K/\alpha)$. Similarly to $\mathrm{GL}_n(\mathcal{O}_K, \alpha)$, we will construct a $\mathrm{VIC}(\mathcal{O}_K/\alpha, 1)$ -module $\mathcal{H}_k(\mathrm{SL}(\mathcal{O}_K, \alpha); \mathbf{k})$ with

$$\mathcal{H}_k(\mathrm{SL}(\mathcal{O}_K, \alpha); \mathbf{k})_{\mathcal{O}_K^n} = \mathrm{H}_k(\mathrm{SL}_n(\mathcal{O}_K, \alpha); \mathbf{k}) \quad (n \geq 0)$$

and prove the following variant of Theorem G.

Theorem H. *Let \mathcal{O}_K be the ring of integers in an algebraic number field K , let $\alpha \subset \mathcal{O}_K$ be a proper nonzero ideal, and let \mathbf{k} be a Noetherian ring. Then the $\mathrm{VIC}(\mathcal{O}_K/\alpha, 1)$ -module $\mathcal{H}_k(\mathrm{SL}(\mathcal{O}_K, \alpha); \mathbf{k})$ is finitely generated for all $k \geq 0$.*

Remark 1.23. Just like for Theorem G, this could be deduced from work of Church–Ellenberg–Farb–Nagpal [ChEFaNag], though we will give a proof using our machinery. \square

Automorphism groups of free groups. Our second example of a $\mathrm{VIC}(R, \mathfrak{U})$ -module comes from the automorphism group $\mathrm{Aut}(F_n)$ of the free group F_n of rank n . Just like for $\mathrm{GL}_n(\mathcal{O}_K)$, work of Hatcher [Hat] and Hatcher–Vogtmann [HatVol] shows that $\mathrm{Aut}(F_n)$ satisfies homological stability. These stable values were computed by Galatius [Ga], who proved in particular that they rationally vanish. For $\ell \geq 2$, the **level ℓ congruence subgroup** of $\mathrm{Aut}(F_n)$, denoted $\mathrm{Aut}(F_n, \ell)$, is the kernel of the map $\mathrm{Aut}(F_n) \rightarrow \mathrm{SL}_n^{\pm 1}(\mathbb{Z}/\ell)$ arising from the action of $\mathrm{Aut}(F_n)$ on $\mathrm{H}_1(F_n; \mathbb{Z}/\ell)$. Satoh [Sat] proved that $\mathrm{H}_1(\mathrm{Aut}(F_n, \ell); \mathbb{Z})$ does not stabilize. Similarly to $\mathrm{SL}_n(\mathcal{O}_K, \alpha)$, few concrete computations are known.

Remark 1.24. Unlike for $\mathrm{SL}_n(\mathcal{O}_K, \alpha)$ it is not known in general if $\mathrm{H}_k(\mathrm{Aut}(F_n, \ell); \mathbb{Q})$ stabilizes, though this does hold for $k \leq 2$ (see [DPu]). \square

The conjugation action of $\mathrm{Aut}(F_n)$ on $\mathrm{Aut}(F_n, \ell)$ descends to an action of $\mathrm{SL}_n^{\pm 1}(\mathbb{Z}/\ell)$ on $\mathrm{H}_k(\mathrm{Aut}(F_n, \ell); \mathbf{k})$. In §6.2, we will construct a $\mathrm{VIC}(\mathbb{Z}/\ell, \pm 1)$ -module $\mathcal{H}_k(\mathrm{Aut}(\ell); \mathbf{k})$ with

$$\mathcal{H}_k(\mathrm{Aut}(\ell); \mathbf{k})_{(\mathbb{Z}/\ell)^n} = \mathrm{H}_k(\mathrm{Aut}(F_n, \ell); \mathbf{k}) \quad (n \geq 0).$$

Our main theorem about this module is as follows.

Theorem I. Fix some $\ell \geq 2$, and let \mathbf{k} be a Noetherian ring. Then the $\mathrm{VIC}(\mathbb{Z}/\ell, \pm 1)$ -module $\mathcal{H}_k(\mathrm{Aut}(\ell); \mathbf{k})$ is finitely generated for all $k \geq 0$.

Applying Theorem F, we get an asymptotic structure theorem for $\mathrm{H}_k(\mathrm{Aut}(F_n, \ell); \mathbf{k})$, partially verifying a conjecture of Church–Farb (see the sentence after [ChFa, Conjecture 8.5]).

Remark 1.25. The kernel of $\mathrm{Aut}(F_n) \rightarrow \mathrm{GL}_n(\mathbb{Z})$ is denoted IA_n and its homology forms a module over $\mathrm{VIC}(\mathbb{Z})$. It would be interesting to understand this module; however, this seems to be a very difficult problem. Our techniques cannot handle this example because \mathbb{Z} is an infinite ring (c.f. Theorem N). \square

Symplectic groups over number rings. Let \mathcal{O}_K and K and $\alpha \subset \mathcal{O}_K$ be as above. The **level α congruence subgroup** of $\mathrm{Sp}_{2n}(\mathcal{O}_K)$, denoted $\mathrm{Sp}_{2n}(\mathcal{O}_K, \alpha)$, is the kernel of the map $\mathrm{Sp}_{2n}(\mathcal{O}_K) \rightarrow \mathrm{Sp}_{2n}(\mathcal{O}_K/\alpha)$. The groups $\mathrm{Sp}_{2n}(\mathcal{O}_K)$ and $\mathrm{Sp}_{2n}(\mathcal{O}_K, \alpha)$ are similar to $\mathrm{GL}_n(\mathcal{O}_K)$ and $\mathrm{GL}_n(\mathcal{O}_K, \alpha)$. In particular, the following hold.

- Charney [Cha3] proved that $\mathrm{Sp}_{2n}(\mathcal{O}_K)$ satisfies homological stability.
- Borel [Bo] calculated the stable rational homology of $\mathrm{Sp}_{2n}(\mathcal{O}_K)$. He also proved that $\mathrm{H}_k(\mathrm{Sp}_{2n}(\mathcal{O}_K, \alpha); \mathbf{k})$ stabilizes when $\mathbf{k} = \mathbb{Q}$.
- For general \mathbf{k} , an argument similar to the one that Lee–Szczarba [LeSz] used for $\mathrm{SL}_n(\mathcal{O}_K, \alpha)$ shows that $\mathrm{H}_k(\mathrm{Sp}_{2n}(\mathcal{O}_K, \alpha); \mathbf{k})$ does not stabilize, even for $k = 1$.
- The conjugation action of $\mathrm{Sp}_{2n}(\mathcal{O}_K)$ on $\mathrm{Sp}_{2n}(\mathcal{O}_K, \alpha)$ induces an action of $\mathrm{Sp}_{2n}(\mathcal{O}_K/\alpha)$ on $\mathrm{H}_k(\mathrm{Sp}_{2n}(\mathcal{O}_K, \alpha); \mathbf{k})$.

In §6.3, we will construct an $\mathrm{SI}(\mathcal{O}_K/\alpha)$ -module $\mathcal{H}_k(\mathrm{Sp}(\mathcal{O}_K, \alpha); \mathbf{k})$ such that

$$\mathcal{H}_k(\mathrm{Sp}(\mathcal{O}_K, \alpha); \mathbf{k})_{(\mathcal{O}_K/\alpha)^{2n}} = \mathrm{H}_k(\mathrm{Sp}_{2n}(\mathcal{O}_K, \alpha); \mathbf{k}) \quad (n \geq 0).$$

Our main result concerning this functor is as follows.

Theorem J. Let \mathcal{O}_K be the ring of integers in an algebraic number field K , let $\alpha \subset \mathcal{O}_K$ be a proper nonzero ideal, and let \mathbf{k} be a Noetherian ring. Then the $\mathrm{SI}(\mathcal{O}_K/\alpha)$ -module $\mathcal{H}_k(\mathrm{Sp}(\mathcal{O}_K, \alpha); \mathbf{k})$ is finitely generated for all $k \geq 0$.

Applying Theorem F, we obtain an asymptotic structure theorem for $\mathrm{H}_k(\mathrm{Sp}_{2n}(\mathcal{O}_K, \alpha); \mathbf{k})$. This partially verifies a conjecture of Church–Farb [ChFa, Conjecture 8.2.2].

The mapping class group. Let \mathcal{S}_g^b be a compact oriented genus g surface with b boundary components. The **mapping class group** of \mathcal{S}_g^b , denoted MCG_g^b , is the group of homotopy classes of orientation-preserving homeomorphisms of \mathcal{S}_g^b that restrict to the identity on $\partial\mathcal{S}_g^b$. This is one of the basic objects in low-dimensional topology; see [FaMar] for a survey. Harer [Har] proved that the mapping class group satisfies a form of homological stability. If $i: \mathcal{S}_g^b \hookrightarrow \mathcal{S}_{g'}^{b'}$ is a subsurface inclusion, then there is a map $i_*: \mathrm{MCG}_g^b \rightarrow \mathrm{MCG}_{g'}^{b'}$ that extends mapping classes by the identity. Harer’s theorem says that for all $k \geq 1$, the induced map $\mathrm{H}_k(\mathrm{MCG}_g^b; \mathbf{k}) \rightarrow \mathrm{H}_k(\mathrm{MCG}_{g'}^{b'}; \mathbf{k})$ is an isomorphism for $g \gg 0$. For $\mathbf{k} = \mathbb{Q}$, the stable homology of the mapping class group was calculated by Madsen–Weiss [MadWe].

Congruence subgroups of MCG . Fix some $\ell \geq 2$. The group MCG_g^b acts on $\mathrm{H}_1(\mathcal{S}_g^b; \mathbb{Z}/\ell)$. This action preserves the algebraic intersection pairing $\hat{i}(\cdot, \cdot)$. If $b = 0$, then Poincaré duality implies that $\hat{i}(\cdot, \cdot)$ is symplectic. If instead $b = 1$, then gluing a disc to the boundary component does not change $\mathrm{H}_1(\mathcal{S}_g^b; \mathbb{Z}/\ell)$, so $\hat{i}(\cdot, \cdot)$ is still symplectic. For $g \geq 0$ and $0 \leq b \leq 1$, the action of MCG_g^b on $\mathrm{H}_1(\mathcal{S}_g^b; \mathbb{Z}/\ell) \cong (\mathbb{Z}/\ell)^{2g}$ thus induces a representation $\mathrm{MCG}_g^b \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. It is classical [FaMar, §6.3.2] that this is surjective. Its kernel is the **level ℓ congruence subgroup** $\mathrm{Mod}_g^b(\ell)$ of MCG_g^b . It is known that $\mathrm{H}_k(\mathrm{MCG}_g^b(\ell); \mathbf{k})$ does not stabilize, even for

$k = 1$ (see [Pu1, Theorem H]). Also, the conjugation action of MCG_g^b on $\mathrm{MCG}_g^b(\ell)$ induces an action of $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ on $\mathrm{H}_k(\mathrm{MCG}_g^b(\ell); \mathbf{k})$.

Remark 1.26. For $b > 1$, the algebraic intersection pairing is not symplectic, so we do not get a symplectic representation. We will not discuss this case. \square

Stability. We will restrict ourselves to the case where $b = 1$; the problem with closed surfaces is that they cannot be embedded into larger surfaces, so one cannot “stabilize” them. In §6.5, we will construct an $\mathrm{SI}(\mathbb{Z}/\ell)$ -module $\mathcal{H}_k(\mathrm{MCG}(\ell); \mathbf{k})$ such that

$$\mathcal{H}_k(\mathrm{MCG}(\ell); \mathbf{k})_{(\mathbb{Z}/\ell)^{2g}} = \mathrm{H}_k(\mathrm{MCG}_g^1(\ell); \mathbf{k}) \quad (g \geq 0).$$

Our main theorem concerning this module is as follows.

Theorem K. *Fix some $\ell \geq 2$, and let \mathbf{k} be a Noetherian ring. Then the $\mathrm{SI}(\mathbb{Z}/\ell)$ -module $\mathcal{H}_k(\mathrm{MCG}(\ell); \mathbf{k})$ is finitely generated for all $k \geq 0$.*

Applying Theorem F, we get an asymptotic structure theorem for $\mathrm{H}_k(\mathrm{MCG}_g^1(\ell); \mathbf{k})$. This partially verifies a conjecture of Church–Farb [ChFa, Conjecture 8.5].

Remark 1.27. The kernel of $\mathrm{MCG}_g^1 \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$ is the Torelli group, denoted \mathcal{T}_g^1 , and its homology forms a module over $\mathrm{SI}(\mathbb{Z})$. Just like for IA_n , we would like to understand this module but cannot do so with our techniques since \mathbb{Z} is an infinite ring. \square

1.6 Twisted homological stability

We now discuss twisted homological stability. We wish to thank Aurélien Djament for his help with this section.

Motivation: the symmetric group. A classical theorem of Nakaoka [Nak] says that \mathfrak{S}_n satisfies classical homological stability. Church [Ch2] proved a very general version of this with twisted coefficients. Namely, he proved that if M is a finitely generated FI-module and $M_n = M_{[n]}$, then for all $k \geq 0$ the map $\mathrm{H}_k(\mathfrak{S}_n; M_n) \rightarrow \mathrm{H}_k(\mathfrak{S}_{n+1}; M_{n+1})$ is an isomorphism for $n \gg 0$ (in fact, he obtained linear bounds for when stability occurs). We remark that this applies to much more general coefficient systems than earlier work of Betley [Be]. The key to Church’s theorem is the fact that the category of FI-modules is locally Noetherian. We will abstract Church’s argument to other categories (though our results will be weaker in that we will not give bounds for when stability occurs); the possibility of doing so was one of Church’s motivations for conjecturing Theorem D.

General linear group. In [Va], van der Kallen proved that for rings R satisfying a mild condition (for instance, R can be a finite ring), $\mathrm{GL}_n(R)$ satisfies homological stability. The paper [Va] also shows that this holds for certain twisted coefficient systems (those satisfying a “polynomial” condition introduced by Dwyer [Dw]; see [WahRW] for a general framework for arguments like Dwyer’s using the language of functor categories). The arguments in [Va] also work for $\mathrm{SL}_n^{\mathfrak{U}}(R)$ for subgroups $\mathfrak{U} \subset R$ of the group of units. We will prove the following generalization of van der Kallen’s result (at least for finite commutative rings). If M is a $\mathrm{VIC}(R, \mathfrak{U})$ -module, then denote by M_n the value M_{R^n} .

Theorem L. *Let R be a finite ring, let $\mathfrak{U} \subset R$ be a subgroup of the group of units, and let M be a finitely generated $\mathrm{VIC}(R, \mathfrak{U})$ -module over a Noetherian ring. Then for all $k \geq 0$, the map $\mathrm{H}_k(\mathrm{SL}_n^{\mathfrak{U}}(R); M_n) \rightarrow \mathrm{H}_k(\mathrm{SL}_{n+1}^{\mathfrak{U}}(R); M_{n+1})$ is an isomorphism for $n \gg 0$.*

Remark 1.28. This is more general than van der Kallen’s result: if \mathbf{k} is a field, then his hypotheses on the coefficient systems M_n imply that $\dim_{\mathbf{k}}(M_n)$ grows like a polynomial in n , which need not hold for finitely generated $\mathrm{VIC}(R, \mathfrak{U})$ -modules. For instance, one can define a finitely generated $\mathrm{VIC}(R, \mathfrak{U})$ -module whose dimensions grow exponentially via

$$M_n = \mathbf{k}[\mathrm{Hom}_{\mathrm{VIC}(R, \mathfrak{U})}(R^1, R^n)]. \quad \square$$

The symplectic group. Building on work of Charney [Cha3], Mirzaii–van der Kallen [MirVa] proved that for many rings R (including all finite rings), the groups $\mathrm{Sp}_{2n}(R)$ satisfy homological stability. In fact, though Mirzaii–van der Kallen do not explicitly say this, Charney’s paper shows that the results of [MirVa] also apply to twisted coefficients satisfying an appropriate analogue of Dwyer’s polynomial condition. We will prove the following generalization of this. If M is an $\mathrm{SI}(R)$ -module, then denote by M_n the value $M_{R^{2n}}$.

Theorem M. *Let R be a finite ring and let M be a finitely generated $\mathrm{SI}(R)$ -module over a Noetherian ring. Then for all $k \geq 0$, the map $\mathrm{H}_k(\mathrm{Sp}_{2n}(R); M_n) \rightarrow \mathrm{H}_k(\mathrm{Sp}_{2n+2}(R); M_{n+1})$ is an isomorphism for $n \gg 0$.*

Remark 1.29. Examples similar to the one we gave after Theorem L show that this is more general than Mirzaii–van der Kallen’s result. \square

1.7 Comments and future directions

We now discuss some related results and open questions.

Remark 1.30 (Homological stability). The examples in Theorems G, H, I, J, K can be interpreted when $\ell = 1$ or when α is the unit ideal. In this case, we interpret the categories $\mathrm{VIC}(0) = \mathrm{SI}(0)$ as follows: there is one object for each nonnegative integer $n \geq 0$ and there is a unique morphism $n \rightarrow n'$ whenever $n' \geq n$. Then representation stability of the relevant homology groups is the usual homological stability statements: for example, for each $i \geq 0$, and $n \gg 0$ the map $\mathrm{H}_i(\mathrm{SL}_n(\mathbb{Z}); \mathbf{k}) \rightarrow \mathrm{H}_i(\mathrm{SL}_{n+1}(\mathbb{Z}); \mathbf{k})$ is an isomorphism. \square

Infinite rings. We have repeatedly emphasized that it is necessary for our rings R to be finite. Our final theorem explains why this is necessary.

Theorem N. *Let R be an infinite ring. For $\mathfrak{C} \in \{\mathrm{VI}(R), \mathrm{VIC}(R), \mathrm{SI}(R)\}$, the category of \mathfrak{C} -modules is not locally Noetherian.*

This naturally leads to the question of how to develop representation stability results for categories such as $\mathrm{VIC}(R)$, etc. when R is an infinite ring.

Bounds. Our asymptotic structure theorem does not provide bounds for when it begins. This is an unavoidable artifact of our proof, which make essential use of local Noetherianity. The results of the first author in [Pu2] concerning central stability for the homology groups of congruence subgroups (as representations of \mathfrak{S}_n) are different and do give explicit bounds. It would be interesting to prove versions of our theorems with explicit bounds.

Orthogonal and unitary categories. It is also possible to include orthogonal versions of our categories, i.e., finite rank free R -modules equipped with a nondegenerate orthogonal form. This is a richer category than the previous ones because the rank of an orthogonal module no longer determines its isomorphism type. While many of the ideas and constructions of this paper should be relevant, it will be necessary to generalize them further to include the

example of categories of orthogonal modules. Similarly, one can consider unitary versions. We have omitted discussion of these variations for reasons of space.

Twisted commutative algebras. The category of FI-modules fits into the more general framework of modules over twisted commutative algebras (these are commutative algebras in the category of functors from the groupoid of finite sets) and this perspective is used in [SamSn1, SamSn2, SamSn3] to develop the basic properties of these categories. Similarly, the categories in this paper fit into a more general framework of modules over algebras which generalize twisted commutative algebras where the groupoid of finite sets is replaced by another groupoid (see Remark 3.5). When \mathbf{k} is a field of characteristic 0, twisted commutative algebras have alternative concrete descriptions in terms of $\mathrm{GL}_\infty(\mathbf{k})$ -equivariant commutative algebras (in the usual sense) thanks to Schur–Weyl duality. For the other cases in our paper, no such alternative description seems to be available, so it would be interesting to understand these more exotic algebraic structures.

1.8 Outline and strategy of proof

We now give an outline of the paper and comment on our proofs. Let \mathcal{C} be one of the categories we discussed above, i.e., $\mathrm{VI}(R)$, $\mathrm{V}(R)$, $\mathrm{V}'(R)$, $\mathrm{VIC}(R, \mathfrak{U})$, or $\mathrm{SI}(R)$. The first step is to establish that the category of \mathcal{C} -modules is locally Noetherian, i.e., that the finite generation property is inherited by submodules. This is done in §2, where we prove Theorems A, B, C, D, E, and N. We then prove our asymptotic structure result (Theorem F) in §3. The brief §4 shows how to use our locally Noetherian results to prove our twisted homological stability theorems (Theorems L and M). We then give, in §5, a framework (adapted from the standard proof of homological stability due to Quillen) for proving that \mathcal{C} -modules arising from congruence subgroups are finitely generated. Finally, §6 applies the above framework to prove our results that assert that modules arising from the homology groups of various congruence subgroups are finitely generated, namely Theorems G, H, I, J, and K.

Acknowledgments. We wish to thank Tom Church, Yves Cornulier, Aurélien Djament, Jordan Ellenberg, Nicholas Kuhn, Amritanshu Prasad, Andrew Snowden, and Nathalie Wahl for helpful conversations and correspondence. We also thank some anonymous referees for their careful reading of the paper and numerous suggestions and improvements.

2 Noetherian results

The goal of this section is to prove Theorems A, B, C, D, E, and N, which assert that modules over the various categories discussed in the introduction are locally Noetherian. We begin in §2.1 with some preliminary results. Next, in §2.2 we introduce the category $\mathrm{OVIC}(R)$ and in §2.3 we prove that the category of $\mathrm{OVIC}(R)$ -modules is locally Noetherian. We will use this in §2.4 to prove Theorems A, B, C, and D. These cover all of our categories except $\mathrm{SI}(R)$, which we deal with in §2.5. We close in §2.6 by proving Theorem N, which asserts that our categories are **not** locally Noetherian when R is an infinite ring.

Since it presents no additional difficulties, in this section we allow \mathbf{k} to be an arbitrary ring (unital, but not necessarily commutative). By a \mathbf{k} -module, we mean a left \mathbf{k} -module, and Noetherian means left Noetherian. However, we emphasize that R will always be a (unital) commutative ring. Also, in this section we will denote the effects of functors by parentheses rather than subscripts, so for instance if $M: \mathcal{C} \rightarrow \mathcal{D}$ is a functor and $x \in \mathcal{C}$, then we will write $M(x) \in \mathcal{D}$ for the image of x under M .

2.1 Preliminary results

We first discuss some preliminary results. Let \mathbf{k} be a fixed Noetherian ring.

Functor categories. Let \mathcal{C} be a category. A \mathcal{C} -module (over \mathbf{k}) is a functor $\mathcal{C} \rightarrow \text{Mod}_{\mathbf{k}}$ where $\text{Mod}_{\mathbf{k}}$ is the category of left \mathbf{k} -modules. A morphism of \mathcal{C} -modules is a natural transformation. For $x \in \mathcal{C}$, let $P_{\mathcal{C},x}$ denote the representable \mathcal{C} -module generated at x , i.e., $P_{\mathcal{C},x}(y) = \mathbf{k}[\text{Hom}_{\mathcal{C}}(x, y)]$. If M is a \mathcal{C} -module, then a morphism $P_{\mathcal{C},x} \rightarrow M$ is equivalent to a choice of element in $M(x)$. The category of \mathcal{C} -modules is an Abelian category, with kernels, cokernels, exact sequences, etc. calculated pointwise. A \mathcal{C} -module is **finitely generated** if it is a quotient of a finite direct sum of $P_{\mathcal{C},x}$ (allowing different choices of x). This agrees with the definition given in the introduction.

Lemma 2.1. *Let \mathcal{C} be a category. The category of \mathcal{C} -modules is locally Noetherian if and only if for all $x \in \mathcal{C}$, every submodule of $P_{\mathcal{C},x}$ is finitely generated.*

Proof. Immediate since Noetherianity is preserved by quotients and finite direct sums. \square

Functors between functor categories. Let $\Phi: \mathcal{C} \rightarrow \mathcal{D}$ a functor. Given a \mathcal{D} -module $M: \mathcal{D} \rightarrow \text{Mod}_{\mathbf{k}}$, we get a \mathcal{C} -module $\Phi^*(M) = M \circ \Phi$. Note that Φ^* is an exact functor from \mathcal{D} -modules to \mathcal{C} -modules. We say that Φ is **finite** if $\Phi^*(P_{\mathcal{D},y})$ is finitely generated for all $y \in \mathcal{D}$. Recall that Φ is **essentially surjective** if every object of \mathcal{D} is isomorphic to an object of the form $\Phi(x)$ for $x \in \mathcal{C}$.

Lemma 2.2. *Let $\Phi: \mathcal{C} \rightarrow \mathcal{D}$ be an essentially surjective and finite functor. Assume that the category of \mathcal{C} -modules is locally Noetherian. Then the category of \mathcal{D} -modules is locally Noetherian.*

Proof. Pick $y \in \mathcal{D}$ and let $M_1 \subseteq M_2 \subseteq \dots$ be a chain of submodules of $P_{\mathcal{D},y}$. By Lemma 2.1, it is enough to show that the M_n stabilize. Since Φ^* is exact, we get a chain of submodules $\Phi^*(M_1) \subseteq \Phi^*(M_2) \subseteq \dots$ of $\Phi^*(P_{\mathcal{D},y})$. Since Φ is finite, $\Phi^*(P_{\mathcal{D},y})$ is finitely generated, so since \mathcal{C} is locally Noetherian there is some N such that $\Phi^*(M_n) = \Phi^*(M_{n+1})$ for $n \geq N$. Since Φ is essentially surjective, it preserves strict inclusions of submodules: if $M_n \subsetneq M_{n+1}$, then there is some $z \in \mathcal{D}$ such that $M_n(z) \subsetneq M_{n+1}(z)$ and we can find $x \in \mathcal{C}$ such that $\Phi(x) \cong z$, which means that $\Phi^*(M_n)(x) \subsetneq \Phi^*(M_{n+1})(x)$. Thus $M_n = M_{n+1}$ for $n \geq N$, as desired. \square

Remark 2.3. In all of our applications of Lemma 2.2 in this paper, essential surjectivity will be obvious, as the objects of our categories are indexed by natural numbers. \square

Well partial orderings. A poset \mathcal{P} is **well partially ordered** if for any sequence x_1, x_2, \dots in \mathcal{P} , there exists $i < j$ such that $x_i \leq x_j$. See [Kr] for a survey. The direct product $\mathcal{P}_1 \times \mathcal{P}_2$ of posets is in a natural way a poset with $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. The following lemma is well-known, see for example [SamSn4, §2] for a proof.

- Lemma 2.4.**
- (a) *A subposet of a well partially ordered poset is well partially ordered.*
 - (b) *If \mathcal{P} is well partially ordered and x_1, x_2, \dots is a sequence, then there is an infinite increasing sequence $i_1 < i_2 < \dots$ such that $x_{i_1} \leq x_{i_2} \leq \dots$.*
 - (c) *A finite direct product of well partially ordered posets is well partially ordered.*

Posets of words. Let Σ be a finite set. Let Σ^* be the set of words $s_1 \cdots s_n$ whose letters come from Σ . Define a poset structure on Σ^* by saying that $s_1 \cdots s_n \leq s'_1 \cdots s'_m$ if there is an increasing function $f: [n] \rightarrow [m]$ such that $s_i = s'_{f(i)}$ for all i . The following is a special case of Higman's lemma.

Lemma 2.5 (Higman, [Hi]). Σ^\star is a well partially ordered poset.

We will also need a variant of Higman’s lemma. Define a partially ordered set $\tilde{\Sigma}^\star$ whose objects are the same as Σ^\star by saying that $s_1 \cdots s_n \leq s'_1 \cdots s'_m$ if there is an increasing function $f: [n] \rightarrow [m]$ such that $s'_{f(i)} = s_i$ for $i = 1, \dots, n$ and for each $j = 1, \dots, m$, there exists i such that $f(i) \leq j$ and $s'_j = s'_{f(i)}$.

Lemma 2.6. $\tilde{\Sigma}^\star$ is a well partially ordered poset.

See [SamSn4, Prop. 8.2.1]; a different proof of this is in [DrKu, Proof of Prop. 7.5].

2.2 The category OVIC

Fix a finite commutative ring R .

Motivation. Let $\mathfrak{U} \subset R^\times$ be a subgroup of the group of multiplicative units. For the purpose of proving that the category of $\text{VIC}(R, \mathfrak{U})$ -modules is locally Noetherian, we will use the “cheap” definition of $\text{VIC}(R, \mathfrak{U})$ from the introduction and use the alternate description of its morphisms from Remark 1.11. Thus $\text{VIC}(R, \mathfrak{U})$ is the category whose objects are $\{R^n \mid n \geq 0\}$ and whose morphisms from R^n to $R^{n'}$ are pairs (f, f') , where $f: R^n \rightarrow R^{n'}$ and $f': R^{n'} \rightarrow R^n$ satisfy $f'f = 1_{R^n}$ and $\det(f) \in \mathfrak{U}$ if $n = n'$.

We will introduce yet another category $\text{OVIC}(R)$ that satisfies $\text{OVIC}(R) \subset \text{VIC}(R, \mathfrak{U})$ (the “O” stands for “Ordered”; see Lemma 2.13 below). We emphasize that $\text{OVIC}(R)$ does **not** depend on \mathfrak{U} . The objects of $\text{OVIC}(R)$ are $\{R^n \mid n \geq 0\}$, and the $\text{OVIC}(R)$ -morphisms from R^n to $R^{n'}$ are $\text{VIC}(R, \mathfrak{U})$ -morphisms (f, f') such that f' is what we will call a *column-adapted map* (see below for the definition). The key property of $\text{OVIC}(R)$ is that every $\text{VIC}(R)$ -morphism $(f, f'): R^n \rightarrow R^{n'}$ (notice that we are using $\text{VIC}(R)$ and not $\text{VIC}(R, \mathfrak{U})$) can be uniquely factored as $(f, f') = (f_1, f'_1)(f_2, f'_2)$, where $(f_1, f'_1): R^n \rightarrow R^{n'}$ is an $\text{OVIC}(R)$ -morphism and $(f_2, f'_2): R^{n'} \rightarrow R^{n'}$ is an isomorphism; see Lemma 2.11 below. This implies in particular that the category $\text{OVIC}(R)$ has no non-identity automorphisms. We will prove that the category of $\text{OVIC}(R)$ -modules is locally Noetherian and that the inclusion functor $\text{OVIC}(R) \hookrightarrow \text{VIC}(R, \mathfrak{U})$ is finite. The key point here is the above factorization and the fact that $\text{Aut}_{\text{VIC}(R)}(R^n)$ is finite. By Lemma 2.2, this will imply that the category of $\text{VIC}(R, \mathfrak{U})$ -modules is locally Noetherian.

Column-adapted maps, local case. In this paragraph, assume that R is a commutative local ring. Recall that the set of non-invertible elements of R forms the unique maximal ideal of R . An R -linear map $f: R^{n'} \rightarrow R^n$ is **column-adapted** if there is a subset $S = \{s_1 < \cdots < s_n\} \subseteq \{1, \dots, n'\}$ with the following two properties. Regard f as an $n \times n'$ matrix and let $\mathbf{b}_1, \dots, \mathbf{b}_n$ be the standard basis for R^n .

- For $1 \leq i \leq n$, the s_i^{th} column of f is \mathbf{b}_i .
- For $1 \leq i \leq n$ and $1 \leq j < s_i$, the entry in position (i, j) of f is non-invertible.

The set S is unique and will be written $S_c(f)$.

Example 2.7. An example of a column-adapted map $f: R^6 \rightarrow R^3$ is the linear map represented by the matrix

$$\begin{pmatrix} \star & 1 & 0 & \star & 0 & \star \\ \star & 0 & 1 & \star & 0 & \star \\ \star & 0 & 0 & \star & 1 & \star \end{pmatrix}.$$

Here \star can be any element of R , while \star can be any non-invertible element of R . □

The following observation might clarify the nature of column-adapted maps. It follows immediately from our definitions.

Observation 2.8. *All column-adapted maps are surjective, and the only column-adapted map $f: R^n \rightarrow R^n$ is the identity.*

Column-adapted maps, general case. Now assume that R is an arbitrary finite commutative ring. This implies that R is Artinian, so there exists an isomorphism $R \cong R_1 \times \cdots \times R_q$ where the R_i are finite commutative local rings [AtMac, Theorem 8.7]. For the remainder of this section, we fix one such isomorphism. Observe that

$$\mathrm{Hom}_R(R^{n'}, R^n) = \mathrm{Hom}_{R_1}(R_1^{n'}, R_1^n) \times \cdots \times \mathrm{Hom}_{R_q}(R_q^{n'}, R_q^n),$$

so given $f \in \mathrm{Hom}_R(R^{n'}, R^n)$ we can write $f = (f_1, \dots, f_q)$ with $f_i \in \mathrm{Hom}_{R_i}(R_i^{n'}, R_i^n)$. We will say that f is **column-adapted** if each f_i is column-adapted.

Lemma 2.9. *Let R be a finite commutative ring and let $f: R^a \rightarrow R^b$ and $g: R^b \rightarrow R^c$ be column-adapted maps. Then $gf: R^a \rightarrow R^c$ is column-adapted.*

Proof. It is enough to deal with the case where R is a local ring. Write $S_c(f) = \{s_1 < \cdots < s_b\}$ and $S_c(g) = \{t_1 < \cdots < t_c\}$. For $1 \leq i \leq c$, let $u_i = s_{t_i}$. We claim that gf is a column-adapted map with $S_c(gf) = \{u_1 < \cdots < u_c\}$. The first condition is clear, so we must verify the second condition. Consider $1 \leq i \leq n$ and $1 \leq j < u_i$. We then have

$$(gf)_{ij} = g_{i1}f_{1j} + g_{i2}f_{2j} + \cdots + g_{ib}f_{bj}.$$

By definition, g_{ik} is non-invertible for $1 \leq k < t_i$. Also, for $t_i \leq k \leq b$ the element f_{kj} is non-invertible since $j < u_i = s_{t_i} \leq s_k$. It follows that all the terms in our expression for $(gf)_{ij}$ are non-invertible, so since the non-invertible elements of R form an ideal we conclude that $(gf)_{ij}$ is non-invertible, as desired. \square

Lemma 2.10. *Let R be a finite commutative ring and let $f: R^{n'} \rightarrow R^n$ be a surjection. Then we can uniquely factor f as $f = f_2 f_1$, where $f_1: R^{n'} \rightarrow R^n$ is column-adapted and $f_2: R^n \rightarrow R^n$ is an isomorphism.*

Proof. It is enough to deal with the case where R is a local ring. Since R^n is projective, we can find $g: R^n \rightarrow R^{n'}$ such that $fg = 1_{R^n}$. We can view f and g as matrices, and the Cauchy–Binet formula says that

$$1 = \det(fg) = \sum_I \det(f_{[n],I}) \det(g_{I,[n]})$$

where I ranges over all n -element subsets of $[n'] = \{1, \dots, n'\}$ and $f_{I,J}$ means the minor of f with rows I and columns J . Since R is local, the non-units form an ideal, so there exists at least one I such that $\det(f_{[n],I})$ is a unit. Let I be the lexicographically minimal subset such that this holds. There is a unique $h \in \mathrm{GL}(R^n)$ with $(hf)_{[n],I} = 1_{R^n}$. This implies that hf is column-adapted with $S_c(hf) = I$, and the desired factorization is $f_1 = hf$, $f_2 = h^{-1}$. \square

The category OVIC. Continue to let R be a general finite commutative ring. Define $\mathrm{OVIC}(R)$ to be the category whose objects are $\{R^n \mid n \geq 0\}$ and where $\mathrm{Hom}_{\mathrm{OVIC}(R)}(R^n, R^{n'})$ consists of pairs (f, f') as follows.

- $f \in \mathrm{Hom}_R(R^n, R^{n'})$ and $f' \in \mathrm{Hom}_R(R^{n'}, R^n)$.
- $f'f = 1_{R^n}$, so f is an injection and f' is a surjection.
- f' is column-adapted.

For $(f, f') \in \text{Hom}_{\text{OVIC}(R)}(R^n, R^{n'})$ and $(g, g') \in \text{Hom}_{\text{OVIC}(R)}(R^{n'}, R^{n''})$, we define $(g, g')(f, f')$ to be $(gf, f'g') \in \text{Hom}_{\text{OVIC}(R)}(R^n, R^{n''})$. Lemma 2.9 implies that this makes sense. The key property of $\text{OVIC}(R)$ is as follows.

Lemma 2.11. *Let R be a finite commutative ring and let $(f, f') \in \text{Hom}_{\text{VIC}(R)}(R^n, R^{n'})$. Then we can uniquely write $(f, f') = (f_1, f'_1)(f_2, f'_2)$, where $(f_2, f'_2) \in \text{Aut}_{\text{VIC}(R)}(R^n)$ and $(f_1, f'_1) \in \text{Hom}_{\text{OVIC}(R)}(R^n, R^{n'})$.*

Proof. Letting $f' = f'_2 f'_1$ be the factorization from Lemma 2.10, the desired factorization is $(f, f') = (f f'_2, f'_1)((f'_2)^{-1}, f'_2)$. To see that this first factor is in $\text{Hom}_{\text{OVIC}(R)}(R^n, R^{n'})$, observe that $f'_1 f f'_2 = (f'_2)^{-1} f' f f'_2 = (f'_2)^{-1} f'_2 = 1_{R^n}$. \square

Free and dependent rows. Assume that R is local. The condition $f'f = 1_{R^n}$ in the definition of an $\text{OVIC}(R)$ -morphism implies that the rows of f indexed by $S_c(f')$ are determined by the other rows, which can be chosen freely. So we will call the rows of f indexed by $S_c(f')$ the **dependent rows** and the other rows the **free rows**.

Example 2.12. Assume that R is a local ring, that $f': R^6 \rightarrow R^3$ is a column-adapted map with $S_c(f') = \{2, 3, 5\}$, and that $f: R^3 \rightarrow R^6$ satisfies $f'f = 1_{R^3}$. The possible matrices representing f' and f are of the form

$$f' = \begin{pmatrix} \star & 1 & 0 & \star & 0 & \star \\ \star & 0 & 1 & \star & 0 & \star \\ \star & 0 & 0 & \star & 1 & \star \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} \ast & \ast & \ast \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \\ \ast & \ast & \ast \\ \diamond & \diamond & \diamond \\ \ast & \ast & \ast \end{pmatrix}.$$

Here \ast can be any element of R , while \star can be any non-invertible element of R and each \diamond is completely determined by the choices of \ast and \star plus the fact that $f'f = 1_{R^3}$. \square

2.3 The category of OVIC -modules is locally Noetherian

The main result of this section is Theorem 2.14 below, which says that the category of $\text{OVIC}(R)$ -modules is locally Noetherian when R is a finite commutative ring.

A partial order. We first need the existence of a certain partial ordering. For $d \geq 0$, define

$$\mathcal{P}_R(d) = \bigsqcup_{n=0}^{\infty} \text{Hom}_{\text{OVIC}(R)}(R^d, R^n).$$

To apply the results of Sam–Snowden in [SamSn4], we would need to find a total ordering of the set $\mathcal{P}_R(d)$ which is invariant under left composition. However, constructing such an ordering seems difficult. Our solution is to construct a total ordering which is invariant under left composition with “enough” morphisms to make an argument in the style of [SamSn4] go through (see Remark 2.16 for some discussion of this). The following lemma gives this ordering and is the key to the proof of Theorem 2.14, which asserts that the category of $\text{OVIC}(R)$ -modules is locally Noetherian when R is a finite commutative ring.

Lemma 2.13. *Let R be a finite commutative ring, and fix $d \geq 0$. There exists a well partial ordering \preceq on $\mathcal{P}_R(d)$ together with an extension \leq of \preceq to a total ordering such that the following holds. Consider $(f, f') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^n)$ and $(g, g') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^{n'})$ with $(f, f') \preceq (g, g')$. Then there is some $(\phi, \phi') \in \text{Hom}_{\text{OVIC}(R)}(R^n, R^{n'})$ with the following two properties.*

1. We have $(g, g') = (\phi, \phi')(f, f')$.
2. If $(f_1, f'_1) \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^{n'})$ satisfies $(f_1, f'_1) < (f, f')$, then $(\phi, \phi')(f_1, f'_1) < (g, g')$.

Since the proof of Lemma 2.13 is lengthy, we postpone it until the end of this section and instead show how it can be used to prove local Noetherianity.

Noetherianity. We now come to our main result.

Theorem 2.14. *Let R be a finite commutative ring. Then the category of $\text{OVIC}(R)$ -modules is locally Noetherian.*

Proof. Fix a Noetherian ring \mathbf{k} . For $d \geq 0$, define P_d to be the $\text{OVIC}(R)$ -module $P_{\text{OVIC}(R), R^d}$, where the notation is as in §2.1, so $P_d(R^n) = \mathbf{k}[\text{Hom}_{\text{OVIC}(R)}(R^d, R^n)]$. By Lemma 2.1, it is enough to show that every submodule of P_d is finitely generated.

Let \preceq and \leq be the orderings on $\mathcal{P}_R(d)$ given by Lemma 2.13. For an element $(f, f') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^n)$, denote by $e_{f, f'}$ the associated basis element of $P_d(R^n)$. Given nonzero $x \in P_d(R^n)$, define its **initial term** $\text{init}(x)$ as follows. First, set $\text{init}(0) = 0$. For $x \neq 0$, write $x = \sum \alpha_{f, f'} e_{f, f'}$ with $\alpha_{f, f'} \in \mathbf{k}$, and let (f_0, f'_0) be the \leq -largest element such that $\alpha_{f_0, f'_0} \neq 0$. We then define $\text{init}(x) = \alpha_{f_0, f'_0} e_{f_0, f'_0}$. Next, given a submodule $M \subset P_d$, define its **initial module** $\text{init}(M)$ to be the function that takes $n \geq 0$ to the \mathbf{k} -module $\mathbf{k}\{\text{init}(x) \mid x \in M(R^n)\}$. Warning: $\text{init}(M)$ need not be an $\text{OVIC}(R)$ -submodule of P_d , see Remark 2.16.

Claim. *If $N, M \subset P_d$ are submodules with $N \subset M$ and $\text{init}(N) = \text{init}(M)$, then $N = M$.*

Proof of Claim. Assume that $M \neq N$. Let $n \geq 0$ be such that $M(R^n) \neq N(R^n)$. Pick $y \in M(R^n) \setminus N(R^n)$ such that $\text{init}(y) = \alpha e_t$ where t is \leq -minimal. By assumption, there exists $z \in N(R^n)$ with $\text{init}(z) = \text{init}(y)$. But then $y - z \in M(R^n) \setminus N(R^n)$ and $\text{init}(y - z) = \beta e_{t'}$ where $t' < t$, a contradiction. \square

We now turn to the proof that every submodule of P_d is finitely generated. Assume otherwise, so there is a strictly increasing sequence $M_0 \subsetneq M_1 \subsetneq \dots$ of submodules. By the claim, we must have $\text{init}(M_i) \neq \text{init}(M_{i-1})$ for all $i \geq 1$, so we can find some $n_i \geq 0$ and some $\lambda_i e_{f_i, f'_i} \in \text{init}(M_i)(R^{n_i})$ with $\lambda_i e_{f_i, f'_i} \notin \text{init}(M_{i-1})(R^{n_i})$. Since \preceq is a well partial ordering, we can apply Lemma 2.4(b) and find $i_0 < i_1 < \dots$ such that

$$(f_{i_0}, f'_{i_0}) \preceq (f_{i_1}, f'_{i_1}) \preceq (f_{i_2}, f'_{i_2}) \preceq \dots$$

Since \mathbf{k} is Noetherian, there exists some $m \geq 0$ such that $\lambda_{i_{m+1}}$ is in the left \mathbf{k} -ideal generated by $\lambda_{i_0}, \dots, \lambda_{i_m}$, i.e., we can write $\lambda_{i_{m+1}} = \sum_{j=0}^m c_j \lambda_{i_j}$ with $c_j \in \mathbf{k}$. Consider some $0 \leq j \leq m$. Let $x_j \in M_{i_j}(R^{n_{i_j}})$ be such that $\text{init}(x_j) = \lambda_{i_j} e_{f_{i_j}, f'_{i_j}}$. By Lemma 2.13, there exists some $(\phi_j, \phi'_j) \in \text{Hom}_{\text{OVIC}(R)}(R^{n_{i_j}}, R^{n_{i_{m+1}}})$ such that $(f_{i_{m+1}}, f'_{i_{m+1}}) = (\phi_j, \phi'_j)(f_{i_j}, f'_{i_j})$. Setting $y = \sum_{j=0}^m c_j (\phi_j, \phi'_j) x_j$, an element of $M_{i_{m+1}-1}(R^{n_{i_{m+1}}})$, Lemma 2.13 implies that $\text{init}(y) = \lambda_{i_{m+1}} e_{f_{i_{m+1}}, f'_{i_{m+1}}}$, a contradiction. \square

Insertion maps. For the proof of Lemma 2.13, we will need the following lemma.

Lemma 2.15. *Let R be a commutative local ring. Fix some $1 \leq k \leq \ell \leq n$ and some $S = \{s_1 < s_2 < \dots < s_d\} \subset \{1, \dots, n\}$. Set $\hat{\ell} = \max(\{i \mid s_i < \ell\} \cup \{0\})$. Let $v = (v_1, \dots, v_d) \in R^d$ be such that v_i is non-invertible for all $i > \hat{\ell}$. Define*

$$T = \{s_1 < \dots < s_{\hat{\ell}} < s_{\hat{\ell}} + 1 < \dots < s_d + 1\} \subset \{1, \dots, n + 1\}.$$

There exists $(\phi, \phi') \in \text{Hom}_{\text{OVIC}(R)}(R^n, R^{n+1})$ with the following properties.

- Consider $(f, f') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^n)$ with $S_c(f') = S$.
 - The matrix $f'\phi'$ is obtained from f' by inserting the column vector v between the $(\ell - 1)^{\text{st}}$ and ℓ^{th} columns. Observe that $S_c(f'\phi') = T$.
 - Let $w \in R^d$ be the k^{th} row of f . Then ϕf is obtained from f by inserting w between the $(\ell - 1)^{\text{st}}$ and ℓ^{th} rows (observe that this new row is a free row!) and then modifying the dependent rows so that $(\phi f, f'\phi') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^{n+1})$. More precisely, the $\widehat{s}_i^{\text{th}}$ row of ϕf is the s_i^{th} row of f minus $v_i w$.
- Consider $(g, g') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^n)$ with $S_c(g') < S$ in the lexicographic order. Then $S_c(g'\phi') < T$ in the lexicographic order.

Proof. Define $\widehat{v} \in R^n$ to be the column vector with v_i in position s_i for $1 \leq i \leq d$ and zeros elsewhere. Next, define $\phi': R^{n+1} \rightarrow R^n$ to be the matrix obtained from the $n \times n$ identity matrix Id_{R^n} by inserting \widehat{v} between the $(\ell - 1)^{\text{st}}$ and ℓ^{th} columns. The condition on the v_i ensures that ϕ' is a column-adapted map. The possible choices of $\phi: R^n \rightarrow R^{n+1}$ such that $(\phi, \phi') \in \text{Hom}_{\text{OVIC}(R)}(R^n, R^{n+1})$ have a single free row, namely the ℓ^{th} . Let that row have a 1 in position k and zeros elsewhere. The other dependent rows are determined by this. For a formula for this matrix, letting e be the k^{th} standard basis row vector, the matrix ϕ is the result of inserting e between the ℓ^{th} and $(\ell + 1)^{\text{st}}$ rows of $\text{Id}_{R^n} - \widehat{v}e$. We have thus constructed an element $(\phi, \phi') \in \text{Hom}_{\text{OVIC}(R)}(R^n, R^{n+1})$; that it satisfies the conclusions of the lemma is an easy calculation. \square

Constructing the orders, local case. We now prove Lemma 2.13 in the special case where R is a local ring.

Proof of Lemma 2.13, local case. Let R be a finite commutative local ring. The first step is to construct \preceq . Consider $(f, f') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^n)$ and $(g, g') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^{n'})$. We will say that $(f, f') \preceq (g, g')$ if there exists a sequence

$$(f, f') = (f_0, f'_0), (f_1, f'_1), \dots, (f_{n'-n}, f'_{n'-n}) = (g, g')$$

with the following properties:

- For $0 \leq i \leq n' - n$, we have $(f_i, f'_i) \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^{n+i})$.
- For $0 \leq i < n' - n$, the morphism (f_{i+1}, f'_{i+1}) is obtained from (f_i, f'_i) as follows. Choose some $1 \leq k \leq \ell \leq n + i$ such that $k \notin S_c(f'_i)$. Then f'_{i+1} is obtained from f'_i by inserting a copy of the k^{th} column of f'_i between the $(\ell - 1)^{\text{st}}$ and ℓ^{th} columns of f'_i . Also, f_{i+1} is obtained from f'_i by first inserting a copy of the k^{th} row of f_i (observe that this is a free row!) between the $(\ell - 1)^{\text{st}}$ and ℓ^{th} rows of f_i (this new row is also a free row!) and then modifying the dependent rows to ensure that $(f_{i+1}, f'_{i+1}) \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^{n+i+1})$.

This clearly defines a partial order on $\mathcal{P}_R(d)$.

Claim. *The partially ordered set $(\mathcal{P}_R(d), \preceq)$ is well partially ordered.*

Proof of Claim. Define

$$\Sigma = \left(R^d \amalg \{\clubsuit\} \right) \times \left(R^d \amalg \{\spadesuit\} \right),$$

where \clubsuit and \spadesuit are formal symbols. We will prove that $(\mathcal{P}_R(d), \preceq)$ is isomorphic to a subposet of the well partially ordered poset $\widetilde{\Sigma}^*$ from Lemma 2.6, so by Lemma 2.4(a) is itself well partially ordered. Consider $(f, f') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^n)$. For $1 \leq i \leq n$, let $(r_i, c_i) \in \Sigma$ be as follows.

- If $i \in S_c(f')$, then $(r_i, c_i) = (\clubsuit, \spadesuit)$. Observe that in this case the i^{th} row of f is a dependent row.

- If $i \notin S_c(f')$, then r_i is the i^{th} row of f and c_i is the i^{th} column of f' . Observe that in this case the i^{th} row of f is a free row.

We get an element $(r_1, c_1) \cdots (r_n, c_n) \in \tilde{\Sigma}^*$. We claim that the map taking the pair (f, f') to $(r_1, c_1) \cdots (r_n, c_n)$ is an injection from $\mathcal{P}_R(d)$ to $\tilde{\Sigma}^*$. To see this, consider an element $(r_1, c_1) \cdots (r_n, c_n) \in \tilde{\Sigma}^*$ in its image. Our goal is to reconstruct (f, f') . First, we can read off all the free rows of f . Second, if the i^{th} row of f is free, then we can also read off the i^{th} column of f' . The other columns of f' are determined by the fact that they are precisely the standard basis vectors in R^d , appearing in their natural order. But now that we have all of f' and the free rows of f , the dependent rows of f are determined since $f'f = 1_{R^d}$. The claim follows. That this injection is order-preserving is immediate from the definitions. The only subtle point is the reason we required that $k \notin S_c(f'_i)$ in the second case of the definition of the ordering on $\mathcal{P}_R(d)$. This is needed to ensure that no dependent rows of f are copied when producing a larger element. \square

We now extend \preceq to a total order \leq . Fix some arbitrary total order on R^d , and consider $(f, f') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^n)$ and $(g, g') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^{n'})$. We then determine if $(f, f') \leq (g, g')$ via the following procedure.

- If $n < n'$, then $(f, f') < (g, g')$.
- Otherwise, $n = n'$. If $S_c(f') < S_c(g')$ in lexicographic ordering, set $(f, f') < (g, g')$.
- Otherwise, $S_c(f') = S_c(g')$. If $f' \neq g'$, then compare the sequences of elements of R^d which form the columns of f' and g' using the lexicographic ordering and the fixed total ordering on R^d .
- Finally, if $f' = g'$, then compare the sequences of elements of R^d which form the free rows of f and g using the lexicographic ordering and the fixed total ordering on R^d .

It is clear that this determines a total order \leq on $\mathcal{P}_R(d)$ that extends \preceq .

It remains to check that these orders have the property claimed by the lemma. Consider $(f, f') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^n)$ and $(g, g') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^{n'})$ with $(f, f') \preceq (g, g')$. By repeated applications of Lemma 2.15, there exists $(\phi, \phi') \in \text{Hom}_{\text{OVIC}(R)}(R^n, R^{n'})$ such that $(g, g') = (\phi, \phi')(f, f')$. Now pick $(f_1, f'_1) \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^n)$ such that $(f_1, f'_1) < (f, f')$. We want to show that $(\phi, \phi')(f_1, f'_1) < (\phi, \phi')(f, f')$. If $S_c(f'_1) < S_c(f')$ in the lexicographic order, then this is an immediate consequence of Lemma 2.15, so we can assume that $S_c(f'_1) = S_c(f')$. In this case, it follows from Lemma 2.15 that $f'_1\phi'$ and $f'\phi'$ are obtained by inserting the same columns into f'_1 and f' , respectively. If $f'_1 \neq f'$, then it follows immediately that $(\phi, \phi')(f_1, f'_1) < (\phi, \phi')(f, f')$. Otherwise, it follows from Lemma 2.15 that ϕf_1 and ϕf are obtained by duplicating the same rows in f_1 and f , respectively, and it follows that $(\phi, \phi')(f_1, f'_1) < (\phi, \phi')(f, f')$, as desired. \square

Remark 2.16. The total ordering that we define for the proof of Lemma 2.13 need not be compatible with all compositions in the sense that $f' < f$ implies that $gf' < gf$ for all g . This assumption is in place in [SamSn4] which is why we cannot use the machinery from that paper. To see this, consider the example $R = \mathbb{Z}/16$ and define

$$\begin{aligned} f &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & f'_1 &= \begin{pmatrix} 2 & 1 \end{pmatrix}, & f'_2 &= \begin{pmatrix} 6 & 1 \end{pmatrix}, \\ g &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, & g'_1 &= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & g'_2 &= \begin{pmatrix} 6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Then $(f, f'_i) \in \text{Hom}_{\text{OVIC}(R)}(R^1, R^2)$ and $(g, g'_i) \in \text{Hom}_{\text{OVIC}(R)}(R^2, R^3)$ for $i = 1, 2$. Also,

$$\begin{aligned} g'_1 f'_1 &= \begin{pmatrix} 4 & 2 & 1 \end{pmatrix}, & g'_1 f'_2 &= \begin{pmatrix} 12 & 6 & 1 \end{pmatrix}, \\ g'_2 f'_1 &= \begin{pmatrix} 12 & 2 & 1 \end{pmatrix}, & g'_2 f'_2 &= \begin{pmatrix} 4 & 6 & 1 \end{pmatrix}. \end{aligned}$$

Using our ordering, we need to fix a total ordering on $\mathbb{Z}/16$. If we choose $2 < 6$, then we would have $f'_1 < f'_2$. This forces $4 < 12$ by considering $g'_1 f'_1 < g'_1 f'_2$, but it also forces $12 < 4$ by considering $g'_2 f'_1 < g'_2 f'_2$. Hence our ordering is not compatible with all compositions.

In particular, we cannot guarantee that $\text{init}(M)$ is an $\text{OVIC}(R)$ -submodule of P_d . It is possible that there exist total orderings that are compatible with all compositions. \square

Constructing the orders, general case. We finally come to the general case.

Proof of Lemma 2.13, general case. Let R be an arbitrary finite commutative ring. As we mentioned above, R is Artinian, so we can write $R \cong R_1 \times \cdots \times R_q$ with R_i a finite local ring [AtMac, Theorem 8.7]. Indeed, we have fixed one such isomorphism. For all $n, n' \geq 0$, we then have

$$\text{Hom}_R(R^n, R^{n'}) = \text{Hom}_{R_1}(R_1^n, R_1^{n'}) \times \cdots \times \text{Hom}_{R_q}(R_q^n, R_q^{n'}).$$

The set $\mathcal{P}_R(d)$ can thus be identified with the set of tuples

$$((f_1, f'_1), \dots, (f_q, f'_q)) \in \mathcal{P}_{R_1}(d) \times \cdots \times \mathcal{P}_{R_q}(d)$$

such that there exists some single $n \geq 0$ with $(f_i, f'_i) \in \text{Hom}_{\text{OVIC}(R_i)}(R_i^n, R_i^n)$ for all $1 \leq i \leq q$. Above we constructed the desired orders \preceq and \leq on each $\mathcal{P}_{R_i}(d)$. Define a partial order \preceq on their product using the product partial order and define a total order \leq on their product using the lexicographic order. This restricts to orders \preceq and \leq on $\mathcal{P}_R(d)$. Lemma 2.4 implies that \preceq is well partially ordered, and the remaining conclusions are immediate. \square

2.4 The categories of VI-, V-, V'-, and VIC-modules are locally Noetherian

We now prove several of our main theorems.

Proof of Theorem D. We wish to prove that the category of $\text{VIC}(R, \mathfrak{U})$ -modules is locally Noetherian. By the discussion at the beginning of §2.2 (and Theorem 2.14), it is enough to show that the inclusion functor $\Phi: \text{OVIC}(R) \rightarrow \text{VIC}(R, \mathfrak{U})$ is finite. Fix $d \geq 0$ and set $M = P_{\text{VIC}(R, \mathfrak{U}), R^d}$, so $M(R^n) = \mathbf{k}[\text{Hom}_{\text{VIC}(R, \mathfrak{U})}(R^d, R^n)]$. Our goal is to prove that the $\text{OVIC}(R)$ -module $\Phi^*(M)$ is finitely generated. For every $\varphi \in \text{SL}_d^{\mathfrak{U}}(R)$, we get an element (φ, φ^{-1}) of $\Phi^*(M)(R^d)$, and thus a morphism $P_{\text{OVIC}(R), R^d} \rightarrow \Phi^*(M)$. For $\varphi \in \text{GL}_d(R)$ and $(f, f') \in \text{Hom}_{\text{OVIC}(R)}(R^d, R^{d+1})$, the element $(f, f')(\varphi, \varphi^{-1})$ is in $\Phi^*(M)(R^{d+1})$, and thus determines a morphism $P_{\text{OVIC}(R), R^{d+1}} \rightarrow \Phi^*(M)$. Lemma 2.11 implies that the resulting map

$$\left(\bigoplus_{\text{SL}_d^{\mathfrak{U}}(R)} P_{\text{OVIC}(R), R^d} \right) \oplus \left(\bigoplus_{\text{GL}_d(R) \times \text{Hom}_{\text{OVIC}(R)}(R^d, R^{d+1})} P_{\text{OVIC}(R), R^{d+1}} \right) \longrightarrow \Phi^*(M)$$

is a surjection. Since R is finite, this proves that $\Phi^*(M)$ is finitely generated. \square

Proof of Theorem A. We wish to prove that the category of $\text{VI}(R)$ -modules is locally Noetherian. The forgetful functor $\text{VIC}(R) \rightarrow \text{VI}(R)$ is essentially surjective since it is the identity map on the objects. To see that it is finite, consider the pullback of $P_{\text{VI}(R), R^n}$ to $\text{VIC}(R)$. Since it is generated as a $\text{VI}(R)$ -module by the identity map $\text{id}_{R^n} \in P_{\text{VI}(R), R^n}(R^n) = \mathbf{k}[\text{hom}_{\text{VI}(R)}(R^n, R^n)]$, the same is true as a $\text{VIC}(R)$ -module since the forgetful map is surjective on morphism sets. This surjectivity uses the fact that R is finite. Indeed, this implies that every stably free R -module is free, so the cokernel of a splittable R -linear map $R^m \rightarrow R^n$ is free. The desired statement now follows from Lemma 2.2 and Theorem D. \square

Proof of Theorem B. We wish to prove that the category of $\text{V}(R)$ -modules is locally Noetherian. By Lemma 2.2 and Theorem A, it is enough to prove that the inclusion functor $\iota: \text{VI}(R) \rightarrow \text{V}(R)$ is essentially surjective and finite. Essential surjectivity is clear, so we must only prove finiteness. Let V be a free R -module. By considering its image, every splittable R -linear map canonically factors as a composition of a surjective map followed by a splittable injective map. So we have the decomposition of $\text{VI}(R)$ -modules

$$\iota^*(P_{\text{V}(R), V}) = \bigoplus_{V \rightarrow W \rightarrow 0} P_{\text{VI}(R), W}$$

where the sum is over all free quotients W of V . \square

Proof of Theorem C. We wish to prove that the category of $\text{V}'(R)$ -modules is locally Noetherian. By Lemma 2.2 and Theorem A, it is enough to prove that the inclusion functor $\iota: \text{VI}(R) \rightarrow \text{V}'(R)$ is essentially surjective and finite. Essential surjectivity is clear, so we must only prove finiteness. Consider some $d \geq 0$. We must prove that $\iota^*(P_{\text{V}'(R), R^d})$ is a finitely generated $\text{VI}(R)$ -module. Let N be the number of elements of R^d . Consider some $n \geq N$ and some $\text{V}'(R)$ -morphism $f: R^d \rightarrow R^n$. Viewing f as an $n \times d$ matrix, there are at most N distinct rows occurring in f . This implies that there exists some $g \in \text{GL}_d(R)$ such that the last $(n - N)$ rows of $f \circ g$ are identically 0. From this, we see that f can be factored as

$$R^d \xrightarrow{f'} R^N \xrightarrow{f''} R^n,$$

where f'' is splittable and f' is arbitrary. This implies that we have the decomposition of $\text{VI}(R)$ -modules

$$\iota^*(P_{\text{V}'(R), R^d}) = \bigoplus_{R^d \rightarrow R^{N'}} P_{\text{VI}(R), R^{N'}},$$

where the sum is over all R -linear maps $R^d \rightarrow R^{N'}$ with $N' \leq N$. \square

Remark 2.17. A shorter proof of Theorem A (and hence of Theorems B and C) is given in [SamSn4, §8.3] that deduces it from the locally Noetherian property of the opposite of the category of finite sets and surjective functions. We include the proof above since it follows easily from Theorem D which is the result that we actually need for our applications, and because the ideas for the proof were only made possible by work from both projects. \square

2.5 The category of SI-modules is locally Noetherian

Fix a finite commutative ring R . Our goal in this section is to adapt the proof that the category of $\text{VIC}(R)$ -modules is locally Noetherian to prove that the category of $\text{SI}(R)$ -modules is locally Noetherian. The proof of this is immediately after Theorem 2.20 below.

Row-adapted maps. An R -linear map $f: R^n \rightarrow R^m$ is **row-adapted** if the transpose f^t of the matrix representing f is column-adapted in the sense of §2.2, in which case we

define $S_r(f) = S_c(f^t)$. Lemma 2.9 implies that the composition of two row-adapted maps is row-adapted.

The category OSI. Define $\text{OSI}'(R)$ to be the category whose objects are pairs (R^{2n}, ω) , where ω is a symplectic form on R^{2n} , and whose morphisms from (R^{2n}, ω) to $(R^{2n'}, \omega')$ are R -linear maps $f: R^{2n} \rightarrow R^{2n'}$ which are symplectic and row-adapted. Also, define $\text{OSI}(R)$ to be the full subcategory of $\text{OSI}'(R)$ spanned by the (R^{2n}, ω) such that ω is the standard symplectic form on R^{2n} , i.e., if $\{\mathbf{b}_1, \dots, \mathbf{b}_{2n}\}$ is the standard basis for R^{2n} , then

$$\omega(\mathbf{b}_{2i-1}, \mathbf{b}_{2j-1}) = \omega(\mathbf{b}_{2i}, \mathbf{b}_{2j}) = 0 \quad \text{and} \quad \omega(\mathbf{b}_{2i-1}, \mathbf{b}_{2j}) = \delta_{ij} \quad (1 \leq i, j \leq n),$$

where δ_{ij} is the Kronecker delta. We will frequently omit the ω from objects of $\text{OSI}(R)$.

Factorization of SI-morphisms. The following is the analogue for $\text{SI}(R)$ of Lemma 2.11.

Lemma 2.18. *Let R be a finite commutative ring and $f \in \text{Hom}_{\text{SI}(R)}((R^{2n}, \omega), (R^{2n'}, \omega'))$. Then we can uniquely write $f = f_1 f_2$, where $f_2 \in \text{Iso}_{\text{SI}(R)}((R^{2n}, \omega), (R^{2n}, \lambda))$ for some symplectic form λ on R^{2n} and $f_1 \in \text{Hom}_{\text{OSI}'(R)}((R^{2n}, \lambda), (R^{2n'}, \omega'))$.*

Proof. Applying Lemma 2.10 to the transpose of f and then transposing the result, we can uniquely write $f = f_1 f_2$, where $f_2: R^{2n} \rightarrow R^{2n}$ is an isomorphism and $f_1: R^{2n} \rightarrow R^{2n'}$ is a row-adapted map. There exists a unique symplectic form λ on R^{2n} such that $f_2: (R^{2n}, \omega) \rightarrow (R^{2n}, \lambda)$ is a symplectic map. Since $f: (R^{2n}, \omega) \rightarrow (R^{2n'}, \omega')$ is a symplectic map and $f_1 = f f_2^{-1}$, it follows that $f_1: (R^{2n}, \lambda) \rightarrow (R^{2n'}, \omega')$ is a symplectic map, as desired. \square

A partial order. Our goal now is to prove Theorem 2.20 below, which says that the category of $\text{OSI}(R)$ -modules is locally Noetherian. This requires the existence of a certain partial ordering. For $d \geq 0$ and ω a symplectic form on R^{2d} , define

$$\mathcal{P}_R(d, \omega) = \bigsqcup_{n=0}^{\infty} \text{Hom}_{\text{OSI}'(R)}((R^{2d}, \omega), R^{2n});$$

here we are using our convention that R^{2n} is given the standard symplectic form.

Lemma 2.19. *Let R be a finite commutative ring. Fix $d \geq 0$ and a symplectic form ω on R^{2d} . There exists a well partial ordering \preceq on $\mathcal{P}_R(d, \omega)$ together with an extension \leq of \preceq to a total ordering such that the following holds. Consider $f \in \text{Hom}_{\text{OSI}'(R)}((R^{2d}, \omega), R^{2n})$ and $g \in \text{Hom}_{\text{OSI}'(R)}((R^{2d}, \omega), R^{2n'})$ with $f \preceq g$. Then there is some $\phi \in \text{Hom}_{\text{OSI}(R)}(R^{2n}, R^{2n'})$ with the following two properties.*

1. We have $g = \phi f$.
2. If $f_1 \in \text{Hom}_{\text{OSI}'(R)}((R^{2d}, \omega), R^{2n''})$ satisfies $f_1 < f$, then $\phi f_1 < g$.

We postpone the proof of Lemma 2.19 until the end of this section.

Noetherian theorems. For $d \geq 0$ and ω a symplectic form on R^{2d} , define $Q_{d,\omega}$ to be the $\text{OSI}(R)$ -module obtained by pulling back the $\text{OSI}'(R)$ -module $P_{\text{OSI}'(R), (R^{2d}, \omega)}$, so

$$Q_{d,\omega}(R^{2n}) = \mathbf{k}[\text{Hom}_{\text{OSI}'(R)}((R^{2d}, \omega), R^{2n})].$$

The main consequence of Lemma 2.19 is as follows.

Theorem 2.20. *Let R be a finite commutative ring. Then the category of $\text{OSI}(R)$ -modules is locally Noetherian. Moreover, for $d \geq 0$ and ω a symplectic form on R^{2d} , the $\text{OSI}(R)$ -module $Q_{d,\omega}$ is finitely generated.*

Proof. The proof of Theorem 2.14 can easily be adapted to prove that in fact every submodule of $Q_{d,\omega}$ is finitely generated (replace Lemma 2.13 with Lemma 2.19). If ω is the standard symplectic form on R^{2d} , then $Q_{d,\omega} = P_{\text{OSI}(R), R^{2d}}$. We can thus use Lemma 2.1 to deduce that the category of $\text{OSI}(R)$ -modules is locally Noetherian. \square

We now use Theorem 2.20 to deduce Theorem E, which asserts that the category of $\text{SI}(R)$ -modules is locally Noetherian when R is a finite commutative ring.

Proof of Theorem E. By Lemma 2.2 and Theorem 2.20, it is enough to show that the inclusion functor $\Phi: \text{OSI}(R) \rightarrow \text{SI}(R)$ is finite. Fix $d \geq 0$ and ω a symplectic form on R^{2d} . Set $M = P_{\text{SI}(R), (R^{2d}, \omega)}$, so $M(R^{2n}) = \mathbf{k}[\text{Hom}_{\text{SI}(R)}((R^{2d}, \omega), R^{2n})]$; here we remind the reader of our convention that R^{2n} is endowed with its standard symplectic form. Our goal is to prove that the $\text{OSI}(R)$ -module $\Phi^*(M)$ is finitely generated. For every symplectic form λ on R^{2d} and every element of $\text{Iso}_{\text{SI}(R)}((R^{2d}, \omega), (R^{2d}, \lambda))$, we get a morphism $Q_{d,\lambda} \rightarrow \Phi^*(M)$ in the obvious way. Lemma 2.18 implies that the resulting map

$$\bigoplus_{\lambda \text{ a symplectic form on } R^{2d}} \left(\bigoplus_{\text{Iso}_{\text{SI}(R)}((R^{2d}, \omega), (R^{2d}, \lambda))} Q_{d,\lambda} \right) \longrightarrow \Phi^*(M)$$

is a surjection. Theorem 2.20 implies that each $Q_{d,\lambda}$ is a finitely generated $\text{OSI}(R)$ -module. Since R is finite, this proves that $\Phi^*(M)$ is finitely generated. \square

Insertion maps. For the proof of Lemma 2.19, we will need the following analogue of Lemma 2.15.

Lemma 2.21. *Let R be a commutative local ring. Fix $d \geq 0$, a symplectic form ω on R^{2d} , and some $f \in \text{Hom}_{\text{OSI}'(R)}((R^{2d}, \omega), R^{2n})$ and $g \in \text{Hom}_{\text{OSI}'(R)}((R^{2d}, \omega), R^{2n'})$. Assume that there exists some $I \subset \{1, \dots, n'\} \setminus \{i \mid \text{either } 2i-1 \in S_r(g) \text{ or } 2i \in S_r(g)\}$ such that f can be obtained from g by deleting the rows $J := \{2i-1, 2i \mid i \in I\}$ (observe that this implies that $S_r(f)$ is equal to $S_r(g)$ after removing J from $\{1, \dots, 2n'\}$ and renumbering the rows in order). Let the rows of g be $r_1, \dots, r_{2n'}$. Then there exists $\phi \in \text{Hom}_{\text{OSI}(R)}(R^{2n}, R^{2n'})$ with the following properties.*

- We have $g = \phi f$.
- Let $h \in \text{Hom}_{\text{OSI}(R)}((R^{2d}, \omega), R^{2n})$ be such that $S_r(h) = S_r(f)$. Then ϕh is obtained from h by inserting the row r_j in position j of h for all $j \in J$.
- Let $h \in \text{Hom}_{\text{OSI}(R)}((R^{2d}, \omega), R^{2n})$ be such that $S_r(h) < S_r(f)$ in the lexicographic order. Then $S_r(\phi h) < S_r(g)$ in the lexicographic order.

Proof. Set $J' = \{1, \dots, 2n'\} \setminus J$, and write $J = \{j_1 < \dots < j_{2(n'-n)}\}$ and $J' = \{j'_1 < \dots < j'_{2n}\}$. Next, write $S_r(f) = \{s_1 < \dots < s_{2d}\} \subset \{1, \dots, 2n\}$, and for $1 \leq i \leq 2n'$, let $r_i = (r_i^1, \dots, r_i^{2d})$. Define $\hat{r}_i \in R^{2n}$ to be the element that has r_i^j in position s_j for $1 \leq j \leq 2d$ and zeros elsewhere. Finally, define $\phi: R^{2n} \rightarrow R^{2n'}$ to be the map given by the matrix defined as follows.

- For $1 \leq k \leq 2(n'-n)$, the j_k^{th} row of ϕ is \hat{r}_k .
- For $1 \leq k \leq 2n$, the $(j'_k)^{\text{th}}$ row of ϕ has a 1 in position k and zeros elsewhere.

The fact that g is row-adapted implies that ϕ is row-adapted, and the three listed conclusions of the lemma are easy calculations. The only non-trivial fact that must be proved is that $\phi: R^{2n} \rightarrow R^{2n'}$ is symplectic with respect to the standard symplectic forms ω_n and $\omega_{n'}$ on R^{2n} and $R^{2n'}$, respectively.

Let $x_1, \dots, x_{2n} \in R^{2n'}$ be the columns of ϕ . To prove that ϕ is symplectic with respect to ω_n and $\omega_{n'}$ is equivalent to proving that

$$\omega_{n'}(x_{2i-1}, x_{2j-1}) = \omega_{n'}(x_{2i}, x_{2j}) = 0 \quad \text{and} \quad \omega_{n'}(x_{2i-1}, x_{2j}) = \delta_{ij} \quad (1 \leq i, j \leq n). \quad (2.1)$$

For $1 \leq i \leq 2n$, let $x'_i \in R^{2n'}$ be the element obtained by replacing the j_k^{th} entry of x_i with a 0 for $1 \leq k \leq 2(n' - n)$ and let $x''_i = x_i - x'_i$. By construction, x'_1, \dots, x'_{2n} is the standard basis for $R^{2n'}$ with the basis vectors indexed by J removed, so

$$\omega_{n'}(x'_{2i-1}, x'_{2j-1}) = \omega_{n'}(x'_{2i}, x'_{2j}) = 0 \quad \text{and} \quad \omega_{n'}(x'_{2i-1}, x'_{2j}) = \delta_{ij} \quad (1 \leq i, j \leq n).$$

Also, by construction we have $\omega_{n'}(x'_i, x'_j) = 0$ for all $1 \leq i, j \leq 2n$. It follows that (2.1) is equivalent to showing that $\omega_{n'}(x''_i, x''_j) = 0$ for all $1 \leq i, j \leq 2n$.

The only x''_i that are nonzero are x''_{s_i} for $1 \leq i \leq 2d$, so it is enough to show that $\omega_{n'}(x''_{s_i}, x''_{s_j}) = 0$ for $1 \leq i, j \leq 2d$. Let $y_1, \dots, y_{2d} \in R^{2n'}$ be the columns of g . For $1 \leq i \leq 2d$, let $y'_i \in R^{2n'}$ be the element obtained by replacing the j_k^{th} entry of y_i with a 0 for $1 \leq k \leq 2(n' - n)$ and let $y''_i = y_i - y'_i$. By construction, we have $x''_{s_i} = y''_i$ for $1 \leq i \leq 2d$, so we must show that $\omega_{n'}(y''_i, y''_j) = 0$ for $1 \leq i, j \leq 2d$.

Let $z_1, \dots, z_{2d} \in R^{2n}$ be the columns of f and let $\mathbf{b}_1, \dots, \mathbf{b}_{2d}$ be the standard basis for R^{2d} . Since $f: (R^{2d}, \omega) \rightarrow R^{2n}$ and $g: (R^{2d}, \omega) \rightarrow R^{2n'}$ are symplectic maps, it follows that

$$\omega_n(z_i, z_j) = \omega(\mathbf{b}_i, \mathbf{b}_j) \quad \text{and} \quad \omega_{n'}(y_i, y_j) = \omega(\mathbf{b}_i, \mathbf{b}_j) \quad (1 \leq i, j \leq 2d). \quad (2.2)$$

Using the fact that f is obtained by deleting appropriate pairs of rows of g , we see that

$$\omega_n(z_i, z_j) = \omega_{n'}(y'_i, y'_j) \quad (1 \leq i, j \leq 2d). \quad (2.3)$$

Combining (2.2) and (2.3), we see that $\omega_{n'}(y_i, y_j) = \omega_{n'}(y'_i, y'_j)$ for all $1 \leq i, j \leq 2d$. By construction, we have

$$\omega_{n'}(y_i, y'_j) = \omega_{n'}(y'_i, y_j) = \omega_{n'}(y'_i, y'_j) \quad (1 \leq i, j \leq 2d).$$

We conclude that for $1 \leq i, j \leq 2d$ we have $\omega_{2n'}(y''_i, y''_j)$ equal to

$$\begin{aligned} \omega_{n'}(y_i - y'_i, y_j - y'_j) &= \omega_{n'}(y_i, y_j) - \omega_{n'}(y'_i, y_j) - \omega_{n'}(y_i, y'_j) + \omega_{n'}(y'_i, y'_j) \\ &= 2\omega_{n'}(y'_i, y'_j) - 2\omega_{n'}(y'_i, y'_j) = 0, \end{aligned}$$

as desired. \square

Constructing the orders. We now prove Lemma 2.19.

Proof of Lemma 2.19. The general case of Lemma 2.19 can easily be deduced from the case where R is a local ring (the argument is identical to that for Lemma 2.13), so we can assume that R is a local ring. The first step is to construct \preceq . Consider $f \in \text{Hom}_{\text{OSI}(R)}((R^{2d}, \omega), R^{2n})$ and $g \in \text{Hom}_{\text{OSI}(R)}((R^{2d}, \omega), R^{2n'})$. We will say that $f \preceq g$ if f can be obtained from g by deleting the rows $\{2i - 1, 2i \mid i \in I\}$ for some set

$$I \subset \{1, \dots, n'\} \setminus \{i \mid \text{either } 2i - 1 \in S_r(f) \text{ or } 2i \in S_r(f)\}.$$

This clearly defines a partial order on $\mathcal{P}_R(d, \omega)$.

Claim. *The partially ordered set $(\mathcal{P}_R(d, \omega), \preceq)$ is well partially ordered.*

Proof of Claim. Define

$$\Sigma = \left(R^d \amalg \{\spadesuit\} \right) \times \left(R^d \amalg \{\spadesuit\} \right),$$

where \spadesuit is a formal symbol. We will prove that $(\mathcal{P}_R(d, \omega), \preceq)$ is isomorphic to a subposet of the well partially ordered poset Σ^* from Lemma 2.5, so by Lemma 2.4(a) is itself well partially ordered. Consider $f \in \text{Hom}_{\text{OSI}'(R)}((R^{2d}, \omega), R^{2n})$. For $1 \leq i \leq 2n$, let

$$r_i = \begin{cases} \text{the } i^{\text{th}} \text{ row of } f & \text{if } i \notin S_r(f), \\ \spadesuit & \text{if } i \in S_r(f). \end{cases}$$

Thus $r_i \in R^{2d} \amalg \{\spadesuit\}$. For $1 \leq i \leq n$, let $\theta_i = (r_{2i-1}, r_{2i}) \in \Sigma$. We thus get an element $\theta_1 \cdots \theta_n \in \Sigma^*$. We claim that the map taking f to $\theta_1 \cdots \theta_n$ is an injection from $\mathcal{P}_R(d, \omega)$ to Σ^* . To see this, consider an element $\theta_1 \cdots \theta_n \in \Sigma^*$ in its image. Our goal is to reconstruct f . Write $\theta_i = (r_{2i-1}, r_{2i})$, and set $S = \{1 \leq i \leq 2n \mid r_i = \spadesuit\}$. We can read off all the rows of f except for those whose index lies in S . But we know that $S = S_r(f)$, so rows whose index lies in S are precisely the standard basis vectors in R^d , appearing in their natural order. We have thus reconstructed all of f , as claimed. That this injection is order-preserving is immediate from the definitions. The only subtle point is the reason we used the \spadesuit rather than just using the basis vectors in R^d . The point here is that in the definition of the partial order on $\mathcal{P}_R(d, \omega)$, we were not allowed to delete rows in $S_r(f)$, so we have to distinguish them in Σ . \square

We now extend \preceq to a total order \leq . Fix some arbitrary total order on R^{2d} , and consider $f \in \text{Hom}_{\text{OSI}'(R)}((R^{2d}, \omega), R^{2n})$ and $g \in \text{Hom}_{\text{OSI}'(R)}((R^{2d}, \omega), R^{2n'})$. We then determine if $f \leq g$ via the following procedure. Assume without loss of generality that $f \neq g$.

- If $n < n'$, then $f < g$.
- Otherwise, $n = n'$. If $S_r(f) < S_r(g)$ in the lexicographic ordering, then $f < g$.
- Otherwise, $S_r(f) = S_r(g)$. Compare the sequences of elements of R^{2d} which form the rows of f and g using the lexicographic ordering and the fixed total ordering on R^{2d} .

It is clear that this determines a total order \leq on $\mathcal{P}_R(d, \omega)$ that extends \preceq .

It remains to check that these orders have the property claimed by the lemma. Consider $f \in \text{Hom}_{\text{OSI}'(R)}((R^{2d}, \omega), R^{2n})$ and $g \in \text{Hom}_{\text{OSI}'(R)}((R^{2d}, \omega), R^{2n'})$ with $f \preceq g$. By Lemma 2.21, there exists some $\phi \in \text{Hom}_{\text{OSI}(R)}(R^{2n}, R^{2n'})$ such that $g = \phi f$. Now consider $f_1 \in \text{Hom}_{\text{OSI}'(R)}((R^{2d}, \omega), R^{2n})$ such that $f_1 < f$. We want to show that $\phi f_1 < \phi f$. If $S_r(f_1) < S_r(f)$ in the lexicographic order, then this is an immediate consequence of Lemma 2.21, so we can assume that $S_r(f_1) = S_r(f)$. In this case, it follows from Lemma 2.21 that ϕf_1 and ϕf are obtained by inserting the same rows into the same places in f_1 and f , respectively. It follows immediately that $\phi f_1 < \phi f$, as desired. \square

2.6 Counterexamples

In this section, we prove Theorem N, which asserts that the category of \mathbb{C} -modules is not locally Noetherian for $\mathbb{C} \in \{\text{VI}(R), \text{VIC}(R), \text{SI}(R)\}$ when R is an infinite ring. This requires the following lemma.

Lemma 2.22. *Let \mathbf{k} be a nonzero ring and Γ be a group that contains a non-finitely generated subgroup. Then the group ring $\mathbf{k}[\Gamma]$ is not Noetherian.*

Proof. Given a subgroup $H \subseteq \Gamma$, the kernel of the surjection $\mathbf{k}[\Gamma] \rightarrow \mathbf{k}[\Gamma/H]$ is a left ideal $I_H \subset \mathbf{k}[\Gamma]$. Clearly $H \subsetneq H'$ implies $I_H \subsetneq I_{H'}$, so the existence of an infinite ascending chain

of left ideals in $\mathbf{k}[\Gamma]$ follows from the existence of an infinite ascending chain of subgroups of Γ . The existence of this is an immediate consequence of the existence of a non-finitely generated subgroup of Γ . \square

Proof of Theorem N. For all of our choices of \mathfrak{C} , the group $\Gamma := \text{Aut}_{\mathfrak{C}}(R^2)$ contains $\text{SL}_2(R)$. We claim that $\text{SL}_2(R)$ contains a non-finitely generated subgroup. There are two cases. If $\mathbb{Z} \subseteq R$, then $\text{SL}_2(R)$ contains a rank 2 free subgroup. For instance, one can take the subgroup generated by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

This free subgroup contains a non-finitely generated subgroup. If $\mathbb{Z} \not\subseteq R$, then the additive group of R is annihilated by some positive integer ℓ . Since R is infinite, this implies that the additive group of R is not finitely generated. The claim now follows from the fact that the additive group of R is a subgroup of $\text{SL}_2(R)$ via the embedding

$$r \mapsto \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}.$$

By Lemma 2.22, the claim implies that $\mathbf{k}[\Gamma]$ is not Noetherian. Define a \mathfrak{C} -module M via the formula

$$M_V = \begin{cases} \mathbf{k}[\text{Aut}_{\mathfrak{C}}(V)] & \text{if } V \cong \mathbb{Z}^2, \\ 0 & \text{otherwise.} \end{cases}$$

The morphisms are the obvious ones. It is then clear that M is finitely generated but contains non-finitely generated submodules. \square

3 Partial resolutions and representation stability

This section contains the machinery that we will use to prove our asymptotic structure theorem. It contains three sections. Recall the definition of a (weak) complemented category from §1.4. In §3.1, we will discuss automorphisms of complemented categories. Next, in §3.2 we will state Theorem 3.7, which gives a certain partial resolution for finitely generated modules over a complemented category. Theorem 3.7 is proven in §3.3. At the end of §3.3, we will derive our asymptotic structure theorem (Theorem F) from Theorem 3.7.

The arguments in this section are inspired by [ChEFaNag]. The formal manipulations in §3.1 are also reminiscent of some of those appearing in [WahRW].

3.1 Automorphism groups of complemented categories

Objects in complemented categories equipped with a generator have rich automorphism groups. For instance, we have the following.

Lemma 3.1. *Let (\mathbf{A}, \otimes) be a complemented category with generator X . For $V, V' \in \mathbf{A}$, the group $\text{Aut}_{\mathbf{A}}(V')$ acts transitively on the set $\text{Hom}_{\mathbf{A}}(V, V')$.*

Proof. Without loss of generality, $V' = X^k$ for some $k \geq 0$. Consider morphisms $f, g: V \rightarrow X^k$. Let C and D be the complements to $f(V)$ and $g(V)$, respectively, and let $\phi: V \otimes C \rightarrow X^k$ and $\phi': V \otimes D \rightarrow X^k$ be the associated isomorphisms. Writing $V \cong X^\ell$, we then have $C, D \cong X^{k-\ell}$; in particular, there is an isomorphism $\eta: C \rightarrow D$. The composition

$$X^k \xrightarrow{\phi^{-1}} V \otimes C \xrightarrow{\text{id} \otimes \eta} V \otimes D \xrightarrow{\phi'} X^k$$

is then an isomorphism $\Psi: X^k \rightarrow X^k$ such that $\Psi \circ \phi: V \otimes C \rightarrow X^k$ is an isomorphism that restricts to $g: V \rightarrow X^k$, so $\Psi \circ f = g$. \square

Lemma 3.2. *Let (\mathbf{A}, \otimes) be a complemented category, let $V, V' \in \mathbf{A}$, and define $\varphi: \text{Aut}_{\mathbf{A}}(V) \times \text{Aut}_{\mathbf{A}}(V') \rightarrow \text{Aut}_{\mathbf{A}}(V \otimes V')$ by the formula $\varphi(f, g) = f \otimes g$. Then the following hold.*

- (a) *The homomorphism φ is injective.*
- (b) *Let $i: V \rightarrow V \otimes V'$ and $i': V' \rightarrow V \otimes V'$ be the canonical maps. Then*

$$\begin{aligned} \{\eta \in \text{Aut}_{\mathbf{A}}(V \otimes V') \mid \eta \circ i' = i'\} &= \varphi(\text{Aut}_{\mathbf{A}}(V) \times \text{id}_{V'}) \cong \text{Aut}_{\mathbf{A}}(V), \\ \{\eta \in \text{Aut}_{\mathbf{A}}(V \otimes V') \mid \eta \circ i = i\} &= \varphi(\text{id}_V \times \text{Aut}_{\mathbf{A}}(V')) \cong \text{Aut}_{\mathbf{A}}(V'). \end{aligned}$$

Proof. (a) Consider $f_0 \in \text{Aut}_{\mathbf{A}}(V)$ and $g_0 \in \text{Aut}_{\mathbf{A}}(V')$ such that $\varphi(f_0, g_0) = \text{id}_{V \otimes V'}$. The proofs that $f_0 = \text{id}_V$ and $g_0 = \text{id}_{V'}$ are similar, so we will only give the details for f_0 . Let $\mathbb{1}$ be the identity of (\mathbf{A}, \otimes) and let $u: \mathbb{1} \rightarrow V'$ be the unique map (by assumption, $\mathbb{1}$ is initial). Since $g_0 \circ u = u$, we have a commutative diagram

$$\begin{array}{ccc} V \otimes \mathbb{1} & \xrightarrow{\text{id}_V \otimes u} & V \otimes V' \\ \downarrow f_0 \otimes \text{id}_{\mathbb{1}} & & \downarrow f_0 \otimes g_0 \\ V \otimes \mathbb{1} & \xrightarrow{\text{id}_V \otimes u} & V \otimes V'. \end{array} \quad (3.1)$$

Since $f_0 \otimes g_0 = \text{id}_{V \otimes V'}$, we deduce that $(\text{id}_V \otimes u) \circ (f_0 \otimes \text{id}_{\mathbb{1}}) = \text{id}_V \otimes u$. Like all \mathbf{A} -morphisms, $\text{id}_V \otimes u$ is a monomorphism, so $f_0 \otimes \text{id}_{\mathbb{1}} = \text{id}_{V \otimes \mathbb{1}}$, and hence $f_0 = \text{id}_V$, as desired.

- (b) The isomorphisms are a consequence of (a). We next show that

$$\begin{aligned} \varphi(\text{Aut}_{\mathbf{A}}(V) \times \text{id}_{V'}) &\subset \{\eta \in \text{Aut}_{\mathbf{A}}(V \otimes V') \mid \eta \circ i' = i'\}, \\ \varphi(\text{id}_V \times \text{Aut}_{\mathbf{A}}(V')) &\subset \{\eta \in \text{Aut}_{\mathbf{A}}(V \otimes V') \mid \eta \circ i = i\}. \end{aligned}$$

The proofs are similar, so we will only give the details for $\varphi(\text{id}_V \times \text{Aut}_{\mathbf{A}}(V'))$. Consider $g_1 \in \text{Aut}_{\mathbf{A}}(V')$. The map $\varphi(\text{id}_V, g_1) \circ i$ is the composition of the canonical isomorphism $V \cong V \otimes \mathbb{1}$ with the composition

$$V \otimes \mathbb{1} \xrightarrow{\text{id}_V \otimes u} V \otimes V' \xrightarrow{\text{id}_V \otimes g_1} V \otimes V'.$$

The uniqueness of u implies that this composition equals $\text{id}_V \otimes (g_1 \circ u) = \text{id}_V \otimes u$, so $\varphi(\text{id}_V, g_1) \circ i = i$, as desired.

We next show that

$$\begin{aligned} \varphi(\text{Aut}_{\mathbf{A}}(V) \times \text{id}_{V'}) &\supset \{\eta \in \text{Aut}_{\mathbf{A}}(V \otimes V') \mid \eta \circ i' = i'\}, \\ \varphi(\text{id}_V \times \text{Aut}_{\mathbf{A}}(V')) &\supset \{\eta \in \text{Aut}_{\mathbf{A}}(V \otimes V') \mid \eta \circ i = i\}. \end{aligned}$$

The proofs are similar, so we will only give the details for $\varphi(\text{id}_V \times \text{Aut}_{\mathbf{A}}(V'))$. Consider $h_2 \in \text{Aut}_{\mathbf{A}}(V \otimes V')$ such that $h_2 \circ i = i$. We wish to find some $g_2 \in \text{Aut}_{\mathbf{A}}(V')$ such that $\varphi(\text{id}_V, g_2) = h_2$. Observe that $i'(V')$ is a complement to $i(V)$ and $h_2 \circ i'(V')$ is a complement to $h_2 \circ i(V) = i(V)$. By the uniqueness of complements, there must exist $g_2 \in \text{Aut}_{\mathbf{A}}(V')$ such that $h_2 \circ i' = i' \circ g_2$. Under the injection

$$\text{Hom}_{\mathbf{A}}(V \otimes V', V \otimes V') \hookrightarrow \text{Hom}_{\mathbf{A}}(V, V \otimes V') \times \text{Hom}_{\mathbf{A}}(V', V \otimes V'),$$

both h_2 and $\text{id}_V \otimes g_2$ map to the same thing, namely $(i, i' \circ g_2)$. We conclude that $h_2 = \text{id}_V \otimes g_2 = \varphi(\text{id}_V, g_2)$, as desired. \square

3.2 Recursive presentations and partial resolutions

Fix a complemented category (\mathbf{A}, \otimes) with a generator X and fix an \mathbf{A} -module M . We wish to study several chain complexes associated to M .

Shift functor. Consider some $p \geq 0$ and some $V \in \mathbf{A}$. Define

$$\mathfrak{J}_{V,p} = \text{Hom}_{\mathbf{A}}(X^p, V).$$

This might be the empty set. The groups $\text{Aut}(X)^p$ and \mathfrak{S}_p and the wreath product $\mathfrak{S}_p \wr \text{Aut}(X) = \mathfrak{S}_p \times \text{Aut}(X)^p$ all act on X^p and induce free actions on $\mathfrak{J}_{V,p}$. Set

$$\mathfrak{J}'_{V,p} = \mathfrak{J}_{V,p} / \text{Aut}(X)^p, \quad \mathfrak{J}''_{V,p} = \mathfrak{J}_{V,p} / \mathfrak{S}_p, \quad \mathfrak{J}'''_{V,p} = \mathfrak{J}_{V,p} / (\mathfrak{S}_p \wr \text{Aut}(X)).$$

For $h \in \mathfrak{J}_{V,p}$, let $W_h \subset V$ be the complement of $h(X^p)$. We then define an \mathbf{A} -module $\Sigma_p M$ via the formula

$$(\Sigma_p M)_V = \bigoplus_{h \in \mathfrak{J}_{V,p}} M_{W_h} \quad (V \in \mathbf{A}).$$

As far as morphisms go, consider a morphism $f: V \rightarrow V'$ and some $h \in \mathfrak{J}_{V,p}$. We then have $f \circ h \in \mathfrak{J}_{V',p}$. Moreover, letting $C \subset V'$ be the complement of $f(V)$, there is a natural isomorphism $V \otimes C \cong V'$. Under this isomorphism, we can identify $W_{f \circ h}$ with $W_h \otimes C$. The canonical morphism $W_h \rightarrow W_h \otimes C$ thus induces a map $M_{W_h} \rightarrow M_{W_{f \circ h}}$. This allows us to define (component by component) a morphism $(\Sigma_p M)_f: (\Sigma_p M)_V \rightarrow (\Sigma_p M)_{V'}$. We will identify $\Sigma_0 M$ with M in the obvious way. As notation, we will write $(\Sigma_p M)_{V,h}$ for the term of $(\Sigma_p M)_V$ associated to $h \in \mathfrak{J}_{V,p}$.

Define $\Sigma'_p M$ and $\Sigma''_p M$ and $\Sigma'''_p M$ by summing over $\mathfrak{J}'_{V,p}$ and $\mathfrak{J}''_{V,p}$ and $\mathfrak{J}'''_{V,p}$, rather than $\mathfrak{J}_{V,p}$. These are quotients of $\mathfrak{J}_{V,p}$ as \mathbf{A} -modules by the submodules spanned by $(x, (-1)^g x)$ where $x \in M_{W,h}$ and $(-1)^g x \in M_{W, h \circ g}$ for $g \in \text{Aut}(X)^p$ or $g \in \mathfrak{S}_p$ or $g \in \mathfrak{S}_p \wr \text{Aut}(X)$, respectively, and $(-1)^g$ refers to the sign of the \mathfrak{S}_p -component of g in the second and third case (and is defined to be 1 in the first case).

Lemma 3.3. *If (\mathbf{A}, \otimes) is a complemented category with generator X and if M is a finitely generated \mathbf{A} -module, then $\Sigma_p M$ and $\Sigma'_p M$ and $\Sigma''_p M$ and $\Sigma'''_p M$ are finitely generated \mathbf{A} -modules for all $p \geq 0$.*

Proof. It is enough to prove this for $\Sigma_p M$ since this surjects onto the others.

There exist $V_1, \dots, V_m \in \mathbf{A}$ and $x_i \in M_{V_i}$ for $1 \leq i \leq m$ such that M is generated by $\{x_1, \dots, x_m\}$. For all $V \in \mathbf{A}$, let $h_V: X^p \rightarrow V \otimes X^p$ be the canonical morphism, so $(\Sigma_p M)_{V \otimes X^p, h_V} = M_V$. For $1 \leq i \leq m$, define $\bar{x}_i \in (\Sigma_p M)_{V_i \otimes X^p}$ to be the element

$$x_i \in (\Sigma_p M)_{V_i \otimes X^p, h_{V_i}} \subset (\Sigma_p M)_{V_i \otimes X^p}.$$

We claim that $\Sigma_p M$ is generated by $\bar{x}_1, \dots, \bar{x}_m$. Indeed, let N be the submodule of $\Sigma_p M$ generated by the indicated elements. Consider $V \in \mathbf{A}$. Our goal is to show that $N_V = (\Sigma_p M)_V$. If $(\Sigma_p M)_V \neq 0$, then $V \cong X^q$ for some $q \geq p$, and it is enough to show that $N_{X^q} = (\Sigma_p M)_{X^q}$. It is clear that $(\Sigma_p M)_{X^{q-p} \otimes X^p, h_{X^{q-p}}} \subset N_{X^q}$. Lemma 3.1 implies that $\text{Aut}_{\mathbf{A}}(X^q)$ acts transitively on $\mathfrak{J}_{X^q,p}$. We conclude that N_{X^q} contains $(\Sigma_p M)_{X^q,h}$ for all $h \in \mathfrak{J}_{X^q,p}$, so $N_{X^q} = M_{X^q}$. \square

Remark 3.4. The operation Σ_p is an exact functor on the category of \mathbf{A} -modules: given a morphism $f: M \rightarrow N$, there is a morphism $\Sigma_p f: \Sigma_p M \rightarrow \Sigma_p N$ defined in the obvious way, and the same is true for Σ'_p and Σ''_p and Σ'''_p . Note that Σ_p can be identified with the p^{th} iterate of Σ_1 , and a similar statement holds for Σ'_p , but not for Σ''_p or Σ'''_p . \square

Chain complex. For $p \geq 1$, we now define a morphism $d: \Sigma_p M \rightarrow \Sigma_{p-1} M$ of \mathbf{A} -modules as follows. Consider $V \in \mathbf{A}$. For $1 \leq i \leq p$, let $t_i: X \rightarrow X^p$ be the canonical morphism of the i^{th} term and let $s_i: X^{p-1} \rightarrow X^p$ be the morphism that corresponds to the $(p-1)$ -tuple $(t_1, \dots, \widehat{t_i}, \dots, t_p)$ under the isomorphism

$$\text{Hom}_{\mathbf{A}}(X^{p-1}, X^p) \cong \text{Hom}_{\mathbf{A}}(X, X^p) \times \dots \times \text{Hom}_{\mathbf{A}}(X, X^p)$$

from the definition of a complemented category. For $h \in \mathfrak{J}_{V,p}$, we have $h \circ s_i \in \mathfrak{J}_{V,p-1}$. Moreover, we can identify $W_{h \circ s_i}$ with $W_h \otimes (h \circ t_i(X))$. Define $d_i: \Sigma_p M \rightarrow \Sigma_{p-1} M$ to be the morphism that takes $(\Sigma_p M)_{V,h} = M_{W_h}$ to $(\Sigma_{p-1} M)_{V,h \circ s_i} = M_{W_h \otimes (h \circ t_i(X))}$ via the map induced by the canonical morphism $W_h \rightarrow W_h \otimes (h \circ t_i(X))$. We then define $d = \sum_{i=1}^p (-1)^{i-1} d_i$. The usual argument shows that $d \circ d = 0$, so we have defined a chain complex of \mathbf{A} -modules:

$$\Sigma_* M: \dots \rightarrow \Sigma_3 M \rightarrow \Sigma_2 M \rightarrow \Sigma_1 M \rightarrow M.$$

The differentials factor through the quotients $\Sigma'_p M$ and $\Sigma''_p M$ and $\Sigma'''_p M$, so we get complexes

$$\begin{aligned} \Sigma'_* M &: \dots \rightarrow \Sigma'_3 M \rightarrow \Sigma'_2 M \rightarrow \Sigma'_1 M \rightarrow M, \\ \Sigma''_* M &: \dots \rightarrow \Sigma''_3 M \rightarrow \Sigma''_2 M \rightarrow \Sigma''_1 M \rightarrow M, \\ \Sigma'''_* M &: \dots \rightarrow \Sigma'''_3 M \rightarrow \Sigma'''_2 M \rightarrow \Sigma'''_1 M \rightarrow M. \end{aligned}$$

Remark 3.5. Let \mathbf{A}° be the underlying groupoid of \mathbf{A} , i.e., the subcategory where we take all objects of \mathbf{A} and only keep the isomorphisms. Using the complemented structure of \mathbf{A} , there is a natural symmetric monoidal structure on \mathbf{A}° -modules defined by

$$(F \otimes G)(V) = \bigoplus_{V=V' \otimes V''} F(V') \otimes_{\mathbf{k}} G(V'').$$

So we can define commutative algebras and their modules. Define an \mathbf{A}° -module A_1 by $V \mapsto \mathbf{k}$ if V is isomorphic to the generator of \mathbf{A} and $V \mapsto 0$ otherwise and set $\tilde{A} = \text{Sym}(A_1)$ where Sym denotes the free symmetric algebra. Then

$$\tilde{A}(V) = \bigoplus_{\substack{\{L_1, \dots, L_n\} \\ V=L_1 \otimes \dots \otimes L_n}} \mathbf{k}$$

where the sum is over all unordered decompositions of V into rank 1 subspaces. There is a quotient A of \tilde{A} given by $V \mapsto \mathbf{k}$ for all V (identify all summands above), and the category of \mathbf{A} -modules is equivalent to the category of A -modules: a map $A \otimes M \rightarrow M$ is equivalent to giving a map $M(V') \rightarrow M(V)$ for each decomposition $V = V' \otimes V''$, and the associativity of multiplication is equivalent to these maps being compatible with composition.

Under this interpretation, $\Sigma'_* M$ becomes

$$\dots \rightarrow A_1^{\otimes 3} \otimes M \rightarrow A_1^{\otimes 2} \otimes M \rightarrow A_1 \otimes M \rightarrow M,$$

while $\Sigma'''_* M$ is the Koszul complex of M thought of as an \tilde{A} -module:

$$\dots \rightarrow \bigwedge^3 A_1 \otimes M \rightarrow \bigwedge^2 A_1 \otimes M \rightarrow A_1 \otimes M \rightarrow M. \quad \square$$

Relation to finite generation. We pause now to make the following observation.

Lemma 3.6. *Let (\mathbf{A}, \otimes) be a complemented category with generator X and let M be an \mathbf{A} -module over a ring \mathbf{k} . Assume that M_V is a finitely generated \mathbf{k} -module for all $V \in \mathbf{A}$. Then M is finitely generated if and only if there exists some $N \geq 0$ such that the map $d: (\Sigma_1 M)_V \rightarrow M_V$ is surjective for all $V \in \mathbf{A}$ whose X -rank is at least N .*

Proof. If M is generated by elements x_1, \dots, x_k with $x_i \in M_{V_i}$, then we can take N to be the maximal X -rank of V_1, \dots, V_k plus 1. Indeed, if $V \in \mathbf{A}$ has X -rank at least N , then every morphism $V_i \rightarrow V$ factors through W for some $W \subset V$ whose X -rank is one less than that of V , which implies that the map $d: (\Sigma_1 M)_V \rightarrow M_V$ is surjective. Conversely, if such an N exists, then for a generating set we can combine generating sets for M_{X^i} for $0 \leq i \leq N-1$ to get a generating set for M . \square

Partial resolutions and representation stability. We finally come to our main theorem which will be used in the proof of Theorem F.

Theorem 3.7. *Let (\mathbf{A}, \otimes) be a complemented category with generator X . Assume that the category of \mathbf{A} -modules is locally Noetherian, and let M be a finitely generated \mathbf{A} -module. Fix some $q \geq 1$. If the X -rank of $V \in \mathbf{A}$ is sufficiently large, then the chain complex*

$$(\Sigma_q M)_V \longrightarrow (\Sigma_{q-1} M)_V \longrightarrow \cdots \longrightarrow (\Sigma_1 M)_V \longrightarrow M_V \longrightarrow 0$$

is exact. The same holds if we replace Σ_p by Σ'_p or Σ''_p or Σ'''_p .

The proof of Theorem 3.7 is in §3.3 after some preliminaries.

Remark 3.8. The different resolutions given by Theorem 3.7 are useful in different contexts. The resolution Σ_* will be used in our finite generation machine in §5 (though the other resolutions could also be used at the cost of complicating the necessary spaces). For $\mathbf{A} = \mathbf{FI}$, the resolution Σ''_* is a version of the “central stability chain complex” from [Pu2]; this also played an important role in [ChEFaNag]. Finally, the resolution Σ'''_* is the “most efficient” of our resolutions in the sense that the terms are smallest. \square

3.3 Proof of Theorem 3.7

We begin with some preliminary results needed for the proof of Theorem 3.7.

Torsion submodule. If M is an \mathbf{A} -module, then define the **torsion submodule** of M , denoted $T(M)$, to be the submodule defined via the formula

$$T(M)_V = \{x \in M_V \mid \text{there exists a morphism } f: V \rightarrow W \text{ with } M_f(x) = 0\}.$$

Lemma 3.9. *Let (\mathbf{A}, \otimes) be a complemented category with generator X and let M be an \mathbf{A} -module. Then $T(M)$ is an \mathbf{A} -submodule of M .*

Proof. Fix $V \in \mathbf{A}$. It suffices to show that $T(M)_V$ is closed under addition. Given $x, x' \in T(M)_V$, pick morphisms $f: V \rightarrow W$ and $f': V \rightarrow W'$ such that $M_f(x) = 0$ and $M_{f'}(x') = 0$. Let $g: W \rightarrow W \otimes W'$ and $g': W' \rightarrow W \otimes W'$ be the canonical morphisms. Lemma 3.1 implies that there exists some $h \in \text{Aut}_{\mathbf{A}}(W \otimes W')$ such that $h \circ g \circ f = g' \circ f'$. This morphism kills both x and x' , and hence also $x + x'$. \square

Lemma 3.10. *Let (\mathbf{A}, \otimes) be a complemented category with generator X . Assume that the category of \mathbf{A} -modules is locally Noetherian and let M be a finitely generated \mathbf{A} -module. Then there exists some $N \geq 0$ such that if $V \in \mathbf{A}$ has X -rank at least N , then $T(M)_V = 0$.*

Proof. Combining Lemma 3.9 with our locally Noetherian assumption, we get that $T(M)$ is finitely generated, say by elements x_1, \dots, x_p with $x_i \in T(M)_{V_i}$ for $1 \leq i \leq p$. Let $f_i: V_i \rightarrow W_i$ be a morphism such that $M_{f_i}(x_i) = 0$. Define $W = W_1 \otimes \cdots \otimes W_p$, and consider some $1 \leq i \leq p$. Define $\tilde{f}_i: V_i \rightarrow W$ to be the composition of f_i with the canonical morphism $W_i \hookrightarrow W$. It is then clear that $M_{\tilde{f}_i}(x_i) = 0$. If N is the X -rank of W , we claim that $T(M)_V = 0$ whenever $V \in \mathbf{A}$ has X -rank at least N . Indeed, every morphism $g: V_i \rightarrow V$ can be factored as

$$V_i \xrightarrow{\tilde{f}_i} W \longrightarrow V,$$

and thus $M_g(x_i) = 0$. Since the x_i generate $T(M)$, it follows that $T(M)_V = 0$. \square

Torsion homology. Since $\Sigma_* M$ is a chain complex of \mathbf{A} -modules, we can take its homology and obtain an \mathbf{A} -module $H_q(\Sigma_* M)$ for each $q \geq 0$. We then have the following.

Lemma 3.11. *Let (\mathbf{A}, \otimes) be a complemented category with generator X . Fix some $q \geq 0$, and consider $V \in \mathbf{A}$. Let $\iota: V \rightarrow V \otimes X$ be the canonical morphism. Then for all $x \in (H_q(\Sigma_* M))_V$, we have $(H_q(\Sigma_* M))_\iota(x) = 0$. In particular, the same is true for any morphism f that maps V to W with larger X -rank. The same holds if we replace Σ_* by Σ'_* or Σ''_* or Σ'''_* .*

Proof. For Σ_* , the relevant map on homology is induced by a map

$$\begin{array}{ccccccc} \cdots & \longrightarrow & (\Sigma_2 M)_{V \otimes X} & \longrightarrow & (\Sigma_1 M)_{V \otimes X} & \longrightarrow & M_{V \otimes X} \longrightarrow 0 \\ & & \uparrow I & & \uparrow I & & \uparrow I \\ \cdots & \longrightarrow & (\Sigma_2 M)_V & \longrightarrow & (\Sigma_1 M)_V & \longrightarrow & M_V \longrightarrow 0 \end{array}$$

of chain complexes. We will prove that I is chain homotopic to the zero map. The chain homotopy will descend to a similar chain homotopy on Σ'_* and Σ''_* and Σ'''_* , so this will imply the lemma.

Let $\iota: V \rightarrow V \otimes X$ be the canonical morphism. Recall that

$$(\Sigma_p M)_V = \bigoplus_{h \in \mathcal{J}_{V,p}} M_{W_h} \quad \text{and} \quad (\Sigma_p M)_{V \otimes X} = \bigoplus_{h \in \mathcal{J}_{V \otimes X,p}} M_{W_h}.$$

The map $I: (\Sigma_p M)_V \rightarrow (\Sigma_p M)_{V \otimes X}$ takes $(\Sigma_p M)_{V,h} = M_{W_h}$ to $(\Sigma_p M)_{V \otimes X, \iota \circ h} = M_{W_h \otimes X}$ via the map induced by the canonical morphism $W \rightarrow W \otimes X$.

We now define a chain homotopy map

$$G: (\Sigma_p M)_V \rightarrow (\Sigma_{p+1} M)_{V \otimes X}$$

as follows. Given $h \in \mathcal{J}_{V,p} = \text{Hom}_{\mathbf{A}}(X^p, V)$, define $\bar{h}: X^{p+1} \rightarrow V \otimes X$ to be the composition

$$X^{p+1} = X \otimes X^p \longrightarrow X^p \otimes X \xrightarrow{h \otimes \text{id}} V \otimes X,$$

where the first arrow flips the two factors using the symmetric monoidal structure on \mathbf{A} . We then have $\bar{h} \in \mathcal{J}_{V \otimes X, p+1}$. Since

$$(\Sigma_p M)_{V,h} = (\Sigma_{p+1} M)_{V \otimes X, \bar{h}} = M_{W_h},$$

we can define G on $(\Sigma_p M)_{V,h}$ to be the identity map $(\Sigma_p M)_{V,h} = (\Sigma_{p+1} M)_{V \otimes X, \bar{h}}$.

We now claim that $dG + Gd = I$. Indeed, on $(\Sigma_p M)_V$ the map $dG + Gd$ takes the form

$$\left(\sum_{i=1}^p (-1)^{i-1} Gd_i \right) + \left(\sum_{j=1}^{p+1} (-1)^{j-1} d_j G \right).$$

Straightforward calculations show that $d_1 G = I$ and that $Gd_i = d_{i+1} G$ for $1 \leq i \leq p$. \square

Endgame. All the ingredients are now in place for the proof of Theorem 3.7.

Proof of Theorem 3.7. For all $i \geq 0$, Lemma 3.3 implies that the \mathbf{A} -module $\Sigma_i M$ is finitely generated. Our assumption that the category of \mathbf{A} -modules is locally Noetherian then implies that $H_i(\Sigma_* M)$ is finitely generated. Lemma 3.11 implies that for all $i \geq 0$, we have

$$T(H_i(\Sigma_* M)) = H_i(\Sigma_* M).$$

The upshot of all of this is that we can apply Lemma 3.10 to obtain some $N_q \geq 0$ such that if $V \in \mathbf{A}$ has X -rank at least N_q and $0 \leq i \leq q$, then $H_i(\Sigma_* M)_V = 0$. In other words, the chain complex

$$(\Sigma_q M)_V \longrightarrow (\Sigma_{q-1} M)_V \longrightarrow \cdots \longrightarrow (\Sigma_1 M)_V \longrightarrow M_V \longrightarrow 0$$

is exact, as desired. For Σ'_* and Σ''_* and Σ'''_* , the proof is the same. \square

Asymptotic structure theorem. We conclude by proving Theorem F (the main argument here is very similar to that of [ChPu, Lemma 2.23]).

Proof of Theorem F. Injective representation stability follows from Lemma 3.10, and surjective representation stability is immediate from finite generation. All that remains to prove is central stability. Using Theorem 3.7, choose N large enough so that the chain complex

$$(\Sigma_2 M)_V \longrightarrow (\Sigma_1 M)_V \longrightarrow M_V \longrightarrow 0$$

is exact whenever the X -rank of $V \in \mathbf{A}$ is at least N . Define M' to be the left Kan extension to \mathbf{A} of the restriction of M to \mathbf{A}^N . The universal property of the left Kan extension gives a natural transformation $\theta: M' \rightarrow M$ such that $\theta_V: (M')_V \rightarrow M_V$ is the identity for all $V \in \mathbf{A}$ whose X -rank is at most N . We must prove that θ_V is an isomorphism for all $V \in \mathbf{A}$. Letting r be the X -rank of V , the proof is by induction on r . The base cases are when $0 \leq r \leq N$, where the claim is trivial. Assume now that $r > N$ and that the claim is true for all smaller r . The natural transformation θ induces natural transformations $\theta_i: \Sigma_i M' \rightarrow \Sigma_i M$ for all $i \geq 0$, and our inductive hypothesis implies that the maps $(\theta_i)_V: (\Sigma_i M')_V \rightarrow (\Sigma_i M)_V$ are isomorphisms for all $i \geq 1$. We have a commutative diagram

$$\begin{array}{ccccccc} (\Sigma_2 M')_V & \longrightarrow & (\Sigma_1 M')_V & \longrightarrow & (M')_V & \longrightarrow & 0 \\ (\theta_2)_V \downarrow \cong & & (\theta_1)_V \downarrow \cong & & \theta_V \downarrow & & \\ (\Sigma_2 M)_V & \longrightarrow & (\Sigma_1 M)_V & \longrightarrow & M_V & \longrightarrow & 0 \end{array}$$

We warn the reader that while the bottom row is exact, the top row is as yet only a chain complex. If $W \in \mathbf{A}$ has X -rank at most N , then every \mathbf{A} -morphism $W \rightarrow V$ factors through an object whose X -rank is $r - 1$ (for instance, this is an easy consequence of Lemma 3.1). Combining this with the usual formula for a left Kan extension as a colimit, we deduce that the map $(\Sigma_1 M')_V \rightarrow (M')_V$ is surjective. The fact that θ_V is an isomorphism now follows from an easy chase of the above diagram. \square

4 Twisted homological stability

In this section, we prove a general theorem which allows us to deduce twisted homological stability from untwisted homological stability and local Noetherianity (see Theorem 4.2 below; this abstracts an argument of Church [Ch2]). Using this, we will prove Theorems L and M, which give very general twisted homological stability theorems for $\mathrm{SL}_n^{\mathbb{A}}(R)$ and $\mathrm{Sp}_{2n}(R)$. We remark that Wahl–Randal-Williams [WahRW] have also recently and independently proved twisted homological stability theorems. They require their coefficients to be either “polynomial” or “abelian”, but they allow much more general rings R . Their proofs are very different from ours.

Lemma 4.1. *Let (\mathbf{A}, \otimes) be a complemented category with a generator X . Then for all $n \geq r$, we have an isomorphism of $\mathrm{Aut}_{\mathbf{A}}(X^n)$ -representations*

$$\mathbf{k}[\mathrm{Hom}_{\mathbf{A}}(X^r, X^n)] \cong \mathbf{k}[\mathrm{Aut}_{\mathbf{A}}(X^n)/\mathrm{Aut}_{\mathbf{A}}(X^{n-r})] \cong \mathrm{Ind}_{\mathrm{Aut}_{\mathbf{A}}(X^{n-r})}^{\mathrm{Aut}_{\mathbf{A}}(X^n)} \mathbf{k}.$$

Consequently, $\mathrm{H}_k(\mathrm{Aut}_{\mathbf{A}}(X^n); \mathbf{k}[\mathrm{Hom}_{\mathbf{A}}(X^r, X^n)]) \cong \mathrm{H}_k(\mathrm{Aut}_{\mathbf{A}}(X^{n-r}); \mathbf{k})$.

Proof. Let $i \in \mathrm{Hom}_{\mathbf{A}}(X^r, X^n)$ be the canonical morphism $X^r \rightarrow X^r \otimes X^{n-r}$. By Lemma 3.1, $\mathrm{Aut}_{\mathbf{A}}(X^n)$ acts transitively on the set $\mathrm{Hom}_{\mathbf{A}}(X^r, X^n)$. By Lemma 3.2(b), the stabilizer of i in $\mathrm{Aut}_{\mathbf{A}}(X^n)$ is identified with $\mathrm{Aut}_{\mathbf{A}}(X^{n-r})$, which establishes the first isomorphism. The second follows from Shapiro’s lemma [Br, Proposition III.6.2]. \square

Our main theorem is then as follows. If (\mathbf{A}, \otimes) is a complemented category with a generator X and M is an \mathbf{A} -module, then denote M_{X^n} by M_n .

Theorem 4.2. *Let (\mathbf{A}, \otimes) be a complemented category with a generator X and let M be a finitely generated \mathbf{A} -module over a ring \mathbf{k} . Assume that the following hold.*

1. *The category of \mathbf{A} -modules over \mathbf{k} is locally Noetherian.*
2. *For all $k \geq 0$, the map $\mathrm{H}_k(\mathrm{Aut}_{\mathbf{A}}(X^n); \mathbf{k}) \rightarrow \mathrm{H}_k(\mathrm{Aut}_{\mathbf{A}}(X^{n+1}); \mathbf{k})$ induced by the map $\mathrm{Aut}_{\mathbf{A}}(X^n) \rightarrow \mathrm{Aut}_{\mathbf{A}}(X^{n+1})$ that takes $f \in \mathrm{Aut}_{\mathbf{A}}(X^n)$ to $f \otimes \mathrm{id}_X \in \mathrm{Aut}_{\mathbf{A}}(X^{n+1})$ is an isomorphism for $n \gg 0$.*

Then for $k \geq 0$, the map $\mathrm{H}_k(\mathrm{Aut}_{\mathbf{A}}(X^n); M_n) \rightarrow \mathrm{H}_k(\mathrm{Aut}_{\mathbf{A}}(X^{n+1}); M_{n+1})$ is an isomorphism for $n \gg 0$.

Proof. Combining Lemma 4.1 with our second assumption, we see that the theorem is true if M is a finite direct sum of representable \mathbf{A} -modules $P_{\mathbf{A},x}$ with $x \in A$ (see §2.1). Our local Noetherian assumption implies that there exists an \mathbf{A} -module resolution

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

where P_i is a finite direct sum of representable \mathbf{A} -modules for all $i \geq 0$. For all $n \geq 0$, we can combine [Br, Proposition VII.5.2] and [Br, Eqn. VII.5.3] to obtain a spectral sequence

$$E_{pq}^1(n) = \mathrm{H}_p(\mathrm{Aut}_{\mathbf{A}}(X^n); (P_q)_n) \implies \mathrm{H}_{p+q}(\mathrm{Aut}_{\mathbf{A}}(X^n); M_n).$$

This spectral sequence is natural, so there is a map $E_{**}^1(n) \rightarrow E_{**}^1(n+1)$ which converges to the map $\mathrm{H}_*(\mathrm{Aut}_{\mathbf{A}}(X^n); M_n) \rightarrow \mathrm{H}_*(\mathrm{Aut}_{\mathbf{A}}(X^{n+1}); M_{n+1})$. By the above, for each pair (p, q) , the map $E_{pq}^1(n) \rightarrow E_{pq}^1(n+1)$ is an isomorphism for $n \gg 0$. This implies that for each $k \geq 0$, the map

$$\mathrm{H}_k(\mathrm{Aut}_{\mathbf{A}}(X^n); M_n) \rightarrow \mathrm{H}_k(\mathrm{Aut}_{\mathbf{A}}(X^{n+1}); M_{n+1})$$

is an isomorphism for $n \gg 0$, as desired. \square

Proof of Theorems L and M. Theorems L and M are obtained by applying Theorem 4.2 to the categories $\text{VIC}(R, \mathfrak{U})$ and $\text{SI}(R)$, respectively. The necessary locally Noetherian theorems are Theorem D, and Theorem E. The necessary untwisted homological stability theorems are due to van der Kallen [Va] and Mirzaii–van der Kallen [MirVa]. \square

5 A machine for finite generation

This section contains the machine we will use to prove our finite generation results. We begin with two sections of preliminaries: §5.1 is devoted to systems of coefficients, and §5.2 is devoted to the basics of equivariant homology. Finally, §5.3 contains our machine. Our machine is based on an unpublished argument of Quillen for proving homological stability (see, e.g., [HatWah]). The insight that that argument interacts well with central stability is due to the first author [Pu2]. This was later reinterpreted in the language of FI-modules in [ChEFaNag]. The arguments in this section abstract and generalize the arguments in [Pu2] and [ChEFaNag]. Throughout this section, \mathbf{k} is a fixed commutative ring.

5.1 Systems of coefficients

Letting $\widehat{\Delta}$ be the category whose objects are the finite sets $[n]_+ = \{0, \dots, n\}$ and whose morphisms are strictly increasing maps, recall that a semisimplicial set X is a contravariant functor from $\widehat{\Delta}$ to the category of sets. The image of $[n]_+$ is called the set of n -simplices and is denoted X^n . A simplex $\sigma' \in X^m$ is a face of a simplex $\sigma \in X^n$ if σ' is the image of σ under the map $X^n \rightarrow X^m$ induced by some map $[m]_+ \rightarrow [n]_+$. The geometric realization of X is denoted $|X|$. For more on semisimplicial sets, see [Fri] (where they are called Δ -sets).

Fix a semisimplicial set X . Observe that the set $\sqcup_{n=0}^{\infty} X^n$ forms the objects of a category with a unique morphism $\sigma' \rightarrow \sigma$ whenever σ' is a face of σ . We will call this the **simplex category** of X .

Definition 5.1. A **coefficient system** on X is a contravariant functor from the simplex category of X to the category of \mathbf{k} -modules. \square

Remark 5.2. In other words, a coefficient system \mathfrak{F} on X consists of \mathbf{k} -modules $\mathfrak{F}(\sigma)$ for simplices σ of X and homomorphisms $\mathfrak{F}(\sigma' \rightarrow \sigma): \mathfrak{F}(\sigma) \rightarrow \mathfrak{F}(\sigma')$ whenever σ' is a face of σ . These homomorphisms must satisfy the obvious compatibility condition. \square

Definition 5.3. Let \mathfrak{F} be a coefficient system on X . The **simplicial chain complex** of X with coefficients in \mathfrak{F} is as follows. Define

$$C_k(X; \mathfrak{F}) = \bigoplus_{\sigma \in X^k} \mathfrak{F}(\sigma).$$

Next, define a differential $\partial: C_k(C; \mathfrak{F}) \rightarrow C_{k-1}(C; \mathfrak{F})$ in the following way. Consider $\sigma \in X^k$. We will denote an element of $\mathfrak{F}(\sigma) \subset C_k(X; \mathfrak{F})$ by $c \cdot \sigma$ for $c \in \mathfrak{F}(\sigma)$. For $0 \leq i \leq k$, let σ_i be the face of σ associated to the unique morphism $[k-1]_+ \rightarrow [k]_+$ of $\widehat{\Delta}$ whose image does not contain i . For $c \in \mathfrak{F}(\sigma)$, we then define

$$\partial(c \cdot \sigma) = \sum_{i=0}^k (-1)^i c_i \cdot \sigma_i,$$

where c_i is the image of c under the homomorphism $\mathfrak{F}(\sigma_i \rightarrow \sigma): \mathfrak{F}(\sigma) \rightarrow \mathfrak{F}(\sigma_i)$. Taking the homology of $C_*(X; \mathfrak{F})$ yields the **homology groups of X with coefficients in \mathfrak{F}** , which we will denote by $H_*(X; \mathfrak{F})$. \square

Remark 5.4. If \mathfrak{F} is the coefficient system that assigns \mathbf{k} to every simplex and the identity map to every inclusion of a face, then $H_*(X; \mathfrak{F}) \cong H_*(X; \mathbf{k})$. \square

5.2 Equivariant homology

A good reference that contains proofs of everything we state in this section is [Br, §VII].

Semisimplicial sets form a category \mathbf{SS} whose morphisms are natural transformations. Given a semisimplicial set X and a group G , an action of G on X consists of a homomorphism $G \rightarrow \text{Aut}_{\mathbf{SS}}(X)$. Unpacking this, an action of G on X consists of actions of G on X^n for all $n \geq 0$ which satisfy the obvious compatibility condition. Observe that this induces an action of G on X . Also, there is a natural quotient semisimplicial set X/G with $(X/G)^n = X^n/G$.

Remark 5.5. If G is a group acting on a semisimplicial set X , then there is a natural continuous map $|X| \rightarrow |X/G|$ that factors through $|X|/G$. In fact, it is easy to see that $|X|/G = |X/G|$. \square

Definition 5.6. Consider a group G acting on a semisimplicial set X . Let EG be a contractible CW complex on which G acts properly discontinuously and freely, so EG/G is a classifying space for G . Define $EG \times_G X$ to be the quotient of $EG \times |X|$ by the diagonal action of G . The G -equivariant homology groups of X , denoted $H_*^G(X; \mathbf{k})$, are defined to be $H_*(EG \times_G X; \mathbf{k})$. This definition does not depend on the choice of EG . The construction of $EG \times_G X$ is known as the **Borel construction**. \square

The following lemma summarizes two key properties of these homology groups.

Lemma 5.7. *Consider a group G acting on a semisimplicial set X . There is a canonical map $H_*^G(X; \mathbf{k}) \rightarrow H_*(G; \mathbf{k})$ such that if X is k -acyclic, then the map $H_i^G(X; \mathbf{k}) \rightarrow H_i(G; \mathbf{k})$ is an isomorphism for $i \leq k$.*

Proof. The map $H_*^G(X; \mathbf{k}) \rightarrow H_*(G; \mathbf{k})$ comes from the map $EG \times_G X \rightarrow EG/G$ induced by the projection of $EG \times X$ onto its first factor. The second claim is an immediate consequence of the spectral sequence whose E^2 page is [Br, (7.2), §VII.7]. \square

To calculate equivariant homology, we use a certain spectral sequence. First, a definition.

Definition 5.8. Consider a group G acting on a semisimplicial set X . Define a coefficient system $\mathfrak{H}_q(G, X; \mathbf{k})$ on X/G as follows. Consider a simplex σ of X/G . Let $\tilde{\sigma}$ be any lift of σ to X . Set

$$\mathfrak{H}_q(G, X; \mathbf{k})(\sigma) = H_q(G_{\tilde{\sigma}}; \mathbf{k}),$$

where $G_{\tilde{\sigma}}$ is the stabilizer of $\tilde{\sigma}$. This does not depend on the choice of $\tilde{\sigma}$ since conjugation induces the identity on group homology, so it defines a coefficient system on X/G . \square

Our spectral sequence is then as follows. It can be easily extracted from [Br, §VII.8].

Theorem 5.9. *Let G be a group acting on a semisimplicial set X . There is a spectral sequence*

$$E_{p,q}^1 = C_p(X/G; \mathfrak{H}_q(G, X; \mathbf{k})) \implies H_{p+q}^G(X; \mathbf{k})$$

with $d^1: E_{p,q}^1 \rightarrow E_{p-1,q}^1$ equal to the differential of $C_*(X/G; \mathfrak{H}_q(G, X; \mathbf{k}))$.

5.3 A machine for finite generation

We now discuss a machine for proving that the homology groups of certain congruence subgroups form finitely generated modules over a complemented category.

Congruence subgroups. Let (\mathbf{B}, \otimes) be a weak complemented category with a generator Y . The assignment $V \mapsto \text{Aut}_{\mathbf{B}}(\tilde{V})$ for $V \in \mathbf{B}$ is functorial. Indeed, given a \mathbf{B} -morphism $\tilde{f}: \tilde{V}_1 \rightarrow \tilde{V}_2$, let \tilde{C} be the complement to $\tilde{f}(\tilde{V}_1) \subset \tilde{V}_2$ and write $\tilde{V}_2 \cong \tilde{V}_1 \otimes \tilde{C}$; we can then extend automorphisms of \tilde{V}_1 to \tilde{V}_2 by letting them act as the identity on \tilde{C} . It is clear that this is well-defined. Next, let (\mathbf{A}, \otimes) be a complemented category and let $\Psi: \mathbf{B} \rightarrow \mathbf{A}$ be a strong monoidal functor (recall that this is a monoidal functor whose coherence maps are invertible). An argument similar to the above shows that the assignment

$$\tilde{V} \mapsto \ker(\text{Aut}_{\mathbf{B}}(\tilde{V}) \rightarrow \text{Aut}_{\mathbf{A}}(\Psi(\tilde{V}))) \quad (\tilde{V} \in \mathbf{B})$$

is functorial. We will call this functor the **level Ψ congruence subgroup** of $\text{Aut}_{\mathbf{B}}$ and denote its value on \tilde{V} by $\Gamma_{\tilde{V}}(\Psi)$.

Highly surjective functors. The above construction is particularly well-behaved on certain kinds of functors that we now define. Let (\mathbf{A}, \otimes) be a complemented category with a generator X and let (\mathbf{B}, \otimes) be a weak complemented category with a generator Y . A **highly surjective functor** from \mathbf{B} to \mathbf{A} is a strong monoidal functor $\Psi: \mathbf{B} \rightarrow \mathbf{A}$ satisfying:

- $\Psi(Y) = X$.
- For $\tilde{V} \in \mathbf{B}$, the map $\Psi_*: \text{Aut}_{\mathbf{B}}(\tilde{V}) \rightarrow \text{Aut}_{\mathbf{A}}(\Psi(\tilde{V}))$ is surjective.

Remark 5.10. We only assume that Ψ is a strong monoidal functor (as opposed to a strict monoidal functor), so for all $q \geq 0$ we only know that $\Psi(Y^q)$ and X^q are naturally isomorphic rather than identical. To simplify notation, we will simply identify $\Psi(Y^q)$ and X^q henceforth. The careful reader can insert appropriate natural isomorphisms as needed. \square

Lemma 5.11. *Let (\mathbf{A}, \otimes) be a complemented category with a generator X , let (\mathbf{B}, \otimes) be a weak complemented category with a generator Y , and let $\Psi: \mathbf{B} \rightarrow \mathbf{A}$ be a highly surjective functor. Then the following hold.*

- (a) *For all $V \in \mathbf{A}$, there exists some $\tilde{V} \in \mathbf{B}$ such that $\Psi(\tilde{V}) \cong V$.*
- (b) *For all $\tilde{V}_1, \tilde{V}_2 \in \mathbf{B}$, the map $\Psi_*: \text{Hom}_{\mathbf{B}}(\tilde{V}_1, \tilde{V}_2) \rightarrow \text{Hom}_{\mathbf{A}}(\Psi(\tilde{V}_1), \Psi(\tilde{V}_2))$ is surjective.*

Proof. For the first claim, we have $V \cong X^q$ for some $q \geq 0$, so $\Psi(Y^q) = X^q \cong V$.

For the second claim, write $\tilde{V}_1 \cong Y^q$ and $\tilde{V}_2 \cong Y^r$, so $\Psi(\tilde{V}_1) \cong X^q$ and $\Psi(\tilde{V}_2) \cong X^r$. If $q > r$, then $\text{Hom}_{\mathbf{A}}(X^q, X^r) = \emptyset$ and the claim is trivial, so we can assume that $q \leq r$. Consider some $f \in \text{Hom}_{\mathbf{A}}(X^q, X^r)$. Let $\tilde{f}_1 \in \text{Hom}_{\mathbf{B}}(Y^q, Y^r)$ be arbitrary (this set is nonempty; for instance, it contains the composition of the canonical map $Y^q \rightarrow Y^q \otimes Y^{r-q}$ with an isomorphism $Y^q \otimes Y^{r-q} \cong Y^r$). Lemma 3.1 implies that there exists some $f_2 \in \text{Aut}_{\mathbf{A}}(X^r)$ such that $f = f_2 \circ \Psi(\tilde{f}_1)$. Since Ψ is highly surjective, we can find $\tilde{f}_2 \in \text{Aut}_{\mathbf{B}}(Y^r)$ such that $\Psi(\tilde{f}_2) = f_2$. Setting $\tilde{f} = \tilde{f}_2 \circ \tilde{f}_1$, we then have $\Psi(\tilde{f}) = f$, as desired. \square

Lemma 5.12. *Let (\mathbf{A}, \otimes) be a complemented category with generator X , let (\mathbf{B}, \otimes) be a weak complemented category with generator Y , and let $\Psi: \mathbf{B} \rightarrow \mathbf{A}$ be a highly surjective functor. Consider $\tilde{f}, \tilde{f}' \in \text{Hom}_{\mathbf{B}}(Y^q, Y^r)$ such that $\Psi(\tilde{f}) = \Psi(\tilde{f}')$. Then there exists some $\tilde{g} \in \Gamma_{Y^r}(\Psi)$ such that $\tilde{f}' = \tilde{g} \circ \tilde{f}$.*

Proof. Let \tilde{C} be the complement to $\tilde{f}(Y^q) \subset Y^r$ and $\tilde{h}: Y^r \rightarrow Y^q \otimes \tilde{C}$ be the isomorphism such that $\tilde{h} \circ \tilde{f}: Y^q \rightarrow Y^q \otimes \tilde{C}$ is the canonical map. Define $C = \Psi(\tilde{C})$, $h = \Psi(\tilde{h})$, and $f = \Psi(\tilde{f})$, so $h \circ f: X^q \rightarrow X^q \otimes C$ is the canonical map. Lemma 3.1 implies that there exists

some $\tilde{g}_1 \in \text{Aut}_{\mathbf{B}}(Y^q \otimes \tilde{C})$ such that $\tilde{h} \circ \tilde{f}' = \tilde{g}_1 \circ \tilde{h} \circ \tilde{f}$. Define $g_1 = \Psi(\tilde{g}_1)$, so $h \circ f = g_1 \circ h \circ f$. Lemma 3.2(b) implies that there exists some $g_2 \in \text{Aut}_{\mathbf{A}}(C)$ such that $g_1 = \text{id} \otimes g_2$. Since Ψ is highly surjective, we can find some $\tilde{g}_2 \in \text{Aut}_{\mathbf{B}}(\tilde{C})$ such that $\Psi(\tilde{g}_2) = g_2$.

Define $\tilde{g} \in \text{Aut}_{\mathbf{B}}(Y^r)$ to equal $\tilde{h}^{-1} \circ \tilde{g}_1 \circ (\text{id} \otimes \tilde{g}_2^{-1}) \circ \tilde{h}$. Then $\Psi(\tilde{g}) = h^{-1} \circ g_1 \circ (\text{id} \otimes g_2^{-1}) \circ h$, which is the identity since $g_1 = \text{id} \otimes g_2$. Thus $\tilde{g} \in \Gamma_{Y^r}(\Psi)$. Finally, since $\tilde{h} \circ f$ is the canonical map, it follows that $\tilde{h} \circ \tilde{f} \circ (\text{id} \otimes \tilde{g}_2^{-1}) = \tilde{h} \circ \tilde{f}$. Thus

$$\tilde{g} \circ \tilde{f} = \tilde{h}^{-1} \circ \tilde{g}_1 \circ (\text{id} \otimes \tilde{g}_2^{-1}) \circ \tilde{h} \circ \tilde{f} = \tilde{h}^{-1} \circ \tilde{g}_1 \circ \tilde{h} \circ \tilde{f} = \tilde{h}^{-1} \circ \tilde{h} \circ \tilde{f}' = \tilde{f}'. \quad \square$$

Homology of congruence subgroups. Let (\mathbf{A}, \otimes) be a complemented category with generator X , let (\mathbf{B}, \otimes) be a weak complemented category with generator Y , and let $\Psi: \mathbf{B} \rightarrow \mathbf{A}$ be a highly surjective functor. Fixing a ring \mathbf{k} and some $k \geq 0$, we define an \mathbf{A} -module $\mathcal{H}_k(\Psi; \mathbf{k})$ as follows. Since \mathbf{A} is equivalent to the full subcategory spanned by the X^q , it is enough to define $\mathcal{H}_k(\Psi; \mathbf{k})$ on these. Set

$$\mathcal{H}_k(\Psi; \mathbf{k})_{X^q} = \text{H}_k(\Gamma_{Y^q}(\Psi); \mathbf{k}).$$

To define $\mathcal{H}_k(\Psi; \mathbf{k})$ on morphisms, consider $f \in \text{Hom}_{\mathbf{A}}(X^q, X^r)$. By Lemma 5.11, there exists some $\tilde{f} \in \text{Hom}_{\mathbf{B}}(Y^q, Y^r)$ such that $\Psi(\tilde{f}) = f$, and we define

$$\mathcal{H}_k(\Psi; \mathbf{k})_f: \text{H}_k(\Gamma_{Y^q}(\Psi); \mathbf{k}) \rightarrow \text{H}_k(\Gamma_{Y^r}(\Psi); \mathbf{k})$$

to be the map on homology induced by the map $\Gamma_{Y^q}(\Psi) \rightarrow \Gamma_{Y^r}(\Psi)$ induced by \tilde{f} . To see that this is well-defined, observe that if $\tilde{f}' \in \text{Hom}_{\mathbf{B}}(Y^q, Y^r)$ also satisfies $\Psi(\tilde{f}') = f$, then Lemma 5.12 implies that there exists some $\tilde{g} \in \Gamma_{Y^r}(\Psi)$ such that $\tilde{f}' = \tilde{g} \circ \tilde{f}$. Since inner automorphisms act trivially on group homology, we conclude that \tilde{f} and \tilde{f}' induce the same map $\text{H}_k(\Gamma_{Y^q}(\Psi); \mathbf{k}) \rightarrow \text{H}_k(\Gamma_{Y^r}(\Psi); \mathbf{k})$.

Space of Y -embeddings. We wish to give a sufficient condition for the \mathbf{A} -module $\mathcal{H}_k(\Psi; \mathbf{k})$ to be finitely generated. This requires the following definition. Consider $\tilde{V} \in \mathbf{B}$. Recall that for all $k \geq 0$, we defined

$$\mathfrak{J}_{\tilde{V}, k} = \text{Hom}_{\mathbf{B}}(Y^k, \tilde{V}).$$

The **space of Y -embeddings** is the semisimplicial set $\mathfrak{J}_{\tilde{V}}$ defined as follows. First, define $(\mathfrak{J}_{\tilde{V}})^k = \mathfrak{J}_{\tilde{V}, k+1}$. For $f \in \mathfrak{J}_{\tilde{V}, k+1}$, we will denote the associated k -simplex of $\mathfrak{J}_{\tilde{V}}$ by $[f]$. Next, if $[f] \in (\mathfrak{J}_{\tilde{V}})^k$ and $\phi: [\ell]_+ \rightarrow [k]_+$ is a morphism of $\hat{\Delta}$ (i.e., a strictly increasing map from $[\ell]_+ = \{0, \dots, \ell\}$ to $[k]_+ = \{0, \dots, k\}$), then there is a natural induced map $\phi_*: Y^{\ell+1} \rightarrow Y^{k+1}$ (here we are indexing the Y -factors starting at 0). We define the face of $[f]$ associated to ϕ to be $[f \circ \phi_*]$.

Finite generation, statement. We can now state our main theorem.

Theorem 5.13. *Let (\mathbf{A}, \otimes) be a complemented category with a generator X , let (\mathbf{B}, \otimes) be a weak complemented category with a generator Y , and let $\Psi: \mathbf{B} \rightarrow \mathbf{A}$ be a highly surjective functor. Fix a Noetherian ring \mathbf{k} , and assume that the following hold.*

1. *The category of \mathbf{A} -modules is locally Noetherian.*
2. *For all $V \in \mathbf{A}$ and $k \geq 0$, the \mathbf{k} -module $\mathcal{H}_k(\Psi; \mathbf{k})_V$ is finitely generated.*
3. *For all $p \geq 0$, there exists some $N_p \geq 0$ such that for all $\tilde{V} \in \mathbf{B}$ whose Y -rank is at least N_p , the space $\mathfrak{J}_{\tilde{V}}$ is p -acyclic.*

Then $\mathcal{H}_k(\Psi; \mathbf{k})$ is a finitely generated \mathbf{A} -module for all $k \geq 0$.

We prove Theorem 5.13 at the end of this section after some preliminary results.

Identifying the spectral sequence. Let (\mathbf{B}, \otimes) be a weak complemented category with generator Y . For all $\tilde{V} \in \mathbf{B}$, the group $\text{Aut}_{\mathbf{B}}(\tilde{V})$ acts on $\mathfrak{J}_{\tilde{V}}$. The following lemma describes the restriction of this action to a congruence subgroup.

Lemma 5.14. *Let (\mathbf{A}, \otimes) be a complemented category with a generator X , let (\mathbf{B}, \otimes) be a weak complemented category with a generator Y , and let $\Psi: \mathbf{B} \rightarrow \mathbf{A}$ be a highly surjective functor. Fix some $\tilde{V} \in \mathbf{B}$, and set $V = \Psi(\tilde{V}) \in \mathbf{A}$. Then the group $\Gamma_{\tilde{V}}(\Psi)$ acts on $\mathfrak{J}_{\tilde{V}}$ and we have an isomorphism $\mathfrak{J}_{\tilde{V}}/\Gamma_{\tilde{V}}(\Psi) \cong \mathfrak{J}_V$ of semisimplicial sets.*

Proof. The claimed action of $\Gamma_{\tilde{V}}(\Psi)$ is the restriction of the obvious action of $\text{Aut}_{\mathbf{B}}(\tilde{V})$. We have a $\Gamma_{\tilde{V}}(\Psi)$ -invariant map $\mathfrak{J}_{\tilde{V}} \rightarrow \mathfrak{J}_V$ of semisimplicial sets. This induces a map $\rho: \mathfrak{J}_{\tilde{V}}/\Gamma_{\tilde{V}}(\Psi) \rightarrow \mathfrak{J}_V$. For all $k \geq 0$, Lemma 5.11 implies that ρ is surjective on k -simplices and Lemma 5.12 implies that ρ is injective on k -simplices. We conclude that ρ is an isomorphism of semisimplicial sets, as desired. \square

Let (\mathbf{A}, \otimes) be a complemented category with a generator X and let $\Psi: \mathbf{B} \rightarrow \mathbf{A}$ be a highly surjective functor. Fix some $\tilde{V} \in \mathbf{B}$. Our next goal is to understand the spectral sequence given by Theorem 5.9 for the action of $\Gamma_{\tilde{V}}(\Psi)$ on $\mathfrak{J}_{\tilde{V}}$. Set $V = \Psi(\tilde{V}) \in \mathbf{A}$. Using Lemma 5.14, this spectral sequence is of the form

$$E_{p,q}^1 = C_p(\mathfrak{J}_V; \mathfrak{H}_q(\Gamma_{\tilde{V}}(\Psi), \mathfrak{J}_{\tilde{V}}; \mathbf{k})) \implies H_{p+q}^{\Gamma_{\tilde{V}}(\Psi)}(\mathfrak{J}_{\tilde{V}}; \mathbf{k}).$$

Recall the definition of Σ_p from §3.2.

Lemma 5.15. *Let (\mathbf{A}, \otimes) be a complemented category with a generator X , let (\mathbf{B}, \otimes) be a weak complemented category with a generator Y , and let $\Psi: \mathbf{B} \rightarrow \mathbf{A}$ be a highly surjective functor. Fix $\tilde{V} \in \mathbf{B}$, and set $V = \Psi(\tilde{V})$. Finally, fix a ring \mathbf{k} . Then for $p, q \geq 0$ we have*

$$C_{p-1}(\mathfrak{J}_V; \mathfrak{H}_q(\Gamma_{\tilde{V}}(\Psi), \mathfrak{J}_{\tilde{V}}; \mathbf{k})) \cong (\Sigma_p \mathcal{H}_q(\Psi; \mathbf{k}))_V.$$

Proof. Recall that the $(p-1)$ -simplices of \mathfrak{J}_V are in bijection with the set $\mathfrak{J}_{V,p}$ and that the simplex associated to $h \in \mathfrak{J}_{V,p}$ is denoted $[h]$. It follows that

$$C_{p-1}(\mathfrak{J}_V; \mathfrak{H}_q(\Gamma_{\tilde{V}}(\Psi), \mathfrak{J}_{\tilde{V}}; \mathbf{k})) = \bigoplus_{h \in \mathfrak{J}_{V,p}} \mathfrak{H}_q(\Gamma_{\tilde{V}}(\Psi), \mathfrak{J}_{\tilde{V}}; \mathbf{k})([h]).$$

Fix some $h \in \mathfrak{J}_{V,p}$, and let $\tilde{h} \in \mathfrak{J}_{\tilde{V},p}$ be a lift of h . By definition, we have

$$\mathfrak{H}_q(\Gamma_{\tilde{V}}(\Psi), \mathfrak{J}_{\tilde{V}}; \mathbf{k})([h]) = H_q((\Gamma_{\tilde{V}}(\Psi))_{[\tilde{h}]}; \mathbf{k}).$$

We have

$$(\text{Aut}_{\mathbf{B}}(\tilde{V}))_{[\tilde{h}]} = \{\tilde{g} \in \text{Aut}_{\mathbf{B}}(\tilde{V}) \mid \tilde{g} \circ \tilde{h} = \tilde{h}\} \quad \text{and} \quad (\Gamma_{\tilde{V}}(\Psi))_{[\tilde{h}]} = \{\tilde{g} \in \Gamma_{\tilde{V}}(\Psi) \mid \tilde{g} \circ \tilde{h} = \tilde{h}\}.$$

Letting $\tilde{W}_{\tilde{h}} \subset \tilde{V}$ be a complement to $\tilde{h}(Y^p)$, Lemma 3.2(b) implies that $(\text{Aut}_{\mathbf{B}}(\tilde{V}))_{[\tilde{h}]} \cong \text{Aut}_{\mathbf{B}}(\tilde{W}_{\tilde{h}})$. Under this isomorphism, $(\Gamma_{\tilde{V}}(\Psi))_{[\tilde{h}]}$ goes to $\Gamma_{\tilde{W}_{\tilde{h}}}(\Psi)$. Letting $W_h = \Psi(\tilde{W}_{\tilde{h}}) \subset V$, we have that W_h is a complement to $h(X^p)$. By definition, we have

$$H_q((\Gamma_{\tilde{V}}(\Psi))_{[\tilde{h}]}; \mathbf{k}) = H_q(\Gamma_{\tilde{W}_{\tilde{h}}}(\Psi); \mathbf{k}) = (\mathcal{H}_q(\Psi; \mathbf{k}))_{W_h}.$$

The upshot of all of this is that

$$C_{p-1}(\mathfrak{J}_V; \mathfrak{H}_q(\Gamma_{\tilde{V}}(\Psi), \mathfrak{J}_{\tilde{V}}; \mathbf{k})) = \bigoplus_{h \in \mathfrak{J}_{V,p}} (\mathcal{H}_q(\Psi; \mathbf{k}))_{W_h} = (\Sigma_p \mathcal{H}_q(\Psi; \mathbf{k}))_V. \quad \square$$

Finite generation, proof. We finally prove Theorem 5.13.

Proof of Theorem 5.13. The proof is by induction on k . The base case $k = 0$ is trivial, so assume that $k > 0$ and that $\mathcal{H}_{k'}(\Psi; \mathbf{k})$ is a finitely generated \mathbf{A} -module for all $0 \leq k' < k$. Since we are assuming that the \mathbf{k} -module $\mathcal{H}_k(\Psi; \mathbf{k})_V$ is finitely generated for all $V \in \mathbf{A}$ and all $k \geq 0$, we can apply Lemma 3.6 and deduce that it is enough to prove that if $V \in \mathbf{A}$ has X -rank sufficiently large, then the map

$$(\Sigma_1 \mathcal{H}_k(\Psi; \mathbf{k}))_V \rightarrow \mathcal{H}_k(\Psi; \mathbf{k})_V \quad (5.1)$$

is surjective. Letting $\tilde{V} \in \mathbf{B}$ satisfy $\Psi(\tilde{V}) = V$, we have $\mathcal{H}_k(\Psi; \mathbf{k})_V = \mathbf{H}_k(\Gamma_{\tilde{V}}(\Psi); \mathbf{k})$. To prove that (5.1) is surjective, we will use the action of $\Gamma_{\tilde{V}}(\Psi)$ on $\mathcal{J}_{\tilde{V}}$.

By Lemma 5.7 and our assumption about the acyclicity of $\mathcal{J}_{\tilde{V}}$, the map

$$\mathbf{H}_k^{\Gamma_{\tilde{V}}(\Psi)}(\mathcal{J}_{\tilde{V}}; \mathbf{k}) \rightarrow \mathbf{H}_k(\Gamma_{\tilde{V}}; \mathbf{k})$$

is an isomorphism if the Y -rank of \tilde{V} (which equals the X -rank of V) is sufficiently large. Combining Theorem 5.9 with Lemma 5.15, we get a spectral sequence

$$E_{p,q}^1 \cong (\Sigma_{p+1} \mathcal{H}_q(\Psi; \mathbf{k}))_V \implies \mathbf{H}_{p+q}^{\Gamma_{\tilde{V}}(\Psi)}(\mathcal{J}_{\tilde{V}}; \mathbf{k}).$$

Moreover, examining the isomorphism in Lemma 5.15, we see that the differential $d: E_{p,q}^1 \rightarrow E_{p-1,q}^1$ is identified with the differential

$$d: (\Sigma_{p+1} \mathcal{H}_q(\Psi; \mathbf{k}))_V \rightarrow (\Sigma_p \mathcal{H}_q(\Psi; \mathbf{k}))_V.$$

Our inductive hypothesis says that $\mathcal{H}_q(\Psi; \mathbf{k})$ is a finitely generated \mathbf{A} -module for $0 \leq q < k$. Theorem 3.7 therefore implies that if the X -rank of V is sufficiently large, we have $E_{p,q}^2 = 0$ for $0 \leq q < k$ and $1 \leq p \leq k + 1$. This implies that the map

$$(\Sigma_1 \mathcal{H}_k(\Psi; \mathbf{k}))_V = E_{0,k}^1 \rightarrow \mathbf{H}_k^{\Gamma_{\tilde{V}}(\Psi)}(\mathcal{J}_{\tilde{V}}; \mathbf{k}) \cong \mathbf{H}_k(\Gamma_{\tilde{V}}; \mathbf{k})$$

is a surjection, as desired. \square

6 Finite generation theorems

In this section, we construct the modules claimed in the introduction and prove Theorems G, H, I, J, and K; they are proved in §6.1, §6.1, §6.2, §6.3, and §6.5, respectively. These theorems assert that the homology groups of various congruence subgroups form finitely generated modules over the appropriate categories. They will all be deduced from Theorem 5.13. We also introduce the weak complemented category **Surf** in §6.4.

There is an interesting parallel between the complexes that appear in this section and those that appear in the paper [WahRW] of Wahl–Randal-Williams, which proves a variety of more traditional homological stability theorems. Our paper and theirs were written independently (though in later revisions to save space we removed some proofs that duplicated work of theirs).

6.1 The general and special linear groups: Theorems G and H

Let \mathcal{O}_K be the ring of integers in an algebraic number field K , let $\alpha \subset \mathcal{O}_K$ be a proper nonzero ideal, let $\mathfrak{U} \subset \mathcal{O}_K/\alpha$ be the image of the group of units $\widehat{\mathfrak{U}} \subset \mathcal{O}_K$, and let \mathbf{k} be a Noetherian ring. In this section, we construct $\mathrm{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$ -modules $\mathcal{H}_k(\mathrm{GL}(\mathcal{O}_K, \alpha); \mathbf{k})$ as in Theorem G for all $k \geq 0$; see Corollary 6.2. We then prove Theorem G, which says that $\mathcal{H}_k(\mathrm{GL}(\mathcal{O}_K, \alpha); \mathbf{k})$ is a finitely generated $\mathrm{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$ -module for all $k \geq 0$. Theorem H is a similar result for the special linear group. Its proof is similar to that of the general linear group, and we leave it as an exercise to the reader.

Categories and functors. As was noted in Example 1.16, for any ring R the category $\mathrm{VIC}(R)$ is a complemented category whose monoidal structure is direct sum. We have

$$\mathrm{Aut}_{\mathrm{VIC}(R)}(R^n) = \mathrm{GL}_n(R) \quad (n \geq 0). \quad (6.1)$$

Define a strong monoidal functor $\Psi: \mathrm{VIC}(\mathcal{O}_K) \rightarrow \mathrm{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$ by $\Psi(V) = (\mathcal{O}_K/\alpha) \otimes_{\mathcal{O}_K} V$. The reason for the presence of \mathfrak{U} in the target of Ψ is that objects V of $\mathrm{VIC}(\mathcal{O}_K)$ are equipped with $\widehat{\mathfrak{U}}$ -orientations (a vacuous condition since $\widehat{\mathfrak{U}}$ acts transitively on the set of orientations!), and these induce \mathfrak{U} -orientations on $(\mathcal{O}_K/\alpha) \otimes_{\mathcal{O}_K} V$. The functor Ψ takes the generator \mathcal{O}_K^1 of $\mathrm{VIC}(\mathcal{O}_K)$ to the generator $(\mathcal{O}_K/\alpha)^1$ of $\mathrm{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$. Define $\mathrm{GL}(\mathcal{O}_K, \alpha) = \Gamma(\Psi)$. Using the identification in (6.1), we have $\mathrm{GL}(\mathcal{O}_K, \alpha)_{\mathcal{O}_K^n} = \mathrm{GL}_n(\mathcal{O}_K, \alpha)$.

Lemma 6.1. *The functor Ψ is highly surjective.*

Proof. This is equivalent to the fact that the map $\mathrm{GL}_n(\mathcal{O}_K) \rightarrow \mathrm{SL}_n^{\mathfrak{U}}(\mathcal{O}_K/\alpha)$ is surjective for all $n \geq 0$. This follows from the fact that the map $\mathrm{SL}_n(\mathcal{O}_K) \rightarrow \mathrm{SL}_n(\mathcal{O}_K/\alpha)$ is surjective, which follows from strong approximation (see, e.g., [PIRa, Chapter 7]; this is a special feature of rings of integers). \square

As we discussed in §5.3, Lemma 6.1 has the following corollary.

Corollary 6.2. *For all $k \geq 0$, there exists a $\mathrm{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$ -module $\mathcal{H}_k(\mathrm{GL}(\mathcal{O}_K, \alpha); \mathbf{k})$ such that $\mathcal{H}_k(\mathrm{GL}(\mathcal{O}_K, \alpha); \mathbf{k})_{(\mathcal{O}_K/\alpha)^n} = \mathrm{H}_k(\mathrm{GL}_n(\mathcal{O}_K, \alpha); \mathbf{k})$.*

Identifying the space of embeddings. We now wish to give a concrete description of the space of \mathcal{O}_K^1 -embeddings of \mathcal{O}_K^n (recall the definition of the space of Y -embeddings from §5.3). A **split partial basis** for \mathcal{O}_K^n consists of elements $((v_0, w_0), \dots, (v_p, w_p))$ as follows.

- The v_i are elements of \mathcal{O}_K^n that form part of a free basis for the free \mathcal{O}_K -module \mathcal{O}_K^n .
- The w_i are elements of $\mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K^n, \mathcal{O}_K)$.
- For all $0 \leq i, j \leq p$, we have $w_i(v_j) = \delta_{i,j}$ (Kronecker delta).

These conditions imply that the w_i also form part of a free basis for the free \mathcal{O}_K -module $\mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K^n, \mathcal{O}_K)$. The **space of split partial bases** for \mathcal{O}_K^n , denoted $\mathrm{PB}_n(\mathcal{O}_K)$, is the semisimplicial set whose p -simplices are split partial bases $((v_0, w_0), \dots, (v_p, w_p))$ for \mathcal{O}_K^n .

Lemma 6.3. *The space of \mathcal{O}_K^1 -embeddings of \mathcal{O}_K^n is isomorphic to $\mathrm{PB}_n(\mathcal{O}_K)$.*

Proof. Fix some $p \geq 0$. The p -simplices of the space of \mathcal{O}_K^1 -embeddings of \mathcal{O}_K^n are in bijection with elements of $\mathrm{Hom}_{\mathrm{VIC}(\mathcal{O}_K)}(\mathcal{O}_K^{p+1}, \mathcal{O}_K^n)$. We will define set maps

$$\eta_1: \mathrm{Hom}_{\mathrm{VIC}(\mathcal{O}_K)}(\mathcal{O}_K^{p+1}, \mathcal{O}_K^n) \rightarrow (\mathrm{PB}_n(\mathcal{O}_K))^p, \quad \eta_2: (\mathrm{PB}_n(\mathcal{O}_K))^p \rightarrow \mathrm{Hom}_{\mathrm{VIC}(\mathcal{O}_K)}(\mathcal{O}_K^{p+1}, \mathcal{O}_K^n)$$

that are compatible with the semisimplicial structure and satisfy $\eta_1 \circ \eta_2 = \mathrm{id}$ and $\eta_2 \circ \eta_1 = \mathrm{id}$. The constructions are as follows.

- Consider $(f, C) \in \text{Hom}_{\text{VIC}(\mathcal{O}_K)}(\mathcal{O}_K^{p+1}, \mathcal{O}_K^n)$, so by definition $f: \mathcal{O}_K^{p+1} \rightarrow \mathcal{O}_K^n$ is an \mathcal{O}_K -linear injection and $C \subset \mathcal{O}_K^n$ is a complement to $f(\mathcal{O}_K^{p+1})$. For $0 \leq i \leq p$, define $v_i \in \mathcal{O}_K^n$ to be the image under f of the $(i-1)^{\text{st}}$ basis element of \mathcal{O}_K^{p+1} and define $w_i \in \text{Hom}_{\mathcal{O}_K}(\mathcal{O}_K^n, \mathcal{O}_K)$ via the formulas

$$w_i(v_j) = \delta_{ij} \quad \text{and} \quad w_i|_C = 0 \quad (0 \leq j \leq p).$$

Then $\eta_1(f, C) := ((v_0, w_0), \dots, (v_p, w_p))$ is a split partial basis.

- Consider a split partial basis $((v_0, w_0), \dots, (v_{p+1}, w_{p+1}))$. Define $f: \mathcal{O}_K^{p+1} \rightarrow \mathcal{O}_K^n$ to be the map that takes the i^{th} basis element to v_{i-1} . Also, define $C = \bigcap_{i=0}^p \ker(w_i)$. Then $\eta_2(((v_0, w_0), \dots, (v_p, w_p))) := (f, C)$ is an element of $\text{Hom}_{\text{VIC}(\mathcal{O}_K)}(\mathcal{O}_K^{p+1}, \mathcal{O}_K^n)$. \square

The space $\text{PB}_n(\mathcal{O}_K)$ was introduced by Charney [Cha2], who proved the following theorem.

Theorem 6.4 ([Cha2, Theorem 3.5]). *The space $\text{PB}_n(\mathcal{O}_K)$ is $\frac{n-4}{2}$ -acyclic.*

Remark 6.5. Charney actually worked over rings that satisfy Bass’s stable range condition SR_r ; the above follows from her work since \mathcal{O}_K satisfies SR_3 . Also, [Cha2, Theorem 3.5] contains the data of an ideal $\alpha \subset \mathcal{O}_K$ (not the same as the ideal we are working with!); the reference reduces to Theorem 6.4 when $\alpha = 0$. Finally, Charney’s definition of $\text{PB}_n(R)$ is different from ours, but for Dedekind domains the definitions are equivalent; see [Re]. \square

Putting it all together. We now prove Theorem G, which asserts that the $\text{VIC}(\mathcal{O}/\alpha, \mathfrak{U})$ -module $\mathcal{H}_k(\text{GL}(\mathcal{O}_K, \alpha); \mathbf{k})$ is finitely generated.

Proof of Theorem G. We apply Theorem 5.13 to the highly surjective functor Ψ . This theorem has three hypotheses. The first is that the category of $\text{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$ -modules is locally Noetherian, which is Theorem D. The second is that $\mathcal{H}_k(\text{GL}_n(\mathcal{O}_K, \alpha); \mathbf{k})$ is a finitely generated \mathbf{k} -module for all $k \geq 0$, which is a theorem of Borel–Serre [BoSe]. The third is that the space of \mathcal{O}_K^1 -embeddings is highly acyclic, which follows from Lemma 6.3 and Theorem 6.4. We conclude that we can apply Theorem 5.13 and deduce that $\mathcal{H}_k(\text{GL}(\mathcal{O}_K, \alpha); \mathbf{k})$ is a finitely generated $\text{VIC}(\mathcal{O}_K/\alpha, \mathfrak{U})$ -module for all $k \geq 0$, as desired. \square

6.2 The automorphism group of a free group: Theorem I

Fix $\ell \geq 2$, and let \mathbf{k} be a Noetherian ring. In this section, we construct $\text{VIC}(\mathbb{Z}/\ell, \pm 1)$ -modules $\mathcal{H}_k(\text{Aut}(\ell); \mathbf{k})$ as in Theorem I for all $k \geq 0$; see Corollary 6.7. We then prove Theorem I, which says that $\mathcal{H}_k(\text{Aut}(\ell); \mathbf{k})$ is a finitely generated $\text{VIC}(\mathbb{Z}/\ell, \pm 1)$ -module for all $k \geq 0$.

The category of free groups. Define \mathbf{Fr} to be a category whose objects are the finite-rank free groups and whose monoidal structure is the free product $*$. The \mathbf{Fr} -morphisms from F to F' are pairs $(f, C): F \rightarrow F'$, where $f: F \rightarrow F'$ is an injective homomorphism and $C \subseteq F'$ is a subgroup such that $F' = f(F) * C$. The category \mathbf{Fr} is a complemented category. The subobject attached to a morphism $(f, C): F \rightarrow F'$ can be thought of as the pair $(f(F), C)$ and its complement is the subobject $(C, f(F))$. A generator is the rank 1 free group $F_1 \cong \mathbb{Z}$, and

$$\text{Aut}_{\mathbf{Fr}}(F_n) = \text{Aut}(F_n) \quad (n \geq 0). \quad (6.2)$$

The category \mathbf{Fr} was first introduced by Djament–Vespa [DjVe2] to study the twisted homology of $\text{Aut}(F_n)$.

Functor. The category $\text{VIC}(\mathbb{Z}/\ell, \pm 1)$ is a complemented category whose monoidal structure is given by the direct sum. There is a strong monoidal functor $\Psi: \mathbf{Fr} \rightarrow \text{VIC}(\mathbb{Z}/\ell, \pm 1)$ defined

via the formula $\Psi(F) = H_1(F; \mathbb{Z}/\ell)$. The reason for the ± 1 here is that Ψ factors through the functor $\mathbf{Fr} \rightarrow \mathbf{VIC}(\mathbb{Z})$ that takes F to $H_1(F; \mathbb{Z})$, and elements of $\mathbf{VIC}(\mathbb{Z})$ are equipped with ± 1 -orientations (a vacuous condition since the group of units of \mathbb{Z} is ± 1). The functor Ψ takes the generator $F_1 \cong \mathbb{Z}$ of \mathbf{Fr} to the generator $(\mathbb{Z}/\ell)^1$ of $\mathbf{VIC}(\mathbb{Z}/\ell, \pm 1)$. Define $\text{Aut}(\ell) = \Gamma(\Psi)$. Using the identification in (6.2), we have $\text{Aut}(\ell)_{F_n} = \text{Aut}(F_n, \ell)$.

Lemma 6.6. *The functor Ψ is highly surjective.*

Proof. This is equivalent to the fact that the map $\text{Aut}(F_n) \rightarrow \text{GL}_n(\mathbb{Z})$ is surjective for all $n \geq 0$, which is classical (see, e.g., [LySc, Chapter I, Proposition 4.4]). \square

As we discussed in §5.3, Lemma 6.6 has the following corollary.

Corollary 6.7. *For all $k \geq 0$, there exists a $\mathbf{VIC}(\mathbb{Z}/\ell, \pm 1)$ -module $\mathcal{H}_k(\text{Aut}(\ell); \mathbf{k})$ such that $\mathcal{H}_k(\text{Aut}(\ell); \mathbf{k})_{(\mathbb{Z}/\ell)^n} = H_k(\text{Aut}(F_n, \ell); \mathbf{k})$.*

Identifying the space of embeddings. We now wish to give a concrete description of the space of F_1 -embeddings of F_n . A **split partial basis** for F_n consists of elements $((v_0, C_0), \dots, (v_p, C_p))$ as follows.

- The v_i are elements of F_n .
- The C_i are subgroups of F_n such that $F_n = \langle v_i \rangle * C_i$.
- For all $0 \leq i, j \leq p$ with $i \neq j$, we have $v_j \in C_i$.

We remark that the condition $F_n = \langle v_i \rangle * C_i$ implies that $C_i \cong F_{n-1}$. Also, the conditions together imply that $\bigcap_{i=0}^p C_i \cong F_{n-p-1}$ and that the v_i can be extended to a free basis for F_n . The **space of split partial bases** for F_n , denoted $\text{PB}(F_n)$, is the semisimplicial set whose p -simplices are split partial bases $((v_0, C_0), \dots, (v_p, C_p))$ for F_n . We remark that the C_i in this space play the same role as the kernels of the w_i in $\text{PB}_n(\mathcal{O}_K)$.

Lemma 6.8. *The space of F_1 -embeddings of F_n is isomorphic to $\text{PB}(F_n)$.*

Proof. Fix some $p \geq 0$. The p -simplices of the space of F_1 -embeddings of F_n are in bijection with elements of $\text{Hom}_{\mathbf{Fr}}(F_{p+1}, F_n)$. We will define set maps

$$\eta_1: \text{Hom}_{\mathbf{Fr}}(F_{p+1}, F_n) \rightarrow (\text{PB}(F_n))^p, \quad \eta_2: (\text{PB}(F_n))^p \rightarrow \text{Hom}_{\mathbf{Fr}}(F_{p+1}, F_n)$$

that are compatible with the semisimplicial structure and satisfy $\eta_1 \circ \eta_2 = \text{id}$ and $\eta_2 \circ \eta_1 = \text{id}$. The constructions are as follows.

- Consider $(f, C) \in \text{Hom}_{\mathbf{Fr}}(F_{p+1}, F_n)$, so by definition $f: F_{p+1} \rightarrow F_n$ is an injective homomorphism and $C \subset F_n$ is a subgroup satisfying $F_n = f(F_{p+1}) * C$. For $0 \leq i \leq p$, define $v_i \in F_n$ to be the image under f of the $(i-1)^{\text{st}}$ basis element of F_{p+1} and define $C_i \subset F_n$ to be the subgroup generated by $\{v_j \mid i \neq j\} \cup C$. Then $\eta_1(f, C) := ((v_0, C_0), \dots, (v_p, C_p))$ is a split partial basis.
- Consider a split partial basis $((v_0, C_0), \dots, (v_{p+1}, C_{p+1}))$. Define $f: F_{p+1} \rightarrow F_n$ to be the map that takes the i^{th} basis element to v_{i-1} . Also, define $C = \bigcap_{i=0}^p C_i$. Then $\eta_2(((v_0, C_0), \dots, (v_p, C_p))) := (f, C)$ is an element of $\text{Hom}_{\mathbf{Fr}}(F_{p+1}, F_n)$. \square

The space $\text{PB}(F_n)$ appears in work of Wahl–Randal-Williams, where they prove the following theorem (making essential use of work of Hatcher–Vogtmann [HatVo2]).

Theorem 6.9 ([WahRW, Proposition 5.3]). *The space $\text{PB}(F_n)$ is $\frac{n-3}{2}$ -connected.*

Putting it all together. We now prove Theorem I, which asserts that the $\mathbf{VIC}(\mathbb{Z}/\ell, \pm 1)$ -module $\mathcal{H}_k(\text{Aut}(\ell); \mathbf{k})$ is finitely generated.

Proof of Theorem I. We apply Theorem 5.13 to the highly surjective functor Ψ . This theorem has three hypotheses. The first is that the category of $\text{VIC}(\mathbb{Z}/\ell, \pm 1)$ -modules is locally Noetherian, which is Theorem D. The second is that $H_k(\text{Aut}(F_n); \mathbf{k})$ is a finitely generated \mathbf{k} -module for all $k \geq 0$, which is a theorem of Culler–Vogtmann [CuVo, Corollary on p. 93]. The third is that the space of F_1 -embeddings is highly acyclic, which follows from Lemma 6.8 and Theorem 6.9. We conclude that we can apply Theorem 5.13 and deduce that $\mathcal{H}_k(\text{Aut}(\ell); \mathbf{k})$ is a finitely generated $\text{VIC}(\mathbb{Z}/\ell, \pm 1)$ -module for all $k \geq 0$, as desired. \square

6.3 The symplectic group: Theorem J

Let \mathcal{O}_K be the ring of integers in an algebraic number field K , let $\alpha \subset \mathcal{O}_K$ be a proper nonzero ideal, and let \mathbf{k} be a Noetherian ring. In this section, we construct $\text{SI}(\mathcal{O}_K/\alpha)$ -modules $\mathcal{H}_k(\text{Sp}(\mathcal{O}_K, \alpha); \mathbf{k})$ as in Theorem J for all $k \geq 0$; see Corollary 6.11. We then prove Theorem J, which says that these are finitely generated modules.

Categories and functors. As was noted in Example 1.17, for any ring R the category $\text{SI}(R)$ is a complemented category whose monoidal structure is given by the orthogonal direct sum. We have

$$\text{Aut}_{\text{SI}(R)}(R^n) = \text{Sp}_{2n}(R) \quad (n \geq 0). \quad (6.3)$$

Define a strong monoidal functor $\Psi: \text{SI}(\mathcal{O}_K) \rightarrow \text{SI}(\mathcal{O}_K/\alpha)$ by $\Psi(V) = (\mathcal{O}_K/\alpha) \otimes_{\mathcal{O}_K} V$. This functor takes the generator \mathcal{O}_K^2 of $\text{SI}(\mathcal{O}_K)$ to the generator $(\mathcal{O}_K/\alpha)^2$ of $\text{SI}(\mathcal{O}_K/\alpha)$ (both are equipped with the standard symplectic form). Define $\text{Sp}(\mathcal{O}_K, \alpha) = \Gamma(\Psi)$. Using the identification in (6.3), we have $\text{Sp}(\mathcal{O}_K, \alpha)_{\mathcal{O}_K^2} = \text{Sp}_{2n}(\mathcal{O}_K, \alpha)$.

Lemma 6.10. *The functor Ψ is highly surjective.*

Proof. This is equivalent to the fact that the map $\text{Sp}_{2n}(\mathcal{O}_K) \rightarrow \text{Sp}_{2n}(\mathcal{O}_K/\alpha)$ is surjective for all $n \geq 0$, which follows from strong approximation (see, e.g., [PIRa, Chapter 7]). \square

As we discussed in §5.3, Lemma 6.10 has the following corollary.

Corollary 6.11. *For all $k \geq 0$, there exists an $\text{SI}(\mathcal{O}_K/\alpha)$ -module $\mathcal{H}_k(\text{Sp}(\mathcal{O}_K, \alpha); \mathbf{k})$ such that $\mathcal{H}_k(\text{Sp}(\mathcal{O}_K, \alpha); \mathbf{k})_{(\mathcal{O}_K/\alpha)^{2n}} = H_k(\text{Sp}_{2n}(\mathcal{O}_K, \alpha); \mathbf{k})$.*

Identifying the space of embeddings. We now wish to give a concrete description of the space of \mathcal{O}_K^2 -embeddings of \mathcal{O}_K^{2n} . Letting \hat{i} be the symplectic form on \mathcal{O}_K^{2n} , a **symplectic basis** for \mathcal{O}_K^{2n} is an ordered basis $(a_1, b_1, \dots, a_n, b_n)$ such that $\hat{i}(a_i, a_j) = \hat{i}(b_i, b_j) = 0$ and $\hat{i}(a_i, b_j) = \delta_{ij}$ for $1 \leq i, j \leq n$, where δ_{ij} is the Kronecker delta. A **partial symplectic basis** for \mathcal{O}_K^{2n} is a sequence $(a_1, b_1, \dots, a_p, b_p)$ of elements of \mathcal{O}_K^{2n} that can be extended to a symplectic basis $(a_1, b_1, \dots, a_n, b_n)$. The **space of partial symplectic bases** for \mathcal{O}_K^{2n} , denoted $\text{PSB}_n(\mathcal{O}_K)$, is the semisimplicial set whose $(p-1)$ -cells are partial symplectic bases $(a_1, b_1, \dots, a_p, b_p)$.

Lemma 6.12. *The space of \mathcal{O}_K^2 -embeddings of \mathcal{O}_K^{2n} is isomorphic to $\text{PSB}_n(\mathcal{O}_K)$.*

Proof. Fix some $p \geq 0$. The p -simplices of the space of \mathcal{O}_K^2 -embeddings of \mathcal{O}_K^{2n} are in bijection with elements of $\text{Hom}_{\text{SI}(\mathcal{O}_K)}(\mathcal{O}_K^{p+1}, \mathcal{O}_K^{2n})$. We can define a set map

$$\eta: \text{Hom}_{\text{SI}(\mathcal{O}_K)}(\mathcal{O}_K^{p+1}, \mathcal{O}_K^{2n}) \rightarrow (\text{PSB}_n(\mathcal{O}_K))^p$$

by letting $\eta(f)$ be the image under $f \in \text{Hom}_{\text{SI}(\mathcal{O}_K)}(\mathcal{O}_K^{p+1}, \mathcal{O}_K^{2n})$ of the standard symplectic basis for \mathcal{O}_K^{p+1} . Clearly η is a bijection compatible with the semisimplicial structure. \square

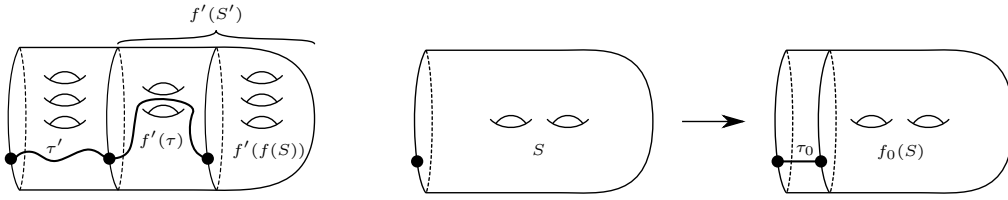


Figure 1: The LHS shows how to compose morphisms $(f, \tau): S \rightarrow S'$ and $(f', \tau'): S' \rightarrow S''$. The picture depicts S'' . The RHS shows the identity morphism $(f_0, \tau_0): S \rightarrow S$, which pushes ∂S across a collar neighborhood.

The space $\text{PSB}_n(\mathcal{O}_K)$ was introduced by Charney [Cha3], who proved the following theorem.

Theorem 6.13 ([Cha3, Corollary 3.3]). *The space $\text{PSB}_n(\mathcal{O}_K)$ is $\frac{n-6}{2}$ -connected.*

Remark 6.14. Just like for $\text{PB}_n(\mathcal{O}_K)$ in §6.1, the definition of $\text{PSB}_n(R)$ is different from Charney's, but can be proved equivalent for Dedekind domains using the ideas from [Re]. \square

Putting it all together. We now prove Theorem J, which asserts that the $\text{SI}(\mathcal{O}/\alpha)$ -module $\mathcal{H}_k(\text{Sp}(\mathcal{O}_K, \alpha); \mathbf{k})$ is finitely generated.

Proof of Theorem J. We apply Theorem 5.13 to the highly surjective functor Ψ . This theorem has three hypotheses. The first is that the category of $\text{SI}(\mathcal{O}_K/\alpha)$ -modules is locally Noetherian, which is Theorem E. The second is that $\mathcal{H}_k(\text{Sp}_{2n}(\mathcal{O}_K, \alpha); \mathbf{k})$ is a finitely generated \mathbf{k} -module for all $k \geq 0$, which is a theorem of Borel–Serre [BoSe]. The third is that the space of \mathcal{O}_K^2 -embeddings is highly acyclic, which follows from Lemma 6.12 and Theorem 6.13. We conclude that we can apply Theorem 5.13 and deduce that $\mathcal{H}_k(\text{Sp}(\mathcal{O}_K, \alpha); \mathbf{k})$ is a finitely generated $\text{SI}(\mathcal{O}_K/\alpha)$ -module for all $k \geq 0$, as desired. \square

6.4 The category of surfaces

In preparation for proving Theorem K (which concerns congruence subgroups of the mapping class group), this section is devoted to a weak complemented category **Surf** of surfaces. This category was introduced by Ivanov [I, §2.5]; our contribution is to endow it with a monoidal structure. It is different from previous monoidal categories of surfaces (for instance, in [Mil]), which do not include morphisms between non-isomorphic surfaces. A related monoidal category of surfaces was constructed independently by Wahl–Randal-Williams [WahRW, §5.6], though their morphisms are different from ours.

The category Surf. The objects of the category **Surf** are pairs (S, η) as follows.

- S is a compact oriented surface with one boundary component.
- $\eta: [0, 1] \rightarrow \partial S$ is an embedding such that S lies to the left of $\text{Im}(\eta)$. We will call η the **boundary arc** of S .

This endows S with a natural basepoint, namely $\eta(1/2) \in \partial S$. To keep our notation from getting out of hand, we will usually refer to (S, η) simply as S , leaving η implicit. For $S, S' \in \text{Surf}$, the set $\text{Hom}_{\text{Surf}}(S, S')$ consists of homotopy classes of pairs (f, τ) as follows.

- $f: S \rightarrow \text{Int}(S')$ is an orientation-preserving embedding.
- τ is a properly embedded arc in $S' \setminus \text{Int}(f(S))$ which goes from the basepoint of S' to the image under f of the basepoint of S .

If $(f, \tau): S \rightarrow S'$ and $(f', \tau'): S' \rightarrow S''$ are morphisms, then

$$(f', \tau') \circ (f, \tau) = (f' \circ f, \tau' \cdot f'(\tau)): S \rightarrow S'',$$

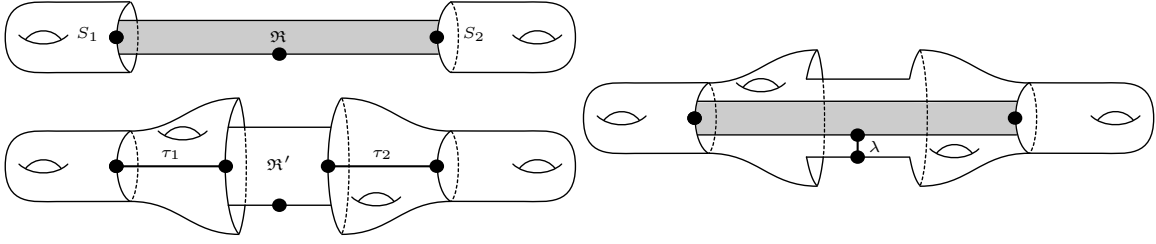


Figure 2: The upper left figure is $S_1 \otimes S_2$, divided into the three subsurfaces S_1 and S_2 and \mathfrak{R} (\mathfrak{R} is shaded). The bottom left figure is $S'_1 \otimes S'_2$; here the rectangle is labeled \mathfrak{R}' . For $i = 1, 2$, the morphism $(f_i, \tau_i): S_i \rightarrow S'_i$ is depicted. The right hand figure shows how $S_1 \otimes S_2$ is embedded in $S'_1 \otimes S'_2$.

where arcs are composed as in the fundamental groupoid (this goes from left to right rather than right to left as in function composition; see Figure 1). The identity morphism of S is the morphism (f_0, τ_0) depicted in Figure 1; the embedding f_0 “pushes” ∂S across a collar neighborhood.

Remark 6.15. The boundary arc of an object of **Surf** is necessary to define the monoidal structure on **Surf**. However, aside from providing a basepoint it plays little role in the structure of **Surf**. Indeed, if $S, S' \in \mathbf{Surf}$ are objects with the same underlying surface, then it is easy to see that there exists an isomorphism $(f, \tau): S \xrightarrow{\cong} S'$. \square

Automorphisms. Ivanov proved the following theorem.

Theorem 6.16 ([I, §2.6]). *Let $S \in \mathbf{Surf}$. Then $\text{Aut}_{\mathbf{Surf}}(S)$ is isomorphic to the mapping class group of S .*

Remark 6.17. One potentially confusing point here is that Ivanov allows his surfaces to have multiple boundary components. The theorem in [I, §2.6] is more complicated than Theorem 6.16, but it reduces to Theorem 6.16 when S has one boundary component. \square

A bifunctor. Let $\mathfrak{R} = [0, 3] \times [0, 1] \subset \mathbb{R}^2$. Consider objects S_1 and S_2 of **Surf**. Define $S_1 \otimes S_2$ to be the following object of **Surf**. For $i = 1, 2$, let $\eta_i: [0, 1] \rightarrow S_i$ be the boundary arc of S_i . Then $S_1 \otimes S_2$ is the result of gluing S_1 and S_2 to \mathfrak{R} as follows.

- For $t \in [0, 1]$, identify $\eta_1(t) \in S_1$ and $(0, t) \in \mathfrak{R}$.
- For $t \in [0, 1]$, identify $\eta_2(t) \in S_2$ and $(3, 1 - t) \in \mathfrak{R}$.

We have $1 - t$ in the above to ensure that the resulting surface is oriented. The boundary arc of $S_1 \otimes S_2$ is $t \mapsto (1 + t, 0) \in \mathfrak{R}$.

Lemma 6.18. *The operation \otimes is a bifunctor on **Surf**.*

Proof. For $i = 1, 2$, let $(f_i, \tau_i): S_i \rightarrow S'_i$ be a **Surf**-morphism. We must construct a canonical morphism

$$(g, \lambda): S_1 \otimes S_2 \longrightarrow S'_1 \otimes S'_2.$$

The based surface $S_1 \otimes S_2$ consists of three pieces, namely S_1 and S_2 and \mathfrak{R} . Identify all three of these pieces with their images in $S_1 \otimes S_2$. Similarly, $S'_1 \otimes S'_2$ consists of three pieces, namely S'_1 and S'_2 and a rectangle which we will call \mathfrak{R}' to distinguish it from $\mathfrak{R} \subset S_1 \otimes S_2$, and we will identify these three pieces with their images in $S'_1 \otimes S'_2$. Let $g: S_1 \otimes S_2 \rightarrow S'_1 \otimes S'_2$ be the embedding depicted in Figure 2, so $g(\mathfrak{R})$ is a regular neighborhood in $S'_1 \otimes S'_2 \setminus \text{Int}(f_1(S_1) \cup f_2(S_2))$ of the union of τ_1 and τ_2 and the arc $t \mapsto (3t, 1/2)$ in $\mathfrak{R}' = [0, 3] \times [0, 1] \subset \mathbb{R}^2$. Also, let λ be the arc shown in Figure 2. It is clear that (g, λ) is well-defined up to homotopy. \square

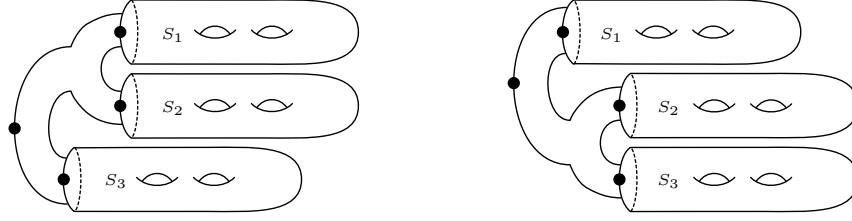


Figure 3: The LHS is $(S_1 \otimes S_2) \otimes S_3$ and the RHS is $S_1 \otimes (S_2 \otimes S_3)$.

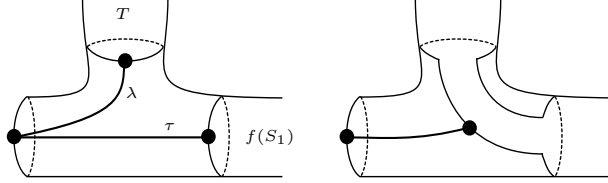


Figure 4: The LHS shows $f(S_1)$ and T together with the arcs τ and λ . The RHS shows the desired morphism $T \otimes S_1 \rightarrow S_2$.

Monoidal structure. We now come to the following result.

Lemma 6.19. *The category \mathbf{Surf} is a monoidal category (not symmetric!) with respect to the bifunctor \otimes .*

Proof. Clearly a disc (with some boundary arc; any two such boundary arcs determine isomorphic objects of \mathbf{Surf}) is a two-sided identity for \otimes . Associativity is clear from Figure 3. The various diagrams that need to commute are all easy exercises. \square

Weak complemented category. We finally come to the main result of this section.

Proposition 6.20. *The monoidal category (\mathbf{Surf}, \otimes) is a weak complemented category with generator a one-holed torus.*

Proof. If \mathcal{S}_1^1 is a one-holed torus with a fixed boundary arc and S is an arbitrary object of \mathbf{Surf} , then letting g be the genus of S we clearly have $S \cong \otimes_{i=1}^g \mathcal{S}_1^1$. It follows that \mathcal{S}_1^1 is a generator of (\mathbf{Surf}, \otimes) . Now let $(f, \tau): S_1 \rightarrow S_2$ be a \mathbf{Surf} -morphism. We must construct a complement to (f, τ) which is unique up to precomposing the inclusion map by an isomorphism. Define T to be the complement of an open regular neighborhood in S_2 of $\partial S_2 \cup \tau \cup f(S_1)$. Thus T is a compact oriented surface with one boundary component. Fix a boundary arc of T . As is shown in Figure 4, there exists a \mathbf{Surf} -morphism $(g, \lambda): T \rightarrow S_2$ such that T is a complement to (f, τ) . Uniqueness of this complement is an easy exercise. \square

6.5 The mapping class group: Theorem K

Fix $\ell \geq 2$, and let \mathbf{k} be a Noetherian ring. In this section, we construct $\mathbf{SI}(\mathbb{Z}/\ell)$ -modules $\mathcal{H}_k(\mathbf{MCG}(\ell); \mathbf{k})$ as in Theorem K for all $k \geq 0$; see Corollary 6.22. We then prove Theorem K, which says that $\mathcal{H}_k(\mathbf{MCG}(\ell); \mathbf{k})$ is a finitely generated $\mathbf{SI}(\mathbb{Z}/\ell)$ -module for all $k \geq 0$.

Categories and functors. Let (\mathbf{Surf}, \otimes) be the weak complemented category of surfaces discussed in §6.4 (see Proposition 6.20). For all $g \geq 0$, fix a boundary arc of \mathcal{S}_g^1 . This allows us to regard each \mathcal{S}_g^1 as an object of \mathbf{Surf} . By Theorem 6.16, we have

$$\mathrm{Aut}_{\mathbf{Surf}}(\mathcal{S}_g^1) = \mathbf{MCG}_g^1 \quad (g \geq 0). \quad (6.4)$$

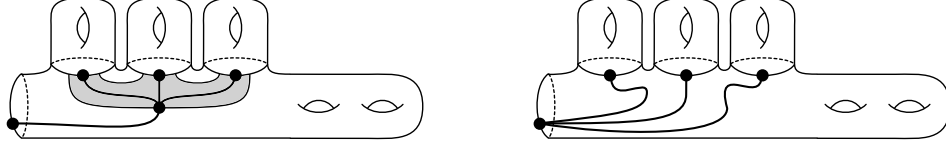


Figure 5: The LHS shows a morphism $(g, \lambda): \otimes_{i=0}^2 \mathcal{S}_1^1 \rightarrow \mathcal{S}_5^1$. The canonical morphisms $(h_j, \delta_j): \mathcal{S}_1^1 \rightarrow \otimes_{i=0}^2 \mathcal{S}_1^1$ are shown, and the union of the rectangles involved in forming $\otimes_{i=0}^2 \mathcal{S}_1^1$ is shaded. The RHS shows the associated tethered torus system.

In Example 1.17 we noted that the category $\text{SI}(\mathbb{Z}/\ell)$ is a complemented category whose monoidal structure is given by the orthogonal direct sum. There is a strong monoidal functor $\Psi: \text{Surf} \rightarrow \text{SI}(\mathbb{Z}/\ell)$ defined via the formula $\Psi(S) = H_1(S; \mathbb{Z}/\ell)$; here $H_1(S; \mathbb{Z}/\ell)$ is equipped with the symplectic algebraic intersection pairing. This functor takes the generator \mathcal{S}_1^1 of Surf to the generator $H_1(\mathcal{S}_1^1; \mathbb{Z}/\ell) \cong (\mathbb{Z}/\ell)^2$ for $\text{SI}(\mathbb{Z}/\ell)$. Define $\text{MCG}(\ell) = \Gamma(\Psi)$. Using the identification in (6.4), we have $\text{MCG}(\ell)_{(\mathbb{Z}/\ell)^{2g}} = \text{MCG}_g^1(\ell)$.

Lemma 6.21. *The functor Ψ is highly surjective.*

Proof. This is equivalent to the fact that the map $\text{MCG}_g^1 \rightarrow \text{Sp}_{2g}(\mathbb{Z}/\ell)$ arising from the action of MCG_g^1 on $H_1(\mathcal{S}_g^1; \mathbb{Z}/\ell)$ is surjective for all $g \geq 0$. This is classical. For instance, it can be factored as $\text{MCG}_g^1 \rightarrow \text{Sp}_{2g}(\mathbb{Z}) \rightarrow \text{Sp}_{2g}(\mathbb{Z}/\ell)$; the map $\text{MCG}_g^1 \rightarrow \text{Sp}_{2g}(\mathbb{Z})$ is surjective by [FaMar, §6.3.2], and the map $\text{Sp}_{2g}(\mathbb{Z}) \rightarrow \text{Sp}_{2g}(\mathbb{Z}/\ell)$ is surjective by strong approximation (see, e.g., [PIRa, Chapter 7]). \square

As we discussed in §5.3, Lemma 6.21 has the following corollary.

Corollary 6.22. *For all $k \geq 0$, there exists an $\text{SI}(\mathbb{Z}/\ell)$ -module $\mathcal{H}_k(\text{MCG}(\ell); \mathbf{k})$ such that $\mathcal{H}_k(\text{MCG}(\ell); \mathbf{k})_{(\mathbb{Z}/\ell)^{2g}} = H_k(\text{MCG}_g^1(\ell); \mathbf{k})$.*

Identifying the space of embeddings. We now wish to give a concrete description of the space of \mathcal{S}_1^1 -embeddings of \mathcal{S}_g^1 . A **tethered torus system** on \mathcal{S}_g^1 is a sequence $((f_0, \tau_0), \dots, (f_p, \tau_p))$ as follows (see the right hand side of Figure 5).

- Each (f_i, τ_i) is a morphism from \mathcal{S}_1^1 to \mathcal{S}_g^1 .
- After homotoping the (f_i, τ_i) , the following hold. Let $* \in \partial \mathcal{S}_g^1$ be the basepoint.
 - $(f_i(\mathcal{S}_1^1) \cup \tau_i) \cap (f_j(\mathcal{S}_1^1) \cup \tau_j) = \{*\}$ for all distinct $0 \leq i, j \leq p$.
 - Going clockwise, the τ_i leave the basepoint $*$ in their natural order (i.e., τ_i leaves before τ_j when $i < j$); see Remark 6.23 below for more on this.

The **space of tethered tori** on \mathcal{S}_g^1 , denoted TT_g , is the semisimplicial set whose p -simplices are tethered torus systems $((f_0, \tau_0), \dots, (f_p, \tau_p))$.

Remark 6.23. The condition on the order in which the τ_i leave the basepoint is needed because of the fact that the monoidal structure on (Surf, \otimes) is not symmetric; in our proof below, a p -simplex of TT_g corresponds to a morphism $\otimes_{i=0}^p \mathcal{S}_1^1 \rightarrow \mathcal{S}_g^1$, and the ordering on the τ_i reflects the ordering on the \otimes -factors. \square

Lemma 6.24. *The space of \mathcal{S}_1^1 -embeddings of \mathcal{S}_g^1 is isomorphic to TT_g .*

Proof. Consider a morphism $(g, \lambda): \otimes_{i=0}^p \mathcal{S}_1^1 \rightarrow \mathcal{S}_g^1$. For $0 \leq j \leq p$, let $(h_j, \delta_j): \mathcal{S}_1^1 \rightarrow \otimes_{i=0}^p \mathcal{S}_1^1$ be the canonical map coming from the j^{th} term. Then $((g \circ h_0, \lambda \cdot g(\delta_0)), \dots, (g \circ h_p, \lambda \cdot g(\delta_p)))$ is a tethered torus system (see Figure 5). This gives an isomorphism between the space of \mathcal{S}_1^1 -embeddings of \mathcal{S}_g^1 and the space of tethered tori. \square

Using results of Hatcher–Vogtmann [HatVo3], we will prove the following in §6.6.

Theorem 6.25. *The space TT_g is $\frac{g-3}{2}$ -connected.*

Putting it all together. We now prove Theorem K, which asserts that the $\mathrm{SI}(\mathbb{Z}/\ell)$ -module $\mathcal{H}_k(\mathrm{MCG}(\ell); \mathbf{k})$ is finitely generated.

Proof of Theorem K. We apply Theorem 5.13 to the highly surjective functor Ψ . This theorem has three hypotheses. The first is that the category of $\mathrm{SI}(\mathbb{Z}/\ell)$ -modules is locally Noetherian, which is Theorem E. The second is that $\mathcal{H}_k(\mathrm{MCG}_g^1(\ell); \mathbf{k})$ is a finitely generated \mathbf{k} -module for all $k \geq 0$. This follows easily from the fact that the moduli space of curves is a quasi-projective variety; see [FaMar, p. 127] for a discussion. The third is that the space of \mathcal{S}_1^1 -embeddings is highly acyclic, which follows from Lemma 6.24 and Theorem 6.25. We conclude that we can apply Theorem 5.13 and deduce that $\mathcal{H}_k(\mathrm{MCG}(\ell); \mathbf{k})$ is a finitely generated $\mathrm{SI}(\mathbb{Z}/\ell)$ -module for all $k \geq 0$, as desired. \square

6.6 The space of tethered tori

Our goal is to prove Theorem 6.25, which asserts that TT_g is $\frac{g-3}{2}$ -connected. We will deduce this from work of Hatcher–Vogtmann. Recall that we have endowed each \mathcal{S}_g^1 with a basepoint $*$ $\in \partial\mathcal{S}_g^1$. A **tethered chain** on \mathcal{S}_g^1 is a triple (α, β, τ) as follows.

- α and β are oriented properly embedded simple closed curves on \mathcal{S}_g^1 that intersect once transversely with a positive orientation.
- τ is a properly embedded arc that starts at $*$, ends at a point of $\beta \setminus \alpha$, and is otherwise disjoint from $\alpha \cup \beta$.

We will identify homotopic tethered chains. The **space of tethered chains**, denoted TC_g , is the semisimplicial set whose p -simplices are sequences $((\alpha_0, \beta_0, \tau_0), \dots, (\alpha_p, \beta_p, \tau_p))$ of tethered chains with the following properties.

- They can be homotoped such that they only intersect at $*$.
- Going clockwise, the τ_i leave the basepoint $*$ in their natural order (i.e., τ_i leaves before τ_j when $i < j$).

We remark that any unordered sequence of tethered chains that satisfies the first condition can be uniquely ordered such that it satisfies the second. The complex TC_g was introduced by Hatcher–Vogtmann [HatVo3], who proved the following theorem.

Theorem 6.26 (Hatcher–Vogtmann, [HatVo3]). *The space TC_g is $\frac{g-3}{2}$ -connected.*

Proof of Theorem 6.25. We begin by defining a map $\pi^0: \mathrm{TC}_g^0 \rightarrow \mathrm{TT}_g^0$. Fix once and for all a tethered chain (A, B, Γ) in \mathcal{S}_1^1 . Consider a tethered chain $(\alpha, \beta, \tau) \in \mathrm{TC}_g^0$. Let T be a closed regular neighborhood of $\alpha \cup \beta$. Choosing T small enough, we can assume that τ only intersects ∂T once. Write $\tau = \tau' \cdot \tau''$, where τ' is the segment of τ from the basepoint of \mathcal{S}_g^1 to ∂T and τ'' is the segment of τ from ∂T to β . Using the change of coordinates principle of [FaMar, §1.3], there exists an orientation-preserving homeomorphism $f: \mathcal{S}_1^1 \rightarrow T$ such that $f(A) = \alpha$ and $f(B) = \beta$ and $f(\Gamma) = \tau''$. By [FaMar, Proposition 2.8] (the “Alexander method”), these conditions determine f up to homotopy. The pair (f, τ') is a tethered torus, and we define $\pi^0((\alpha, \beta, \tau)) = (f, \tau') \in \mathrm{TT}_g^0$. It is clear that this is well-defined and surjective (but not injective).

The map π^0 extends over the higher-dimensional simplices of TC_g in an obvious way to define a map $\pi: \mathrm{TC}_g \rightarrow \mathrm{TT}_g$. Arbitrarily pick a set map $\phi^0: \mathrm{TT}_g^0 \rightarrow \mathrm{TC}_g^0$ such that $\pi^0 \circ \phi^0 = \mathrm{id}$. If $((f_0, \tau_0), \dots, (f_p, \tau_p))$ is a simplex of TT_g and $(\alpha_i, \beta_i, \tau'_i) = \phi^0((f_i, \tau_i))$ for $0 \leq i \leq p$, then by construction we have $\alpha_i, \beta_i \subset \mathrm{Im}(f_i)$. Since the images of the f_i are

disjoint, we conclude that $((\alpha_0, \beta_0, \tau'_0), \dots, (\alpha_p, \beta_p, \tau'_p))$ is a simplex of TC_g . Thus ϕ^0 extends over the higher-dimensional simplices of TT_g to define a map $\phi: \mathrm{TT}_g \rightarrow \mathrm{TC}_g$ satisfying $\pi \circ \phi = \mathrm{id}$. This implies that π induces a surjective map on all homotopy groups, so by Theorem 6.26 we deduce that TT_g is $\frac{g-3}{2}$ -connected. \square

References

- [AtMac] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, MA, 1969.
- [Be] S. Betley, Twisted homology of symmetric groups, *Proc. Amer. Math. Soc.* 130 (2002), no. 12, 3439–3445.
- [Bo] A. Borel, Stable real cohomology of arithmetic groups, *Ann. Sci. École Norm. Sup. (4)* 7 (1974), 235–272 (1975).
- [BoSe] A. Borel and J.-P. Serre, Corners and arithmetic groups, *Comment. Math. Helv.* 48 (1973), 436–491.
- [Br] K. S. Brown, *Cohomology of Groups*, corrected reprint of the 1982 original, Graduate Texts in Mathematics, 87, Springer, New York, 1994.
- [Cha1] R. M. Charney, Homology stability for GL_n of a Dedekind domain, *Invent. Math.* 56 (1980), no. 1, 1–17.
- [Cha2] R. Charney, On the problem of homology stability for congruence subgroups, *Comm. Algebra* 12 (1984), no. 17-18, 2081–2123.
- [Cha3] R. Charney, A generalization of a theorem of Vogtmann, *J. Pure Appl. Algebra* 44 (1987), no. 1-3, 107–125.
- [Ch1] T. Church, Homological stability for configuration spaces of manifolds, *Invent. Math.* 188 (2012), no. 2, 465–504, [arXiv:1103.2441v3](#).
- [Ch2] T. Church, personal communication.
- [ChEFa] T. Church, J. S. Ellenberg and B. Farb, FI-modules and stability for representations of symmetric groups, *Duke Math. J.* 164 (2015), no. 9, 1833–1910, [arXiv:1204.4533v4](#).
- [ChEFaNag] T. Church, J. S. Ellenberg, B. Farb and R. Nagpal, FI-modules over Noetherian rings, *Geom. Topol.* 18 (2014), no. 5, 2951–2984, [arXiv:1210.1854v2](#).
- [ChFa] T. Church and B. Farb, Representation theory and homological stability, *Adv. Math.* 245 (2013), 250–314, [arXiv:1008.1368v3](#).
- [ChPu] T. Church and A. Putman, Generating the Johnson filtration, *Geom. Topol.* 19 (2015), no. 4, 2217–2255, [arXiv:1311.7150v2](#).
- [CuVo] M. Culler and K. Vogtmann, Moduli of graphs and automorphisms of free groups, *Invent. Math.* 84 (1986), no. 1, 91–119.
- [DPu] M. Day and A. Putman, On the second homology group of the Torelli subgroup of $\mathrm{Aut}(F_n)$, preprint 2014, [arXiv:1408.6242v2](#).
- [Dj1] A. Djament, Foncteurs en grassmanniennes, filtration de Krull et cohomologie des foncteurs, *Mém. Soc. Math. Fr. (N.S.)* No. 111 (2007), xxii+213 pp.
- [Dj2] A. Djament, Le foncteur $V \mapsto \mathbb{F}_2[V]^{\otimes 3}$ entre \mathbb{F}_2 -espaces vectoriels est noethérien, *Ann. Inst. Fourier (Grenoble)* 59 (2009), no. 2, 459–490, [arXiv:math/0702667v4](#).
- [Dj3] A. Djament, Sur l’homologie des groupes unitaires à coefficients polynomiaux, *J. K-Theory* 10 (2012), no. 1, 87–139.
- [DjVe1] A. Djament and C. Vespa, Sur l’homologie des groupes orthogonaux et symplectiques à coefficients tordus, *Ann. Sci. Éc. Norm. Supér. (4)* 43 (2010), no. 3, 395–459, [arXiv:0808.4035v4](#).
- [DjVe2] A. Djament and C. Vespa, Sur l’homologie des groupes d’automorphismes des groupes libres à coefficients polynomiaux, *Comment. Math. Helv.* 90 (2015), no. 1, 33–58.
- [DrKu] J. Draisma and J. Kuttler, Bounded-rank tensors are defined in bounded degree, *Duke Math. J.* 163 (2014), no. 1, 35–63, [arXiv:1103.5336v2](#).
- [Dw] W. G. Dwyer, Twisted homological stability for general linear groups, *Ann. of Math. (2)* 111 (1980), no. 2, 239–251.
- [Fa] B. Farb, Representation stability, *Proceedings of the 2014 International Congress of Mathematicians. Volume II*, 1173–1196, [arXiv:1404.4065v1](#).

- [FaMar] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, 49, Princeton Univ. Press, Princeton, NJ, 2012.
- [FraFriPiSch] V. Franjou, E. Friedlander, T. Pirashvili, and L. Schwartz, *Rational representations, the Steenrod algebra and functor homology*, Panoramas et Synthèses, 16, Soc. Math. France, Paris, 2003.
- [FraTo] V. Franjou and A. Touzé (eds.), *Lectures on functor homology*, Progress in Mathematics, 311, Birkhäuser/Springer, Cham, 2015.
- [Fri] G. Friedman, Survey article: an elementary illustrated introduction to simplicial sets, *Rocky Mountain J. Math.* 42 (2012), no. 2, 353–423.
- [Ga] S. Galatius, Stable homology of automorphism groups of free groups, *Ann. of Math. (2)* 173 (2011), no. 2, 705–768, [arXiv:math/0610216v3](#).
- [GanLi] W. L. Gan, L. Li, Noetherian property of infinite EI categories, *New York J. Math.* 21 (2015), 369–382, [arXiv:1407.8235v3](#).
- [Gr] D. Grayson, Higher algebraic K -theory. II (after Daniel Quillen), in *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, 217–240. Lecture Notes in Math., 551, Springer, Berlin.
- [Har] J. L. Harer, Stability of the homology of the mapping class groups of orientable surfaces, *Ann. of Math. (2)* 121 (1985), no. 2, 215–249.
- [Hat] A. Hatcher, Homological stability for automorphism groups of free groups, *Comment. Math. Helv.* 70 (1995), no. 1, 39–62.
- [HatVo1] A. Hatcher and K. Vogtmann, Cerf theory for graphs, *J. London Math. Soc. (2)* 58 (1998), no. 3, 633–655.
- [HatVo2] A. Hatcher and K. Vogtmann, Homology stability for outer automorphism groups of free groups, *Algebr. Geom. Topol.* 4 (2004), 1253–1272, [arXiv:math/0406377v2](#).
- [HatVo3] A. Hatcher and K. Vogtmann, Tethers and homology stability for surfaces, preprint 2015, [arXiv:1508.04334v2](#).
- [HatWah] A. Hatcher and N. Wahl, Stabilization for mapping class groups of 3-manifolds, *Duke Math. J.* 155 (2010), no. 2, 205–269, [arXiv:0709.2173v4](#).
- [Hi] G. Higman, Ordering by divisibility in abstract algebras, *Proc. London Math. Soc. (3)* 2 (1952), 326–336.
- [I] N. V. Ivanov, On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients, in *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, 149–194, *Contemp. Math.*, 150, Amer. Math. Soc., Providence, RI.
- [Kr] J. B. Kruskal, The theory of well-quasi-ordering: A frequently discovered concept, *J. Combinatorial Theory Ser. A* 13 (1972), 297–305.
- [Ku1] N. J. Kuhn, Generic representations of the finite general linear groups and the Steenrod algebra. II, *K-Theory* 8 (1994), no. 4, 395–428.
- [Ku2] N. J. Kuhn, Invariant subspaces of the ring of functions on a vector space over a finite field, *J. Algebra* 191 (1997), no. 1, 212–227.
- [LeSz] R. Lee and R. H. Szczarba, On the homology and cohomology of congruence subgroups, *Invent. Math.* 33 (1976), no. 1, 15–53.
- [LySc] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer, Berlin, 1977.
- [MadWe] I. Madsen and M. Weiss, The stable moduli space of Riemann surfaces: Mumford’s conjecture, *Ann. of Math. (2)* 165 (2007), no. 3, 843–941, [arXiv:math/0212321v3](#).
- [Mc] D. McDuff, Configuration spaces of positive and negative particles, *Topology* 14 (1975), 91–107.
- [Mil] E. Y. Miller, The homology of the mapping class group, *J. Differential Geom.* 24 (1986), no. 1, 1–14.
- [MirVa] B. Mirzaii and W. van der Kallen, Homology stability for unitary groups, *Doc. Math.* 7 (2002), 143–166, [arXiv:math/011117v1](#).
- [Nak] M. Nakaoka, Decomposition theorem for homology groups of symmetric groups, *Ann. of Math. (2)* 71 (1960), 16–42.
- [PlRa] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, translated from the 1991 Russian original by Rachel Rowen, *Pure and Applied Mathematics*, 139, Academic Press, Boston, MA, 1994.

- [Po1] G. M. L. Powell, The Artinian conjecture for $I^{\otimes 2}$, J. Pure Appl. Algebra 128 (1998), no. 3, 291–310.
- [Po2] G. M. L. Powell, The structure of indecomposable injectives in generic representation theory, Trans. Amer. Math. Soc. 350 (1998), no. 10, 4167–4193.
- [Po3] G. M. L. Powell, On Artinian objects in the category of functors between \mathbf{F}_2 -vector spaces, in *Infinite length modules (Bielefeld, 1998)*, 213–228, Trends Math, Birkhäuser, Basel.
- [Pu1] A. Putman, The Picard group of the moduli space of curves with level structures, Duke Math. J. 161 (2012), no. 4, 623–674, [arXiv:0908.0555v3](#).
- [Pu2] A. Putman, Stability in the homology of congruence subgroups, Invent. Math. 202 (2015), no. 3, 987–1027, [arXiv:1201.4876v5](#).
- [Re] I. Reiner, Unimodular complements, Amer. Math. Monthly 63 (1956), 246–247.
- [SamSn1] S. V Sam and A. Snowden, GL-equivariant modules over polynomial rings in infinitely many variables, Trans. Amer. Math. Soc. 368 (2016), 1097–1158, [arXiv:1206.2233v3](#).
- [SamSn2] S. V Sam and A. Snowden, Introduction to twisted commutative algebras, preprint 2012, [arXiv:1209.5122v1](#).
- [SamSn3] S. V Sam and A. Snowden, Stability patterns in representation theory, Forum Math. Sigma 3 (2015), e11, 108 pp., [arXiv:1302.5859v2](#).
- [SamSn4] S. V Sam and A. Snowden, Gröbner methods for representations of combinatorial categories, J. Amer. Math. Soc. 30 (2017), 159–203, [arXiv:1409.1670v3](#).
- [Sat] T. Satoh, The abelianization of the congruence IA-automorphism group of a free group, Math. Proc. Cambridge Philos. Soc. 142 (2007), no. 2, 239–248.
- [Sch] L. Schwartz, *Unstable modules over the Steenrod algebra and Sullivan’s fixed point set conjecture*, Chicago Lectures in Mathematics, Univ. Chicago Press, Chicago, IL, 1994.
- [Sco] A. Scorichenko, Stable K-theory and functor homology over a ring, thesis 2000.
- [Va] W. van der Kallen, Homology stability for linear groups, Invent. Math. 60 (1980), no. 3, 269–295.
- [WahRW] N. Wahl and O. Randal-Williams, Homological stability for automorphism groups, preprint 2015, [arXiv:1409.3541v3](#).
- [Wi] J. Wilson, FI_W -modules and stability criteria for representations of the classical Weyl groups, J. Algebra 420 (2014), 269–332, [arXiv:1309.3817v2](#).

Andrew Putman
 Department of Mathematics
 Rice University, MS 136
 6100 Main St.
 Houston, TX 77005

Steven V Sam
 Department of Mathematics
 University of California, Berkeley
 933 Evans Hall
 Berkeley, CA 94720

Current address:
 Department of Mathematics
 University of Notre Dame
 279 Hurley Hall
 South Bend, IN 46556
andyp@nd.edu

Current address:
 Department of Mathematics
 University of Wisconsin, Madison
 480 Lincoln Drive
 Madison, WI 53706
svs@math.wisc.edu