

# Stability in the homology of congruence subgroups

Andrew Putman\*

February 9, 2015

## Abstract

The homology groups of many natural sequences of groups  $\{G_n\}_{n=1}^\infty$  (e.g. general linear groups, mapping class groups, etc.) stabilize as  $n \rightarrow \infty$ . Indeed, there is a well-known machine for proving such results that goes back to early work of Quillen. Church and Farb discovered that many sequences of groups whose homology groups do not stabilize in the classical sense actually stabilize in some sense as representations. They called this phenomena *representation stability*. We prove that the homology groups of congruence subgroups of  $GL_n(R)$  (for almost any reasonable ring  $R$ ) satisfy a strong version of representation stability that we call *central stability*. The definition of central stability is very different from Church-Farb's definition of representation stability (it is defined via a universal property), but we prove that it implies representation stability. Our main tool is a new machine for proving central stability that is analogous to the classical homological stability machine.

## 1 Introduction

**Arithmetic groups and Borel stability.** The homology groups of arithmetic groups like  $SL_n(\mathbb{Z})$  play important roles in algebraic K-theory, the theory of locally symmetric spaces, and the study of automorphic forms. The fundamental theorem about them is the Borel stability theorem [3], which among other things calculates  $H_k(SL_n(\mathbb{Z}); \mathbb{Q})$  for  $n \gg k$ . The answer turns out to be independent of  $n$  for  $n \gg k$ , so one says that  $H_k(SL_n(\mathbb{Z}); \mathbb{Q})$  *stabilizes*. This stability property was later generalized to  $\mathbb{Z}$ -coefficients by Maazen [17].

For  $\ell \geq 2$ , the *level  $\ell$  congruence subgroup* of  $SL_n(\mathbb{Z})$ , denoted  $SL_n(\mathbb{Z}, \ell)$ , is the kernel of the natural map  $SL_n(\mathbb{Z}) \rightarrow SL_n(\mathbb{Z}/\ell)$ . Both  $SL_n(\mathbb{Z})$  and  $SL_n(\mathbb{Z}, \ell)$  are lattices in  $SL_n(\mathbb{R})$ , and Borel's theorem applies to all such lattices. Amazingly, the output of Borel's theorem does not depend on the lattice one is investigating, so for  $n \gg k$  we have  $H_k(SL_n(\mathbb{Z}, \ell); \mathbb{Q}) \cong H_k(SL_n(\mathbb{Z}); \mathbb{Q})$ . In particular,  $H_k(SL_n(\mathbb{Z}, \ell); \mathbb{Q})$  stabilizes.

**Torsion.** However, the torsion in the homology of  $SL_n(\mathbb{Z}, \ell)$  is far more complicated. A theorem of Charney [7] says that if  $\mathbb{F}$  is a field such that  $\text{char}(\mathbb{F})$  does not divide  $\ell$ , then  $H_k(SL_n(\mathbb{Z}, \ell); \mathbb{F})$  stabilizes. But this restriction on  $\text{char}(\mathbb{F})$  is necessary – Lee and Szczarba [16] proved that  $H_1(SL_n(\mathbb{Z}, \ell); \mathbb{Z}) \cong (\mathbb{Z}/\ell)^{n^2-1}$  for  $n \geq 3$ , which gets larger and larger as  $n$  increases. There are few concrete calculations of any of the higher integral homology groups of  $SL_n(\mathbb{Z}, \ell)$ , and their structure remains largely a mystery. Indeed, aside from certain small values of  $n$  and  $\ell$ , even  $H_2(SL_n(\mathbb{Z}, \ell); \mathbb{Z})$  is not known. Moreover, the sporadic calculations that do exist display no obvious patterns.

In this paper, we give a precise description of how  $H_k(SL_n(\mathbb{Z}, \ell); \mathbb{F})$  changes as  $n$  increases for fields  $\mathbb{F}$  whose characteristic is positive but not too small. An easily stated consequence of our results is the following.

**Theorem A.** *For  $k \geq 1$ , there exists some  $P_k$  such that if  $\mathbb{F}$  is a field with  $\text{char}(\mathbb{F}) \geq P_k$ , then for all  $\ell \geq 2$ , there exists a polynomial  $\phi(n)$  such that  $\phi(n) = \dim_{\mathbb{F}}(H_k(SL_n(\mathbb{Z}, \ell); \mathbb{F}))$  for  $n \gg 0$ .*

This generalizes Lee and Szczarba's theorem, which says that we can take  $\phi(n) = n^2 - 1$  if  $k = 1$  and  $\text{char}(\mathbb{F})$  divides  $\ell$ . Bounds for the constant  $P_k$  can be easily extracted from our results; see below.

*Remark.* Our restriction on  $\text{char}(\mathbb{F})$  depends only on  $k$ , not on  $\ell$ , so Theorem A can be applied in situations where  $\text{char}(\mathbb{F})$  divides  $\ell$ . We conjecture that the restrictions on  $\text{char}(\mathbb{F})$  in Theorem A and in Theorem B below are unnecessary.

---

\*Supported in part by NSF grant DMS-1005318

**More general rings.** In fact, our techniques give results about congruence subgroups of  $\mathrm{GL}_n(R)$  for very general rings  $R$ . It suffices for  $R$  to be a commutative Noetherian ring of finite Krull dimension. This includes rings of integers in algebraic number fields, but it also includes more exotic rings like  $\mathbb{Z}[x_1, \dots, x_m]$  and  $\mathbb{F}_p[t]$ . For a ring  $R$  of this type, van der Kallen [21] proved that  $H_k(\mathrm{GL}_n(R); \mathbb{Z})$  stabilizes as  $n$  increases. Moreover, Charney [7] generalized Borel’s theorem to prove that the  $\mathbb{Q}$ -homology groups of *finite-index* congruence subgroups of  $\mathrm{GL}_n(R)$  stabilize. However, there also exist infinite-index congruence subgroups. Their homology groups can display interesting phenomena even over  $\mathbb{Q}$ , and our results cover these cases as well. There are essentially no known computations of the homology groups of congruence subgroups of  $\mathrm{GL}_n(R)$  for these more general rings.

**Group actions.** The key to understanding the homology groups of congruence subgroups is to observe that they are not just naked abelian groups, but also representations. Indeed,  $H_k(\mathrm{SL}_n(\mathbb{Z}, \ell); \mathbb{Z})$  is acted upon by

$$\mathrm{SL}_n(\mathbb{Z}/\ell) = \mathrm{SL}_n(\mathbb{Z}) / \mathrm{SL}_n(\mathbb{Z}, \ell).$$

This a general phenomena : if  $G$  is a normal subgroup of  $\Gamma$ , then the conjugation action of  $\Gamma$  on  $G$  induces an action of  $\Gamma/G$  on  $H_k(G)$ . Here we are using the fact that the conjugation action of  $G$  on itself induces the trivial action on  $H_k(G)$ . Lee and Szczarba’s theorem actually identifies the  $\mathrm{SL}_n(\mathbb{Z}/\ell)$ -action on  $H_1(\mathrm{SL}_n(\mathbb{Z}, \ell); \mathbb{Z})$ . They prove that there is an  $\mathrm{SL}_n(\mathbb{Z}, \ell)$ -equivariant isomorphism  $H_1(\mathrm{SL}_n(\mathbb{Z}, \ell); \mathbb{Z}) \cong \mathfrak{sl}_n(\mathbb{Z}/\ell)$ , where  $\mathfrak{sl}_n(\mathbb{Z}/\ell)$  is the abelian group of  $n \times n$  matrices over  $\mathbb{Z}/\ell$  with trace 0 and  $\mathrm{SL}_n(\mathbb{Z}/\ell)$  acts on  $\mathfrak{sl}_n(\mathbb{Z}/\ell)$  by conjugation.

**Representation stability.** In summary,  $H_1(\mathrm{SL}_n(\mathbb{Z}, \ell); \mathbb{Z})$  does not stabilize as an abelian group, but in some sense it stabilizes as a representation. In [9], Church and Farb introduced the notion of *representation stability* to make this observation and others like it precise. The basic idea is to give a “stabilization recipe” for producing the next representation in a sequence from the previous ones. Church and Farb proposed stabilization recipes for many different kinds of representations and proved that many natural sequences of representations obeyed their rules.

**New machine.** There is a well-known machine of Quillen which has been used by many people to prove ordinary homological stability for different kinds of groups (see, e.g., [12]). We construct a version of this machine for representation stability. This involves a delicate interplay between equivariant homology and representation theory (in both characteristic 0 and  $p$ ). The key difficulty is that we need a much stronger inductive hypothesis to make the proof work than is provided by representation stability (which, in particular, is not functorial in any natural sense).

The solution to this is the new notion of *central stability*. This is defined in terms of a representation-theoretic universal property. In many ways, its definition is simpler than Church and Farb’s definition of representation stability, which is defined in terms of the structure of the irreducible representations of the groups in question. Nonetheless, it gives much tighter control over the representation theory than does representation stability. In Theorem E below, we will prove that for finite-dimensional representations over a field of characteristic 0, central stability implies representation stability.

*Remark.* It is not obvious from its definition that central stability provides as much control over the representations in question as it does. Indeed, nearly half of this paper is devoted to proving a certain representation-theoretic “regularity” theorem concerning central stability; see Proposition 4.5.

Our machine can be applied in other contexts too. For instance, in [19], the author proves that the homology groups of mapping class groups of arbitrary connected manifolds  $M^n$  ( $n \geq 2$ ) with marked points and nonempty boundary satisfy central stability. We emphasize that these manifolds are completely general – they are not assumed to be compact or even of finite type.

*Remark.* For representations over  $\mathrm{Sp}_{2g}(\mathbb{Z})$ , some of the ideas in the theory of representation stability are also contained in unpublished work of Hain on the cohomology of the Torelli group from the 1990’s.

*Remark.* After a draft of this paper was circulated, we learned that Church, Ellenberg, and Farb [8] have developed a theory of what they call *FI-modules*, which at least in characteristic 0 seem to be closely related to the notion of central stability.

**Representation stability à la Church–Farb.** Before stating our theorems, we need to give a precise definition of central stability. The definition of representation stability introduced by Church and Farb suffers from three defects.

- It is only appropriate for finite-dimensional representations over a field of characteristic 0.
- It is a bit ad-hoc, and requires a “consistent naming scheme” for the irreducible representations.
- It does not pin down the maps between the representations in a sequence.

Central stability overcomes these difficulties; in particular, its definition makes no reference to the characteristic of the field or the dimensions of the representation.

**Central stability, motivation.** Let us return to the example of  $\mathrm{SL}_n(\mathbb{Z}, \ell)$ . Fixing a field  $\mathbb{F}$  and some  $k \geq 1$ , set  $V_n = \mathrm{H}_k(\mathrm{SL}_n(\mathbb{Z}, \ell); \mathbb{F})$ . We will view  $V_n$  as a representation of the symmetric group  $S_n$ , which acts on  $V_n$  via the conjugation action of permutation matrices on  $\mathrm{SL}_n(\mathbb{Z}, \ell)$ .

*Remark.* Of course, one would ideally want a description of  $\mathrm{H}_k(\mathrm{SL}_n(\mathbb{Z}, \ell); \mathbb{F})$  as a representation of the group  $\mathrm{SL}_n(\mathbb{Z}/\ell)$ ; however, especially in finite characteristic the representation theory of  $\mathrm{SL}_n(\mathbb{Z}/\ell)$  is extremely complicated and difficult to work with.

How should we expect the  $S_{n+1}$ -representation  $V_{n+1}$  to be constructed from the  $S_n$ -representation  $V_n$ ? A first guess is that  $V_{n+1}$  is the induced representation  $\mathrm{Ind}_{S_n}^{S_{n+1}} V_n$ . Unfortunately, this cannot be the case. Let  $P \in \mathrm{GL}_{n+1}(\mathbb{Z})$  be the permutation matrix corresponding to the transposition  $(n, n+1) \in S_{n+1}$ . We then have  $P\phi P^{-1} = \phi$  for all  $\phi \in \mathrm{SL}_{n-1}(\mathbb{Z}, \ell) \subset \mathrm{SL}_{n+1}(\mathbb{Z}, \ell)$ . This implies that  $P$  must act trivially on the image of  $V_{n-1}$  in  $V_{n+1}$ . In general, this will not be the case in the induced representation.

**Central stabilization.** It turns out that in a stable range, this is all that goes wrong. To formalize this, we now introduce our “stabilization recipe”. Let  $\phi_{n-1} : V_{n-1} \rightarrow V_n$  be an  $S_{n-1}$ -equivariant map from a representation of  $S_{n-1}$  to a representation of  $S_n$ . The *central stabilization* of  $\phi_{n-1}$ , denoted  $\mathcal{C}(V_{n-1} \xrightarrow{\phi_{n-1}} V_n)$ , is the  $S_{n+1}$ -representation which is the largest quotient of  $\mathrm{Ind}_{S_n}^{S_{n+1}} V_n$  such that  $(n, n+1)$  acts trivially on the image of  $V_{n-1}$ . More precisely, let  $W = \mathrm{Ind}_{S_n}^{S_{n+1}} V_n$ . Composing  $\phi_{n-1}$  with the natural inclusion  $V_n \hookrightarrow W$ , we obtain an  $S_{n-1}$ -equivariant map  $\phi'_{n-1} : V_{n-1} \rightarrow W$ . Then  $\mathcal{C}(V_{n-1} \xrightarrow{\phi_{n-1}} V_n) = W/U$ , where  $U$  is the span of the  $S_{n+1}$ -orbit of the set

$$\{\vec{v} - (n, n+1) \cdot \vec{v} \mid \vec{v} = \phi'_{n-1}(\vec{v}') \text{ for some } \vec{v}' \in V_{n-1}\}.$$

Observe that there is a natural  $S_n$ -equivariant map  $V_n \rightarrow \mathcal{C}(V_{n-1} \xrightarrow{\phi_{n-1}} V_n)$ .

**Examples.** Here are some examples to convince the reader that this is a natural concept.

*Example.* For  $n \geq 1$ , let  $\mathcal{T}_n \cong \mathbb{F}$  be the trivial  $S_n$ -representation. Then  $\mathcal{C}(\mathcal{T}_{n-1} \rightarrow \mathcal{T}_n) = \mathcal{T}_{n+1}$ . Indeed,  $W = \mathrm{Ind}_{S_n}^{S_{n+1}} \mathcal{T}_n$  is the *permutation representation*  $\mathcal{P}_{n+1}$ , i.e. the vector space consisting of  $\mathbb{F}$ -linear combinations of formal symbols  $\{[i] \mid 1 \leq i \leq n+1\}$ . The group  $S_{n+1}$  acts on  $\mathcal{P}_{n+1}$  in the obvious way. The image of  $\mathcal{T}_{n-1}$  in  $W$  is the span of  $[n+1]$ . Defining  $U \subset W$  to be the span of

$$S_{n+1} \cdot \{[n+1] - [n]\} = \{[i] - [j] \mid 1 \leq i, j \leq n+1 \text{ distinct}\},$$

we have  $\mathcal{C}(\mathcal{T}_{n-1} \rightarrow \mathcal{T}_n) = W/U = \mathcal{T}_{n+1}$ .

*Example.* For  $n \geq 2$ , we have  $\mathcal{C}(\mathcal{P}_{n-1} \rightarrow \mathcal{P}_n) = \mathcal{P}_{n+1}$ . Indeed,  $W = \mathrm{Ind}_{S_n}^{S_{n+1}} \mathcal{P}_n$  is the vector space consisting of  $\mathbb{F}$ -linear combinations of formal symbols  $\{[i, j] \mid 1 \leq i, j \leq n+1, i \neq j\}$  and the obvious  $S_{n+1}$ -action. The image of  $\mathcal{P}_{n-1}$  in  $W$  is spanned by  $\{[i, n+1] \mid 1 \leq i \leq n-1\}$ . Defining  $U \subset W$  to be the span of

$$S_{n+1} \cdot \{[i, n+1] - [i, n] \mid 1 \leq i \leq n-1\} = \{[i, j] - [i, k] \mid 1 \leq i, j, k \leq n+1 \text{ distinct}\},$$

we have  $\mathcal{C}(\mathcal{P}_{n-1} \rightarrow \mathcal{P}_n) = W/U = \mathcal{P}_{n+1}$ .

**Central stability, definition.** We finally define central stability. For each  $n$ , let  $V_n$  be a representation of  $S_n$  over  $\mathbb{F}$  and let  $\phi_n : V_n \rightarrow V_{n+1}$  be a linear map which is  $S_n$ -equivariant. We will call the sequence

$$V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} V_4 \xrightarrow{\phi_4} \dots$$

a *coherent sequence* of representations of the symmetric group. We will say that our coherent sequence is *centrally stable* starting at  $N \geq 2$  if for all  $n \geq N$ , we have  $V_{n+1} = \mathcal{C}(V_{n-1} \xrightarrow{\phi_{n-1}} V_n)$  and  $\phi_n$  is the natural map  $V_n \rightarrow \mathcal{C}(V_{n-1} \xrightarrow{\phi_{n-1}} V_n)$ .

**Main theorem.** We now turn to general congruence subgroups. Let  $R$  be a ring with a unit (not necessarily commutative) and let  $q$  be a 2-sided ideal of  $R$ . The *level  $q$  congruence subgroup* of  $\mathrm{GL}_n(R)$ , denoted  $\mathrm{GL}_n(R, q)$ , is the kernel of the map  $\mathrm{GL}_n(R) \rightarrow \mathrm{GL}_n(R/q)$ . The maps  $\mathrm{GL}_n(R) \rightarrow \mathrm{GL}_n(R/q)$  need not be surjective, so there might not exist a  $\mathrm{GL}_n(R/q)$ -action on the homology groups of  $\mathrm{GL}_n(R, q)$ . However,  $S_n$  is embedded in  $\mathrm{GL}_n(R)$  as the group of permutation matrices. Restricting the conjugation action of  $\mathrm{GL}_n(R)$  on  $\mathrm{GL}_n(R, q)$  to  $S_n$ , we get an action of  $S_n$  on  $\mathrm{GL}_n(R, q)$  and thus on  $H_*(\mathrm{GL}_n(R, q); \mathbb{F})$ .

We will need to assume that the pair  $(R, q)$  satisfies the Bass's *stable range condition*  $\mathrm{SR}_{d+2}$  for some  $d \geq 0$  (see §3). This condition depends on  $q$ ; however, Bass also defined a stable range condition  $\mathrm{SR}_{d+2}$  for rings  $R$  and proved that if  $R$  satisfies  $\mathrm{SR}_{d+2}$ , then  $(R, q)$  satisfies  $\mathrm{SR}_{d+2}$  for all 2-sided ideals  $q$ . Almost any reasonable ring  $R$  satisfies  $\mathrm{SR}_{d+2}$  for some  $d$ . For example, fields satisfy  $\mathrm{SR}_2$  and  $\mathbb{Z}$  satisfies  $\mathrm{SR}_3$ . More generally, in [1, Theorem V.3.5] Bass proved that if  $R$  is a commutative Noetherian ring with Krull dimension  $d$ , then  $R$  satisfies  $\mathrm{SR}_{d+2}$ .

Van der Kallen [21] proved that if  $R$  satisfies  $\mathrm{SR}_{d+2}$  for some  $d$ , then the homology groups of  $\mathrm{GL}_n(R)$  are stable in the classical sense.

**Theorem B.** *Let  $R$  be a ring with unit and let  $q$  be a 2-sided ideal of  $R$ . Assume that  $(R, q)$  satisfies  $\mathrm{SR}_{d+2}$ . Fix  $k \geq 1$ , and assume that either  $\mathrm{char}(\mathbb{F}) = 0$  or  $\mathrm{char}(\mathbb{F}) \geq (d+8)2^{k-1} - 3$ . Then the sequence*

$$H_k(\mathrm{GL}_1(R, q); \mathbb{F}) \rightarrow H_k(\mathrm{GL}_2(R, q); \mathbb{F}) \rightarrow H_k(\mathrm{GL}_3(R, q); \mathbb{F}) \rightarrow \dots$$

*of representations of the symmetric group is centrally stable with stability beginning at  $(d+8)2^{k-1} - 4$ .*

*Remark.* We want to emphasize that in Theorem B we are *not* assuming that the homology groups of  $\mathrm{GL}_n(R, q)$  are finite-dimensional.

*Remark.* In [7, §5.4], Charney gives a number of conditions on  $\mathbb{F}$  and  $(R, q)$  which ensure that the groups  $H_k(\mathrm{GL}_n(R, q); \mathbb{F})$  are stable in the classical sense. For instance, she proves that this is true if  $R$  satisfies  $\mathrm{SR}_{d+2}$  and  $R/q$  is finite and  $\mathrm{char}(\mathbb{F}) = 0$ . However, we should emphasize that we are not assuming that  $R/q$  is finite, so our congruence subgroups need not be finite-index and Theorem B has content even if  $\mathrm{char}(\mathbb{F}) = 0$ .

If  $R$  is a commutative ring, then there is a determinant map  $\mathrm{GL}_n(R) \rightarrow R^*$  and we can define  $\mathrm{SL}_n(R)$  and  $\mathrm{SL}_n(R, q)$  in the obvious way. In this notation, the congruence subgroup  $\mathrm{SL}_n(\mathbb{Z}, \ell)$  of  $\mathrm{SL}_n(\mathbb{Z})$  is  $\mathrm{SL}_n(\mathbb{Z}, \ell\mathbb{Z})$ . We then have the following.

**Theorem C.** *Let  $R$  be a commutative ring with unit and let  $q$  be an ideal of  $R$ . Assume that  $(R, q)$  satisfies  $\mathrm{SR}_{d+2}$ . Fix  $k \geq 1$ , and assume that either  $\mathrm{char}(\mathbb{F}) = 0$  or  $\mathrm{char}(\mathbb{F}) \geq (d+8)2^{k-1} - 3$ . Then the sequence*

$$H_k(\mathrm{SL}_1(R, q); \mathbb{F}) \rightarrow H_k(\mathrm{SL}_2(R, q); \mathbb{F}) \rightarrow H_k(\mathrm{SL}_3(R, q); \mathbb{F}) \rightarrow \dots$$

*of representations of the symmetric group is centrally stable with stability beginning at  $(d+8)2^{k-1} - 4$ .*

There is a huge literature on the finiteness properties of groups like  $\mathrm{GL}_n(R, q)$  for special choices of  $R$  and  $q$ . See, for instance, [6]. However, we are not aware of any concrete calculations of even their first homology groups aside from Lee and Szczarba's calculation of  $H_1(\mathrm{SL}_n(\mathbb{Z}, \ell); \mathbb{Z})$ .

**Central stability implies polynomial dimensions.** A fundamental insight of Church, Ellenberg, and Farb [8] is that there is a close relationship between a coherent sequence of representations being representation stable and the dimension of the  $n^{\text{th}}$  term in the sequence being given by a polynomial in  $n$  for  $n \gg 0$ . This inspires the following theorem.

**Theorem D.** *Let*

$$V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow V_4 \longrightarrow \cdots$$

*be a coherent sequence of representations of the symmetric group over a field  $\mathbb{F}$  which is centrally stable with stability starting at  $N$ . Assume that either  $\text{char}(\mathbb{F}) = 0$  or  $\text{char}(\mathbb{F}) \geq 2N + 2$ . Then one of the following holds.*

- $\dim_{\mathbb{F}} V_n = \infty$  for  $n \geq 2N + 1$ .
- *There exists a polynomial  $\phi(n)$  such that  $\phi(n) = \dim_{\mathbb{F}} V_n$  for  $n \geq 2N + 1$ .*

Theorem D can be combined with Theorem B–C to deduce theorems analogous to Theorem A for the homology groups of  $\text{GL}_n(R, q)$  and  $\text{SL}_n(R, q)$ . These results must allow for the possibility that the dimensions of the relevant homology groups are infinite (as in the first possibility in Theorem D). Theorem A does not allow for this possibility; in fact, Borel and Serre [4] proved that  $H_k(\text{SL}_n(\mathbb{Z}, \ell); \mathbb{F})$  is always finite-dimensional.

**Specht stability.** To relate central stability to the fine structure of the representation theory of  $S_n$  (and thus to Church and Farb’s notion of representation stability), we will give in §6 below a definition of what we call *Specht stability*. For finite-dimensional representations over a field of characteristic 0, Specht stability is a strengthening of Church and Farb’s notion of representation stability. This definition is related to (and implies) Church’s notation of *monotonicity* for stable representations, which he defined in [10]. We will prove the following theorem, which shows that sequences of representations which are centrally stable are also Specht stable, and thus also monotone in the sense of Church and representation stable in the sense of Church-Farb.

**Theorem E.** *Let*

$$V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow V_4 \longrightarrow \cdots$$

*be a coherent sequence of representations of the symmetric group over a field  $\mathbb{F}$  which is centrally stable with stability starting at  $N$ . Assume that either  $\text{char}(\mathbb{F}) = 0$  or  $\text{char}(\mathbb{F}) \geq 2N + 2$ . Then the sequence is Specht stable with stability starting at  $2N + 1$ .*

One special case of Theorem E appears in the literature. Assume that  $\text{char}(\mathbb{F}) = 0$ . Fix some  $N \geq 2$  and let  $V_N$  be a finite-dimensional representation of  $S_N$ . Recalling that  $\mathcal{T}_k$  is the trivial representation of  $S_k$ , for  $n \geq N$  define  $V_n = \text{Ind}_{S_N \times S_{n-N}}^{S_n} V_N \boxtimes \mathcal{T}_{n-N}$ . Here  $V_N \boxtimes \mathcal{T}_{n-N}$  is the external tensor product of the  $S_N$ -representation  $V_N$  and the  $S_{n-N}$ -representation  $\mathcal{T}_{n-N}$ , i.e. the  $S_n \times S_{n-N}$ -representation whose underlying vector space is  $V_N \otimes_{\mathbb{F}} \mathcal{T}_{n-N}$  and where  $(g, h) \in S_n \times S_{n-N}$  acts on  $v \otimes w \in V_N \otimes_{\mathbb{F}} \mathcal{T}_{n-N}$  via  $(g, h)(v \otimes w) = (gv) \otimes (hw)$ . There are natural maps  $V_n \hookrightarrow V_{n+1}$ , and it is easy to see that the sequence

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow V_N \longrightarrow V_{N+1} \longrightarrow V_{N+2} \longrightarrow \cdots$$

is centrally stable starting at  $N$ . Hemmer [14] proved that this sequence is representation stable in the sense of Church-Farb starting at  $2N$  and Church [10] proved that it is monotone starting at  $N$ . There is also an alternate proof of both of these results due to Sam-Weyman [20].

**Outline of paper.** We begin in §2 by describing our machine for proving central stability. This is followed by §3, which shows how to apply this machine to congruence subgroups and prove Theorems B and C. Next, in §4 we construct the *central stability chain complex*, which is a technical tool needed for our machine. In §5, we prove that our machine works. This proof depends on a proposition about the central stability chain complex. This proposition is proven in §6, which also defines Specht stability. This proof depends on Theorem E, which is proven in §7. Finally, in §8 we prove Theorem D.

**Acknowledgments.** I wish to thank Ruth Charney, Jordan Ellenberg, Benson Farb, Oscar Randal-Williams, and Ben Webster for their help. I want to offer special thanks to Tom Church for pointing out an error in a previous version of this paper and for his help in figuring out how to patch it.

## 2 Description of central stability machine

We now describe our machine for proving central stability. This machine is similar to the classical homological stability machine as described in, for example, [12, §5] (which we recommend reading as motivation).

Fix a field  $\mathbb{F}$ . Assume that we are given groups  $\{G_n\}_{n=1}^\infty$  and  $\{\tilde{G}_n\}_{n=1}^\infty$  together with splittings  $\tilde{G}_n = G_n \rtimes S_n$ . Moreover, assume that we are given inclusions  $G_n \hookrightarrow G_{n+1}$  and  $\tilde{G}_n \hookrightarrow \tilde{G}_{n+1}$  for all  $n$  which fit into a commutative diagram of the form

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & G_{n-1} & \longrightarrow & G_n & \longrightarrow & G_{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & \tilde{G}_{n-1} & \longrightarrow & \tilde{G}_n & \longrightarrow & \tilde{G}_{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & S_{n-1} & \longrightarrow & S_n & \longrightarrow & S_{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

with the maps  $S_n \rightarrow S_{n+1}$  the standard inclusions. The conjugation action of  $\tilde{G}_n$  on its normal subgroup  $G_n$  induces an action of  $S_n$  on  $H_k(G_n; \mathbb{F})$  for all  $k \geq 0$ . Our goal is to prove that the coherent sequence

$$H_k(G_1; \mathbb{F}) \longrightarrow H_k(G_2; \mathbb{F}) \longrightarrow H_k(G_3; \mathbb{F}) \longrightarrow \cdots$$

of representations of the symmetric group is centrally stable.

Before describing the inputs to our machine, we will need a definition. In this paper, all actions of groups on simplicial complexes are assumed to be simplicial.

**Definition.** A group  $G$  acts on a simplicial complex  $X$  *nicely* if it satisfies the following condition. Consider two vertices  $w$  and  $w'$  of  $X$  which are joined by an edge. Then there does not exist any  $g \in G$  such that  $g \cdot w = w'$ .

*Remark.* If  $G$  acts nicely on a simplicial complex  $X$ , then  $X/G$  can be equipped with the structure of a cell complex whose  $\ell$ -cells are the  $G$ -orbits of  $\ell$ -cells in  $X$ . We remark that this might not be a simplicial complex structure since there might be multiple cells with the same set of vertices (it is what we will call a *weak simplicial complex* in §5.1 below).

For each  $n$ , our machine will require the following inputs.

- A simplicial complex  $X_n$  such that  $G_n$  acts nicely on  $X_n$ . Also, this action should extend to a (not necessarily nice) action of  $\tilde{G}_n$  on  $X_n$ .
- An  $(n-1)$ -simplex  $\Delta_n$  of  $X_n$  and an enumeration  $\{v_1^n, \dots, v_n^n\}$  of the vertices of  $\Delta_n$ .

Of course, we will require these inputs to satisfy a sequence of conditions. First, we will need the  $X_n$  to be highly connected so that we can use these actions to calculate the homology groups of  $G_n$ . In fact, we can get away with assuming that the  $X_n$  are highly acyclic. Recall that a space  $Y$  is *k-acyclic* if  $H_q(Y; \mathbb{Z}) = 0$  for  $0 \leq q \leq k$ . We make the following assumption about the  $X_n$ .

**Assumption 1.** For some constant  $C \geq 1$ , for all  $k \geq 1$  the space  $X_n$  is  $k$ -acyclic for  $n \geq C2^{k-1} - 3$ .

Next, we will need  $\Delta_n$  to be a strict fundamental domain for the action, at least in a stable range. More precisely, we need the following.

**Assumption 2.** For  $k \geq 1$  and  $n \geq C2^{k-1} - 3$ , the  $G_n$ -orbit of every simplex in the  $(k+2)$ -skeleton of  $X_n$  contains a simplex of  $\Delta_n$ . Here  $C$  is the same constant as in Assumption 1.

*Remark.* The niceness of the  $G_n$ -action on  $X_n$  ensures that no two simplices of  $\Delta_n$  are in the same  $G_n$ -orbit.

We need the following assumption on the stabilizers of this action.

**Assumption 3.** For  $0 \leq i \leq n-2$ , the stabilizer in  $G_n$  of the simplex  $\{v_{n-i}^n, \dots, v_n^n\}$  is  $G_{n-i-1}$ .

Our final two assumptions concern the extension of the action to  $\tilde{G}_n$ , and in particular the action of  $S_n \subset \tilde{G}_n$ .

**Assumption 4.** The action of  $S_n$  preserves the set  $\{v_1^n, \dots, v_n^n\}$ . Moreover, the action of  $S_n$  on this set is the usual permutation action.

*Remark.* Assumptions 2 and 4 together imply that  $\tilde{G}_n$  acts transitively on  $k+2$ -simplices for  $n \geq C2^{k-1} - 2$ .

**Assumption 5.** Consider any  $2 \leq m \leq n$ , and denote by  $S_{\{m, \dots, n\}} \subset S_n$  the symmetric group on the set  $\{m, \dots, n\}$ . Then  $S_{\{m, \dots, n\}}$  lies in the centralizer of  $G_{m-1} \subset G_n$ .

With these assumptions, our theorem is as follows. Its proof is in §5.

**Theorem 2.1.** *Let the notation and assumptions be as above, and fix some  $k \geq 1$ . Assume that either  $\text{char}(\mathbb{F}) = 0$  or  $\text{char}(\mathbb{F}) \geq C2^{k-1} - 3$ . Then the sequence  $\{\mathbf{H}_k(G_n; \mathbb{F})\}_{n=1}^\infty$  of representations of the symmetric group is centrally stable with stability starting at  $C2^{k-1} - 4$ .*

### 3 Stability for congruence subgroups

We now show how to apply Theorem 2.1 to congruence subgroups and prove Theorems B and C. The proofs of these results are similar, so for concreteness we will give the details for Theorem B and let the reader make the obvious modifications to prove Theorem C.

**Setup.** Fix a ring  $R$  and a proper 2-sided ideal  $q$  of  $R$ . Recall that

$$\text{GL}_n(R, q) = \ker(\text{GL}_n(R) \rightarrow \text{GL}_n(R/q)).$$

Next, let  $\tilde{\text{GL}}_n(R, q) = \text{GL}_n(R, q) \cdot S_n$ , where  $S_n \subset \text{GL}_n(R)$  is the group of permutation matrices. Clearly  $\tilde{\text{GL}}_n(R, q) = \text{GL}_n(R, q) \rtimes S_n$ .

We will assume that  $(R, q)$  satisfies the stable range condition  $\text{SR}_{d+2}$ , which we now define. See [1, Chapter 5] for more details. Let  $\vec{e}_i \in R^n$  denote the vector with a 1 in position  $i$  and zeros elsewhere.

**Definition.** A set  $\{\vec{v}_1, \dots, \vec{v}_k\}$  of vectors in  $R^n$  is *unimodular* if  $R\vec{v}_1 + \dots + R\vec{v}_k$  is a direct summand of  $R^n$ .

*Remark.* If  $\vec{v} = (a_1, \dots, a_n) \in R^n$  is a vector, then the set  $\{\vec{v}\}$  is unimodular if and only if  $Ra_1 + \dots + Ra_n = R$ . We will then say that the vector  $\vec{v}$  is unimodular.

**Definition.** We will say that  $(R, q)$  satisfies the *stable range condition*  $\text{SR}_{d+2}$  if the following condition is satisfied for all  $n \geq d+2$ . Let  $\vec{v} = (a_1, \dots, a_n) \in R^n$  be a unimodular vector such that  $\vec{v} \equiv \vec{e}_1$  modulo  $q$ . There then exist  $b_1, \dots, b_{n-1} \in q$  such that  $(a_1 + b_1a_n, \dots, a_{n-1} + b_{n-1}a_n) \in R^{n-1}$  is unimodular.

**The simplicial complexes.** We now discuss the simplicial complex we will use, which is a slight variant on a complex introduced by Charney in [7]. Let  $\cdot$  denote the usual dot product on  $R^n$ . We remark that if  $R$  is not commutative, then  $\cdot$  is not commutative.

**Definition.** The  $n$ -dimensional *complex of split partial bases* over  $q$ , denoted  $\mathcal{S}\mathcal{B}_n(R, q)$ , is the simplicial complex whose  $k$ -simplices are sets  $\{(\vec{v}_0, \vec{w}_0), \dots, (\vec{v}_k, \vec{w}_k)\} \subset R^n \times R^n$  satisfying the following conditions.

- The set  $\{\vec{v}_0, \dots, \vec{v}_k\}$  is unimodular.
- For each  $0 \leq i \leq k$ , there exists some  $1 \leq j_i \leq n$  such that  $\vec{v}_i \equiv \vec{w}_i \equiv \vec{e}_{j_i}$  modulo  $q$ .
- For  $0 \leq i, j \leq k$ , we have  $\vec{v}_i \cdot \vec{w}_j = \delta_{ij}$ .

*Remark.* One should think of a vertex  $(\vec{v}, \vec{w})$  of  $\mathcal{S}\mathcal{B}_n(R, q)$  as consisting of a unimodular vector  $\vec{v}$  together with a distinguished splitting  $R^n = \langle \vec{v} \rangle \oplus W$ , where  $W = \{\vec{x} \mid \vec{x} \cdot \vec{w} = 0\}$ .

The group  $\text{GL}_n(R, q)$  acts on  $\mathcal{S}\mathcal{B}_n(R, q)$  via the formula

$$M \cdot \{(\vec{v}_0, \vec{w}_0), \dots, (\vec{v}_k, \vec{w}_k)\} = \{(M\vec{v}_0, (M^{-1})^t \vec{w}_0), \dots, (M\vec{v}_k, (M^{-1})^t \vec{w}_k)\}.$$

This action is clearly nice and extends over  $\tilde{\text{GL}}_n(R, q)$ .

**The distinguished simplex.** Our distinguished simplex in  $\mathcal{S}\mathcal{B}_n(R, q)$  will be  $\Delta_n = \{(\vec{e}_1, \vec{e}_1), \dots, (\vec{e}_n, \vec{e}_n)\}$ .

**Verification of the assumptions.** We now verify the five assumptions from §2. Our constant  $C$  will be  $d + 8$ . Theorem B will then follow from Theorem 2.1. Assumptions 3–5 are trivial, so we will omit the details of their verification. It remains to verify Assumptions 1 and 2.

**Assumption 1.** This assumption says that  $\mathcal{S}\mathcal{B}_n(R, q)$  is  $k$ -acyclic for  $n \geq (d + 8)2^{k-1} - 3$ . In fact, we have the following.

**Lemma 3.1.** *If  $(R, q)$  satisfies  $\text{SR}_{d+2}$ , then the complex  $\mathcal{S}\mathcal{B}_n(R, q)$  is  $\frac{n-d-3}{2}$ -acyclic.*

*Proof.* Let  $\mathcal{P}(\mathcal{S}\mathcal{B}_n(R, q))$  be the face poset of  $\mathcal{S}\mathcal{B}_n(R, q)$ , i.e. the poset whose elements are simplices of  $\mathcal{S}\mathcal{B}_n(R, q)$  and where  $\sigma \leq \sigma'$  if  $\sigma$  is a face of  $\sigma'$ . The geometric realization  $|\mathcal{P}(\mathcal{S}\mathcal{B}_n(R, q))|$  of  $\mathcal{P}(\mathcal{S}\mathcal{B}_n(R, q))$  is the barycentric subdivision of  $\mathcal{S}\mathcal{B}_n(R, q)$ . Next, define  $\mathcal{P}'(\mathcal{S}\mathcal{B}_n(R, q))$  to be the poset whose elements are ordered sequences  $(x_0, \dots, x_k)$  of distinct vertices of  $\mathcal{S}\mathcal{B}_n(R, q)$  such that the unordered set  $\{x_0, \dots, x_k\}$  is a simplex of  $\mathcal{S}\mathcal{B}_n(R, q)$ . Sequences  $s$  and  $s'$  in  $\mathcal{P}'(\mathcal{S}\mathcal{B}_n(R, q))$  satisfy  $s \leq s'$  if  $s$  is a subsequence of  $s'$ . In [7, Theorem 3.5], Charney proved that the geometric realization of  $\mathcal{P}'(\mathcal{S}\mathcal{B}_n(R, q))$  is  $\frac{n-d-3}{2}$ -acyclic.

There is a natural map  $\pi : \mathcal{P}'(\mathcal{S}\mathcal{B}_n(R, q)) \rightarrow \mathcal{P}(\mathcal{S}\mathcal{B}_n(R, q))$  which “forgets” the ordering on a sequence. Choose a total ordering on the vertices of  $\mathcal{S}\mathcal{B}_n(R, q)$ , and define a poset map  $\rho : \mathcal{P}(\mathcal{S}\mathcal{B}_n(R, q)) \rightarrow \mathcal{P}'(\mathcal{S}\mathcal{B}_n(R, q))$  by the formula  $\rho(\{x_0, \dots, x_k\}) = (x_0, \dots, x_k)$ , where the ordering on the  $x_i$  is chosen such that  $x_0 < x_1 < \dots < x_k$ . It is clear that  $\pi \circ \rho = 1$ , which implies that the map on geometric realizations induced by  $\pi$  is surjective on reduced homology. We conclude that the geometric realization of  $\mathcal{P}(\mathcal{S}\mathcal{B}_n(R, q))$  is  $\frac{n-d-3}{2}$ -acyclic.  $\square$

**Assumption 2.** This assumption says that all the  $\text{GL}_n(R, q)$ -orbits of  $(k + 2)$ -simplices of  $\mathcal{S}\mathcal{B}_n(R, q)$  contain simplices in  $\Delta_n$  for  $n \geq (d + 8)2^{k-1} - 3$ . This is an immediate consequence of the following.

**Lemma 3.2.** *If  $(R, q)$  satisfies  $\text{SR}_{d+2}$ , then the group  $\widetilde{\text{GL}}_n(R, q)$  acts transitively on  $k$ -simplices of  $\mathcal{S}\mathcal{B}_n(R, q)$  for  $k \leq n - d$ .*

This lemma can be proven exactly like [7, Proposition on p. 2101].

## 4 The central stability chain complex

In this section, we introduce the *central stability chain complex*, which will play a key role in our proofs. We need a definition and some notation.

**Notation.** If  $Y$  is a set, then let  $S_Y$  denote the symmetric group on  $Y$ .

**Definition.** Let

$$V_n \xrightarrow{\phi_n} V_{n+1} \xrightarrow{\phi_{n+1}} V_{n+2} \xrightarrow{\phi_{n+2}} \dots \xrightarrow{\phi_{m-1}} V_m \quad (1)$$

be a sequence of maps between vector spaces, where  $V_i$  is a representation of  $S_i$  and  $\phi_i$  is  $S_i$ -equivariant for all  $i$ . This sequence is *potentially centrally stable* if the following holds for all  $n \leq i < j \leq m$ . Let  $\vec{v} \in V_j$  be in the image of  $V_i$ . Then  $\sigma \cdot \vec{v} = \vec{v}$  for all  $\sigma \in S_{\{i+1, \dots, j\}}$ .

*Remark.* If  $V_{i+1} = \mathcal{C}(V_i \rightarrow V_{i-1})$  for all  $n < i < m$ , then the sequence is obviously potentially centrally stable.

**Notation.** For  $k \geq 0$ , let  $\mathcal{A}_k$  denote the sign representation of  $S_k$ . Also, if  $V$  is a representation of  $S_n$ , then denote  $\text{Ind}_{S_n \times S_k}^{S_{n+k}} V \boxtimes \mathcal{A}_k$  by  $\text{IA}_k(V)$ . We remark that we discussed the external tensor product  $\boxtimes$  at the end of the introduction.

Given a potentially centrally stable sequence like (1) and some  $M \geq m$ , the associated  $M$ -central stability chain complex will be of the form

$$\text{IA}_{M-n}(V_n) \rightarrow \text{IA}_{M-n-1}(V_{n-1}) \rightarrow \dots \rightarrow \text{IA}_{M-m}(V_m).$$

Observe that each vector space is a representation of  $S_M$ .



**Boundary maps.** We begin by constructing the appropriate boundary maps. Let  $\phi_n : V_n \rightarrow V_{n+1}$  be an  $S_n$ -equivariant map from a representation of  $S_n$  to a representation of  $S_{n+1}$ , and fix some  $M > n$ . We will construct an  $S_M$ -equivariant map  $\partial_n : \mathbf{IA}_{M-n}(V_n) \rightarrow \mathbf{IA}_{M-n-1}(V_{n+1})$ . This map will be called the  $M$ -boundary map associated to  $\phi_n$ .

The construction is as follows. Let  $C_n$  be a set of right coset representatives for  $S_{\{n+2, \dots, M\}}$  in  $S_{\{n+1, \dots, M\}}$ , and define an  $S_n$ -equivariant map  $\partial'_n : V_n \rightarrow \mathbf{IA}_{M-n-1}(V_{n+1})$  via the formula

$$\partial'_n(\vec{v}) = \sum_{\sigma \in C_n} (-1)^{|\sigma|} \sigma \cdot \phi_n(\vec{v}) \quad (\vec{v} \in V_n).$$

Here we are identifying  $V_{n+1}$  with its image in  $\mathbf{IA}_{M-n-1}(V_{n+1})$ .

**Lemma 4.1.** *The map  $\partial'_n$  does not depend on the choice of  $C_n$ .*

*Proof.* Let  $C'_n$  be another set of right coset representatives. For  $s \in C'_n$ , there exists a unique  $\sigma_s \in C_n$  and  $\tau_s \in S_{\{n+2, \dots, M\}}$  such that  $s = \sigma_s \tau_s$ . For  $\vec{v} \in V_n$ , we then have

$$(-1)^{|s|} s \cdot \phi_n(\vec{v}) = (-1)^{|\sigma_s \tau_s|} \sigma_s \tau_s \cdot \vec{v} = (-1)^{|\sigma_s|} (-1)^{|\tau_s|} (-1)^{|\tau_s|} \sigma_s \cdot \vec{v} = (-1)^{|\sigma_s|} \sigma_s \cdot \vec{v}.$$

The lemma follows.  $\square$

**Lemma 4.2.** *For  $\delta \in S_{\{n+1, \dots, M\}}$  and  $\vec{v} \in V_n$ , we have  $\delta \cdot \partial'_n(\vec{v}) = (-1)^{|\delta|} \partial'_n(\vec{v})$ .*

*Proof.* For  $\sigma \in C_n$ , there exists some  $\sigma_\delta \in C_n$  and  $\tau_{\sigma, \delta} \in S_{\{n+2, \dots, M\}}$  such that  $\delta \sigma = \sigma_\delta \tau_{\sigma, \delta}$ . We have  $(-1)^{|\delta|} (-1)^{|\sigma|} = (-1)^{|\sigma_\delta|} (-1)^{|\tau_{\sigma, \delta}|}$ , and thus

$$\begin{aligned} \delta \cdot \partial'_n(\vec{v}) &= \sum_{\sigma \in C_n} (-1)^{|\sigma|} \sigma_\delta \tau_{\sigma, \delta} \cdot \phi_n(\vec{v}) \\ &= \sum_{\sigma \in C_n} (-1)^{|\sigma|} (-1)^{|\tau_{\sigma, \delta}|} \sigma_\delta \cdot \phi_n(\vec{v}) \\ &= (-1)^{|\delta|} \sum_{\sigma \in C_n} (-1)^{|\sigma_\delta|} \sigma_\delta \cdot \phi_n(\vec{v}) = (-1)^{|\delta|} \partial'_n(\vec{v}). \end{aligned}$$

The final equality follows from the fact that the map  $\sigma \mapsto \sigma_\delta$  is a permutation of  $C_n$ .  $\square$

Completing our construction of  $\partial_n$ , Lemma 4.2 implies that  $\partial'_n$  induces an  $S_n \times S_{M-n}$ -equivariant map  $V_n \boxtimes \mathcal{A}_{M-n} \rightarrow \mathbf{IA}_{M-n-1}(V_{n+1})$ , so we obtain a  $S_M$ -equivariant map  $\partial_n : \mathbf{IA}_{M-n}(V_n) \rightarrow \mathbf{IA}_{M-n-1}(V_{n+1})$ .

**First relation to central stabilization.** We pause now to observe that the above gives a presentation for the central stabilization of a map.

**Lemma 4.3.** *Let  $\phi_n : V_n \rightarrow V_{n+1}$  be an  $S_n$ -equivariant map from a representation of  $S_n$  to a representation of  $S_{n+1}$  and let  $\partial_n$  be the  $(n+2)$ -boundary map induced by  $\phi_n$ . Set  $V_{n+2} = \mathcal{C}(V_n \xrightarrow{\phi_n} V_{n+1})$ . There is then an exact sequence*

$$\mathbf{IA}_2(V_n) \xrightarrow{\partial_n} \mathbf{IA}_1(V_{n+1}) \longrightarrow V_{n+2} \longrightarrow 0.$$

*Proof.* By definition, there is a surjection  $\pi : \mathbf{IA}_1(V_{n+1}) \rightarrow V_{n+2}$ . Let  $i : V_n \rightarrow \mathbf{IA}_1(V_{n+1})$  be the  $S_n$ -equivariant map obtained by composing  $V_n \rightarrow V_{n+1}$  with the natural inclusion  $V_{n+1} \hookrightarrow \mathbf{IA}_1(V_{n+1})$ . There is an  $S_n \times S_2$ -equivariant map  $j : V_n \boxtimes \mathcal{A}_2 \rightarrow \mathbf{IA}_1(V_{n+1})$  defined by  $j(v) = v - (n, n+1) \cdot v$ . By the universal property of the induced representation, this extends to a map  $\rho : \mathbf{IA}_2(V_n) \rightarrow \mathbf{IA}_1(V_{n+1})$ . It is easy to see that  $\rho$  is exactly the  $(n+2)$ -boundary map associated to  $\phi_n$ . By definition, the image of  $\rho$  is the kernel of  $\pi$ , and we are done.  $\square$

**The chain complex.** We now prove that the above gives a chain complex, which as we said we will call the *M-central stability chain complex*.

**Lemma 4.4.** *Let*

$$V_n \xrightarrow{\phi_n} V_{n+1} \xrightarrow{\phi_{n+1}} V_{n+2} \xrightarrow{\phi_{n+2}} \cdots \xrightarrow{\phi_{m-1}} V_m$$

*be a potentially centrally stable sequence of representations of the symmetric group and let  $M \geq m$ . For  $n \leq i < m$ , let  $\partial_i$  be the M-boundary map associated to  $\phi_i$ . Then the sequence*

$$IA_{M-n}(V_n) \xrightarrow{\partial_n} IA_{M-n-1}(V_{n+1}) \xrightarrow{\partial_{n+1}} \cdots \xrightarrow{\partial_{m-1}} IA_{M-m}(V_m) \longrightarrow 0$$

*of representations of  $S_M$  is a chain complex.*

*Proof.* Throughout this proof, we will regard  $V_i$  as a subspace of  $IA_{M-i}(V_i)$  for all  $n \leq i \leq m$ . Fix some  $n \leq i < m - 2$ , and consider  $\vec{v} \in V_i$ . It is enough to prove that  $\partial_{i+1}(\partial_i(\vec{v})) = 0$ . Let  $C_i$  and  $C_{i+1}$  be the sets of coset representatives used to construct  $\partial_i$  and  $\partial_{i+1}$ . Set  $\vec{w} = \phi_{i+1}(\phi_i(\vec{v}))$ . Observe that  $\partial_{i+1}(\partial_i(\vec{v}))$  equals

$$\partial_{i+1} \left( \sum_{\sigma \in C_i} (-1)^{|\sigma|} \sigma \cdot \phi_i(\vec{v}) \right) = \sum_{\sigma \in C_i, \sigma' \in C_{i+1}} (-1)^{|\sigma\sigma'|} \sigma\sigma' \cdot \vec{w}.$$

The set  $\{\sigma\sigma' \mid \sigma \in C_i, \sigma' \in C_{i+1}\}$  is a set of right coset representatives for  $S_{\{i+3, \dots, M\}}$  in  $S_{\{i+1, \dots, M\}}$ .

Let  $D$  be a set of right coset representatives for  $S_{\{i+1, i+2\}} \times S_{\{i+3, \dots, M\}}$  in  $S_{\{i+1, \dots, M\}}$ . The set  $\{\sigma \mid \sigma \in D\} \cup \{\sigma(i+1, i+2) \mid \sigma \in D\}$  is thus a set of right coset representatives for  $S_{\{i+3, \dots, M\}}$  in  $S_{\{i+1, \dots, M\}}$ . By an argument similar to that in the proof of Lemma 4.1, we deduce that  $\partial_{i+1}(\partial_i(\vec{v}))$  equals

$$\sum_{\sigma \in D} \left( (-1)^{|\sigma|} \sigma \cdot \vec{w} + (-1)^{|\sigma|+1} \sigma(i+1, i+2) \cdot \vec{w} \right) = 0.$$

Here we have used the fact that  $(i+1, i+2) \cdot \vec{w} = \vec{w}$ , which follows from the potential central stability of our sequence.  $\square$

**Exactness.** The following proposition is perhaps the most important technical result in this paper. Its proof is contained in §6–7. We postpone it because it uses more representation theory than the rest of the paper, and we want to separate as much as possible the representation theoretic parts of this paper from the topological parts. See the beginning of §6 for a road map of its proof.

**Proposition 4.5.** *Let*

$$V_1 \rightarrow V_2 \rightarrow \cdots$$

*be a coherent sequence of representations over a field  $\mathbb{F}$  of the symmetric group which is centrally stable starting at  $N$ . Assume that either  $\text{char}(\mathbb{F}) = 0$  or  $\text{char}(\mathbb{F}) \geq 2N + 2$ . Consider  $n$  and  $m$  and  $M$  such that  $2N + 1 \leq n \leq m \leq M$ . Then the M-central stability chain complex associated to the potentially centrally stable sequence*

$$V_n \rightarrow V_{n+1} \rightarrow \cdots \rightarrow V_m$$

*is exact.*

## 5 Proof that the central stability machine works

In this section, we prove Theorem 2.1. The actual proof is in §5.3. This is preceded by two sections containing necessary background : §5.1 discusses coefficient systems and §5.2 discusses some basic results in equivariant homology theory.

## 5.1 Coefficient systems

Fix a field  $\mathbb{F}$ . For technical reasons, we will need to work in the category of *weak simplicial complexes*, which are defined exactly like simplicial complexes except that they can have more than one simplex spanned by a single set of vertices. Fix a weak simplicial complex  $X$ . Observe that the simplices of  $X$  form the objects of a category with a unique morphism  $\sigma' \rightarrow \sigma$  whenever  $\sigma'$  is a face of  $\sigma$ .

**Definition.** A *coefficient system* on  $X$  is a contravariant functor from the category associated to  $X$  to the category of vector spaces over  $\mathbb{F}$ .

*Remark.* In other words, a coefficient system  $\mathcal{F}$  on  $X$  consists of  $\mathbb{F}$ -vector spaces  $\mathcal{F}(\sigma)$  for simplices  $\sigma$  of  $X$  and linear maps  $\mathcal{F}(\sigma' \rightarrow \sigma) : \mathcal{F}(\sigma) \rightarrow \mathcal{F}(\sigma')$  whenever  $\sigma'$  is a face of  $\sigma$ . These linear maps must satisfy the obvious compatibility condition.

**Definition.** Let  $\mathcal{F}$  be a coefficient system on  $X$ . Fix a total ordering on the elements of  $X^{(0)}$ . The *simplicial chain complex* of  $X$  with coefficients in  $\mathcal{F}$  is as follows. Define

$$C_k(X; \mathcal{F}) = \bigoplus_{\sigma \in X^{(k)}} \mathcal{F}(\sigma).$$

Next, define a differential  $\partial : C_k(C; \mathcal{F}) \rightarrow C_{k-1}(C; \mathcal{F})$  in the following way. Consider  $\sigma \in X^{(k)}$ . We will denote an element of  $\mathcal{F}(\sigma) \subset C_k(X; \mathcal{F})$  by  $c \cdot \sigma$  for  $c \in \mathcal{F}(\sigma)$ . Let  $v_0, \dots, v_k$  be the vertices of  $\sigma$ . Choose the ordering such that  $v_i < v_{i+1}$  for  $0 \leq i < k$ . Denote by  $\sigma_i$  the face of  $\sigma$  opposite the vertex  $v_i$ . For  $c \in \mathcal{F}(\sigma)$ , we then define

$$\partial(c \cdot \sigma) = \sum_{i=0}^k (-1)^i c_i \cdot \sigma_i,$$

where  $c_i$  is the image of  $c$  under the morphism  $\mathcal{F}(\sigma' \rightarrow \sigma) : \mathcal{F}(\sigma) \rightarrow \mathcal{F}(\sigma_i)$ . Taking the homology of  $C_*(X; \mathcal{F})$  yields the *homology groups of  $X$  with coefficients in  $\mathcal{F}$* , which we will denote by  $H_*(X; \mathcal{F})$ .

*Remark.* If  $V$  is an  $\mathbb{F}$ -vector space and  $\mathcal{F}$  is the coefficient system that assigns  $V$  to every simplex and the identity map to every face map, then  $H_*(X; \mathcal{F}) \cong H_*(X; V)$ . We will call this a *constant system of coefficients*.

## 5.2 Equivariant homology

We will need a small portion of the theory of equivariant homology. All the results below are contained (implicitly or explicitly) in [5, §VII]. Recall that if  $G$  acts nicely on a simplicial complex  $X$ , then  $X/G$  is a weak simplicial complex in a natural way.

**Definition.** Consider a group  $G$  acting nicely on a simplicial complex  $X$ . Let  $EG$  be a contractible simplicial complex on which  $G$  acts nicely and freely, so  $EG/G$  is a classifying space for  $G$ . Define  $EG \times_G X$  to be the quotient of  $EG \times X$  by the diagonal action of  $G$ . The  *$G$ -equivariant homology groups of  $X$* , denoted  $H_*^G(X; \mathbb{F})$ , are defined to be  $H_*(EG \times_G X; \mathbb{F})$ .

*Remark.* It is easy to see that  $H_*^G(X; \mathbb{F})$  does not depend on the choice of  $EG$ . The construction of  $EG \times_G X$  is known as the *Borel construction*.

The following lemma summarizes two key properties of these homology groups.

**Lemma 5.1.** *Consider a group  $G$  acting nicely on a simplicial complex  $X$ .*

- *There is a canonical map  $H_*^G(X; \mathbb{F}) \rightarrow H_*(G; \mathbb{F})$ .*
- *If  $X$  is  $k$ -acyclic, then the map  $H_i^G(X; \mathbb{F}) \rightarrow H_i(G; \mathbb{F})$  is an isomorphism for  $i \leq k$ .*

*Remark.* The map  $H_*^G(X; \mathbb{F}) \rightarrow H_*(G; \mathbb{F})$  comes from map  $EG \times_G X \rightarrow EG/G$  induced by the projection of  $EG \times X$  onto its first factor. The second claim is an immediate consequence of the spectral sequence whose  $E^2$  page is (7.2) in [5, §VII.7].

To calculate equivariant homology groups, we will need a certain spectral sequence. First, a definition.

**Definition.** Consider a group  $G$  acting nicely on a simplicial complex  $X$ . Define a coefficient system  $\mathcal{H}_q(G, X; \mathbb{F})$  on  $X/G$  as follows. Consider a simplex  $\sigma$  of  $X/G$ . Let  $\tilde{\sigma}$  be any lift of  $\sigma$  to  $X$ . Set

$$\mathcal{H}_q(G, X; \mathbb{F})(\sigma) = H_q(G_{\tilde{\sigma}}; \mathbb{F}),$$

where  $G_{\tilde{\sigma}}$  is the stabilizer of  $\tilde{\sigma}$ . It is easy to see that this does not depend on the choice of  $\tilde{\sigma}$  and that it defines a coefficient system on  $X/G$ .

Our spectral sequence is then as follows. It can be easily extracted from [5, §VII.8]

**Theorem 5.2.** *Let  $G$  be a group acting nicely on a simplicial complex  $X$ . There is then a spectral sequence converging to  $H_*^G(X; \mathbb{F})$  with*

$$E_{p,q}^2 \cong H_p(X/G; \mathcal{H}_q(G, X; \mathbb{F})).$$

Assume now that  $G$  acts nicely on a simplicial complex  $X$  which is  $k$ -connected, and consider  $v \in X^{(0)}$ . The inclusion map  $G_v \hookrightarrow G$  induces a map  $H_k(G_v; \mathbb{F}) \rightarrow H_k(G; \mathbb{F})$  which is easily described in terms of the spectral sequence in Theorem 5.2. First, Lemma 5.1 says that the spectral sequence in Theorem 5.2 converges to  $H_i(G; \mathbb{F})$  for  $0 \leq i \leq k$ . Next, observe that there is a natural map

$$H_k(G_v; \mathbb{F}) \rightarrow E_{0,k}^2 = H_0(X/G; \mathcal{H}_k(G, X; \mathbb{F}))$$

obtained as the composition

$$H_k(G_v; \mathbb{F}) \hookrightarrow C_0(X/G; \mathcal{H}_k(G, X; \mathbb{F})) \rightarrow H_0(X/G; \mathcal{H}_k(G, X; \mathbb{F})),$$

where the first map is the natural inclusion. The map  $H_k(G_v; \mathbb{F}) \rightarrow H_k(G; \mathbb{F})$  is then the composition

$$H_k(G_v; \mathbb{F}) \rightarrow E_{0,k}^2 \rightarrow E_{0,k}^\infty \rightarrow H_k(G; \mathbb{F}).$$

### 5.3 The proof of Theorem 2.1

We now prove Theorem 2.1. This requires the following standard lemma.

**Lemma 5.3** ([5, Proposition III.5.3]). *Let  $G$  be a group with a subgroup  $H$ . Let  $V$  be a representation of  $G$  and  $W \subset V$  be an  $H$ -subrepresentation. Picking a set  $\{g_i\}_{i \in I}$  of left coset representatives for  $H$  in  $G$ , assume that*

$$V = \bigoplus_{i \in I} g_i \cdot W.$$

Then  $V \cong \text{Ind}_H^G W$ .

*Proof of Theorem 2.1.* Let  $\{G_i\}$  and  $\{\tilde{G}_i\}$  and  $\{X_i\}$  and  $\Delta_i = \{v_1^i, \dots, v_i^i\}$  be as in §2. Fix  $k \geq 1$ . We wish to prove that the sequence

$$H_k(G_1; \mathbb{F}) \longrightarrow H_k(G_2; \mathbb{F}) \longrightarrow H_k(G_3; \mathbb{F}) \longrightarrow \dots$$

of representations of the symmetric group is centrally stable with stability starting at  $C2^{k-1} - 4$ . Assume as an inductive hypothesis that this is true for all smaller nonnegative  $k$  (for  $k = 1$ , this assumption is vacuous). To simplify our notation, we will omit the coefficients  $\mathbb{F}$  from our homology groups and chain groups.

Fix  $n \geq C2^{k-1} - 4$ . We want to prove that there is an  $S_{n+1}$ -equivariant isomorphism

$$H_k(G_{n+1}) \cong \mathcal{C}(H_k(G_{n-1}) \rightarrow H_k(G_n))$$

and that the map  $H_k(G_n) \rightarrow H_k(G_{n+1})$  is as in the definition of central stabilization. To do this, we will use the spectral sequence from Theorem 5.2 for the action of  $G_{n+1}$  on  $X_{n+1}$ . This spectral sequence converges to  $H_*^{G_{n+1}}(X_{n+1})$ . Since  $n+1 \geq C2^{k-1} - 3$ , Assumption 1 implies that  $X_{n+1}$  is  $k$ -acyclic, so by Lemma 5.1 we have  $H_i^{G_{n+1}}(X_{n+1}) \cong H_i(G_{n+1})$  for  $0 \leq i \leq k$ .

To simplify our notation, we will denote  $C_j(X_{n+1}/G_{n+1}; \mathcal{H}_i(G_{n+1}, X_{n+1}))$  by  $C_j^i$ . The action of  $\tilde{G}_{n+1}$  on  $X_{n+1}$  induces an action of  $S_{n+1}$  on  $C_j^i$  which commutes with the boundary map  $C_j^i \rightarrow C_{j-1}^i$  for all  $i$  and  $j$ . The following observation is the key to our proof.

**Claim 1.** Fix some  $1 \leq i \leq k$ . For all  $0 \leq j \leq k+2$ , there exists an  $S_{n+1}$ -equivariant isomorphism  $\eta_j : IA_{j+1}(\mathbf{H}_i(G_{n-j})) \rightarrow C_j^i$  such that the diagram

$$\begin{array}{ccccccc} IA_{k+3}(\mathbf{H}_i(G_{n-k-2})) & \longrightarrow & IA_{k+2}(\mathbf{H}_i(G_{n-k-1})) & \longrightarrow & \cdots & \longrightarrow & IA_1(\mathbf{H}_i(G_n)) \\ \downarrow \eta_{k+2} & & \downarrow \eta_{k+1} & & & & \downarrow \eta_0 \\ C_{k+2}^i & \longrightarrow & C_{k+1}^i & \longrightarrow & \cdots & \longrightarrow & C_0^i \end{array}$$

commutes. Here bottom row is the chain complex computing  $\mathbf{H}_*(X_{n+1}/G_{n+1}; \mathcal{H}_i(G_{n+1}, X_{n+1}))$  and the top row is the  $(n+1)$ -central stability chain complex for the potentially centrally stable sequence

$$\mathbf{H}_i(G_{n-k-2}) \rightarrow \mathbf{H}_i(G_{n-k-1}) \rightarrow \cdots \rightarrow \mathbf{H}_i(G_n).$$

*Proof of claim.* Since  $G_{n+1}$  acts nicely on  $X_{n+1}$ , the simplex  $\Delta_{n+1}$  of  $X_{n+1}$  injects into  $X_{n+1}/G_{n+1}$ . Let  $\bar{\Delta}_{n+1}$  be its image. Assumption 4 says that the action of  $S_{n+1}$  on  $X_{n+1}/G_{n+1}$  preserves  $\bar{\Delta}_{n+1}$ . Letting

$$D_j^i = C_j(\bar{\Delta}_{n+1}; \mathcal{H}_i(G_{n+1}, X_{n+1})),$$

we see that  $D_j^i$  is an  $S_{n+1}$ -representation. There is a natural map  $\kappa_j : D_j^i \rightarrow C_j^i$  induced by the inclusion  $\bar{\Delta}_{n+1} \hookrightarrow X_{n+1}/G_{n+1}$ . Since  $n+1 \geq C2^{k-1} - 3$ , Assumption 2 implies that  $\kappa_j$  is an isomorphism for  $0 \leq j \leq k+2$ . We will prove that

$$D_j^i \cong IA_{j+1}(\mathbf{H}_i(G_{n-j})) \quad (2)$$

as  $S_{n+1}$ -representations. This will give us the desired maps  $\eta_j$ ; proving that the indicated diagram commutes is then an easy exercise in the definitions of the various maps, and is thus omitted.

It remains to prove (2). We want to apply Lemma 5.3. By definition, we have

$$D_j^i = \bigoplus_{1 \leq \ell_0 < \cdots < \ell_j \leq n+1} \mathbf{H}_i((G_{n+1})_{\{v_{\ell_0}^{n+1}, \dots, v_{\ell_j}^{n+1}\}}). \quad (3)$$

By Assumption 5, the subgroup  $S_{n-j} \times S_{j+1} \subset S_{n+1}$  preserves the term  $\mathbf{H}_i((G_{n+1})_{\{v_{n+1-j}^{n+1}, \dots, v_{n+1}^{n+1}\}})$  of (3). Moreover, Assumption 3 says that  $(G_{n+1})_{\{v_{n+1-j}^{n+1}, \dots, v_{n+1}^{n+1}\}} = G_{n-j}$ , so as an  $S_{n-j} \times S_{j+1}$  representation this term is isomorphic to  $(\mathbf{H}_i(G_{n-j})) \boxtimes \mathcal{A}_{j+1}$  (the subgroup  $S_{j+1}$  acts via the sign representation since it is just changing the orientation of the associated simplex). The left cosets of  $S_{n-j} \times S_{j+1}$  in  $S_{n+1}$  are exactly determined by what they do to the unordered set  $\{n+1-j, \dots, n+1\}$ , so letting  $C$  be a complete set of such coset representatives, we obtain that (3) can be rewritten

$$D_j^i = \bigoplus_{\sigma \in C} \sigma \cdot (\mathbf{H}_i((G_{n+1})_{\{v_{n+1-j}^{n+1}, \dots, v_{n+1}^{n+1}\}})).$$

Lemma 5.3 then implies that  $D_j^i \cong IA_{j+1}(\mathbf{H}_i(G_{n-j}))$ , as desired.  $\square$

We can now analyze the  $E^2$ -page of our spectral sequence.

**Claim 2.**  $E_{0,k}^2 = \mathcal{C}(\mathbf{H}_k(G_{n-1}) \rightarrow \mathbf{H}_k(G_n))$ .

*Proof of claim.* Claim 1 implies that

$$E_{0,k}^2 = \text{coker}(C_1^k \rightarrow C_0^k) = \text{coker}(IA_2(\mathbf{H}_k(G_{n-1})) \rightarrow IA_1(\mathbf{H}_k(G_n))),$$

which by Lemma 4.3 equals  $\mathcal{C}(\mathbf{H}_k(G_{n-1}) \rightarrow \mathbf{H}_k(G_n))$ .  $\square$

**Claim 3.**  $E_{j,i}^2 = 0$  for  $1 \leq i < k$  and  $j \geq 1$  such that  $j \leq k-i+1$ .

*Proof of claim.* This is asserting that the sequence

$$C_{k-i+2}^i \rightarrow C_{k-i+1}^i \rightarrow \cdots \rightarrow C_0^i$$

is exact. By Claim 1, this is equivalent to the exactness of the sequence

$$\mathrm{IA}_{k-i+3}(\mathrm{H}_i(G_{n+i-k-2})) \rightarrow \mathrm{IA}_{k-i+2}(\mathrm{H}_i(G_{n+i-k-1})) \rightarrow \cdots \rightarrow \mathrm{IA}_1(\mathrm{H}_i(G_n)) \quad (4)$$

By induction, the sequence

$$\mathrm{H}_i(G_1) \rightarrow \mathrm{H}_i(G_2) \rightarrow \cdots$$

is centrally stable starting at  $C2^{i-1} - 4$ . Using the easily-verified inequality  $2^a - 2^b \geq a - b$  for integers  $a \geq b \geq 0$ , we have

$$\begin{aligned} 2(C2^{i-1} - 4) + 1 &= (C2^{k-1} - 4) - C(2^{k-1} - 2^i) - 3 \\ &\leq (C2^{k-1} - 4) - ((k-1) - i) - 3 \leq n + i - k - 2. \end{aligned}$$

Proposition 4.5 therefore implies that (4) is exact, as desired.  $\square$

**Claim 4.**  $E_{j,0}^2 = 0$  for  $1 \leq j \leq k+1$ .

*Proof of claim.* The coefficient system  $\mathcal{H}_0(G_{n+1}, X_{n+1})$  is the constant coefficient system  $\mathbb{F}$ . As in Claim 1, let  $\bar{\Delta}_{n+1}$  be the image of  $\Delta_{n+1}$  in  $X_{n+1}/G_{n+1}$ . Assumption 2 implies  $\bar{\Delta}_{n+1}$  contains the entire  $(k+2)$ -skeleton of  $X_{n+1}/G_{n+1}$ . Since  $\bar{\Delta}_{n+1}$  is contractible, it follows that  $\mathrm{H}_j(X_{n+1}/G_{n+1}; \mathcal{H}_0(G_{n+1}, X_{n+1})) = 0$  for  $1 \leq j \leq k+1$ , as desired.  $\square$

Summarizing, the part of the  $E^2$ -page of our spectral sequence needed to compute  $\mathrm{H}_k(G_{n+1})$  is as follows.

$$\begin{array}{c} \mathcal{C}(\mathrm{H}_k(G_{n-1}) \rightarrow \mathrm{H}_k(G_n)) \\ \begin{array}{ccccccc} * & & 0 & 0 & & & \\ * & & 0 & 0 & 0 & & \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \\ * & & 0 & 0 & 0 & \cdots & 0 \end{array} \end{array}$$

We conclude that  $\mathrm{H}_k(G_{n+1}) = \mathcal{C}(\mathrm{H}_k(G_{n-1}) \rightarrow \mathrm{H}_k(G_n))$ . The fact that the map  $\mathrm{H}_k(G_n) \rightarrow \mathrm{H}_k(G_{n+1})$  is as in the definition of central stability follows easily from the discussion after Theorem 5.2 together with the fact that  $G_n = (G_{n+1})_{\nu_{n+1}^{n+1}}$ .  $\square$

## 6 Specht stability

In this section, we define a different notion of stability for coherent sequences of representations which we call Specht stability. There are two key results about Specht stability. The first is Theorem E from §1, which says that every centrally stable sequence of representations of the symmetric group is also Specht stable (subject to an assumption on  $\mathrm{char}(\mathbb{F})$ ). Theorem E will be proven in §7. The other key result about Specht stability is the following proposition, which is the analogue of Proposition 4.5 for Specht stability.

**Proposition 6.1.** *Let*

$$V_1 \rightarrow V_2 \rightarrow \cdots$$

*be a coherent sequence of representations of the symmetric group which is Specht stable starting at  $N$ . Consider  $n$  and  $m$  and  $M$  such that  $N \leq n \leq m \leq M$ . Then the  $M$ -central stability chain complex associated to the potentially centrally stable sequence*

$$V_n \rightarrow V_{n+1} \rightarrow \cdots \rightarrow V_m$$

*is exact.*

Proposition 4.5 is an immediate corollary of Proposition 6.1 and Theorem E.

The definition of Specht stability uses the fine structure of the representation theory of the symmetric group, which is briefly recalled in §6.1. In §6.2, we introduce a special filtration on representations of  $S_n$ , and in §6.3 we define Specht stability. Finally, in §6.4 we prove Proposition 6.1, making use of a special case of Proposition 6.1 which is proven in §6.5.

## 6.1 Review of the representation theory of the symmetric group

We begin by quickly reviewing some background material on the representation theory of the symmetric group. There are numerous very different approaches to this material. We will follow the approach of James's book [15], which is the one that seems best suited to working in finite characteristic. Fix a field  $\mathbb{F}$ .

**Partitions and Young diagrams.** A *partition*  $\mu$  of an integer  $n$  is an ordered nonincreasing sequence  $(\mu_1, \dots, \mu_k)$  of positive integers whose sum is  $n$ . We will often write  $\mu \vdash n$  to indicate that  $\mu$  is a partition of  $n$ . A partition  $\mu = (\mu_1, \dots, \mu_k)$  can be visualized as a *Young diagram*, which is a diagram containing  $\mu_1$  empty boxes on the first row,  $\mu_2$  on the second row, etc., with all rows left-justified. For example, the Young diagram for  $(4, 2, 1)$  is



We will frequently confuse a partition with its associated Young diagram; for instance, we will discuss “adding a box to the upper right hand corner” of a partition.

**Tableaux and tabloids.** A *tableau* of shape  $\mu \vdash n$  is obtained by filling in the boxes of the Young diagram of  $\mu$  with the numbers  $\{1, \dots, n\}$  such that each number is used exactly once. A *tabloid* of shape  $\mu$  is similar to a tableau, but the entries in each row are unordered. If  $t$  is a tableau, then we will denote the tabloid obtained by forgetting the ordering on the rows of  $t$  by  $\{t\}$ . Let  $M^\mu(\mathbb{F})$  be the set of  $\mathbb{F}$ -linear combinations of tabloids of shape  $\mu$ . The group  $S_n$  acts on  $M^\mu(\mathbb{F})$  in the obvious way. It is not hard to see that  $M^\mu(\mathbb{F}) = \text{Ind}_{S_{\mu_1} \times S_{\mu_2} \times \dots \times S_{\mu_k}}^{S_n} \mathbb{F}$ , where  $S_{\mu_1} \times \dots \times S_{\mu_k}$  acts trivially on  $\mathbb{F}$  and is embedded in  $S_n$  in the obvious way.

**Polytabloids and Specht modules.** The representations  $M^\mu(\mathbb{F})$  are rarely irreducible. If  $t$  is a tableau of shape  $\mu \vdash n$ , then let  $\text{ColStab}(t)$  be the subgroup of  $S_n$  that preserves the columns of  $t$ . The *polytabloid*  $e_t$  associated to  $t$  is then

$$e_t = \sum_{\sigma \in \text{ColStab}(t)} (-1)^{|\sigma|} \{\sigma \cdot t\} \in M^\mu(\mathbb{F}). \quad (5)$$

The *Specht module* associated to  $\mu$ , denoted  $S^\mu(\mathbb{F})$ , is the span of  $\{e_t \mid t \text{ tableau of shape } \mu\}$  in  $M^\mu(\mathbb{F})$ . The group  $S_n$  clearly acts on  $S^\mu(\mathbb{F})$ . A *standard tableau* is a tableau  $t$  such that the rows and columns of  $t$  are strictly increasing, and a *standard polytabloid* is the polytabloid associated to a standard tableau. The set of standard polytabloids of shape  $\mu$  forms a basis for  $S^\mu(\mathbb{F})$ .

**Decomposing representations.** If  $\text{char}(\mathbb{F}) = 0$  or  $\text{char}(\mathbb{F}) \geq n + 1$  (in other words, if  $\text{char}(\mathbb{F})$  does not divide  $n! = |S_n|$ ), then  $S^\mu(\mathbb{F})$  is an irreducible  $S_n$ -representation, and all irreducible  $S_n$ -representations over  $\mathbb{F}$  arise in this way. Moreover, the above assumption on  $\text{char}(\mathbb{F})$  implies that all representations of  $S_n$  are completely reducible, so we can decompose an  $S_n$ -representation  $V$  over  $\mathbb{F}$  as

$$V = \bigoplus_{i \in I} S^{\mu_i}(\mathbb{F}),$$

where  $\mu_i \vdash n$  for all  $i \in I$ . The isotypic components of this decomposition (that is, the direct sums of isomorphic Specht modules within it) are unique. We emphasize that all of this holds for infinite-dimensional  $V$ , the key point being that if  $V$  is an arbitrary  $S_n$ -representation and  $\vec{v} \in V$ , then the span of the orbit  $S_n \cdot \vec{v}$  is finite-dimensional. If  $0 < \text{char}(\mathbb{F}) \leq n$ , then Specht modules need not be irreducible and  $S_n$ -representations over  $\mathbb{F}$  need not decompose as direct sums of irreducible representations. Nonetheless, the Specht modules still play a basic role in  $S_n$ -representation theory.

**Restricting representations.** Fix  $\mu \vdash n + k$ . We wish to study  $\text{Res}_{S_n}^{S_{n+k}} S^\mu(\mathbb{F})$ . The *deletable rows* of  $\mu$  are the rows from which the right-most box can be deleted to yield a Young diagram (these are the rows that end with a “corner”). A length  $k$  *deletion sequence* for  $\mu$  is an ordered sequence  $\mathfrak{s} = (s_1, \dots, s_k)$  of rows of  $\mu$  such

that  $s_1$  is a deletable row of  $\mu$ , such that  $s_2$  is a deletable row of the Young diagram obtained by deleting the last box in row  $s_1$  of  $\mu$ , etc. For example, if

$$\mu = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & & & \\ \hline \square & & & & \\ \hline \end{array}$$

then  $(1, 2, 1)$  is a deletion sequence but  $(2, 2, 1)$  is not a deletion sequence. Let  $\mu_{\mathfrak{s}}$  denote the Young diagram obtained by performing this sequence of deletions. Thus in the previous example, we would have

$$\mu_{(1,2,1)} = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array}$$

Let  $\mathfrak{S}$  be the set of length  $k$  deletion sequences for  $\mu$ . We then have the following classical *restriction rule*.

**Theorem 6.2** ([15, §9]). *If  $\text{char}(\mathbb{F}) = 0$  or  $\text{char}(\mathbb{F}) \geq n + 1$ , then  $\text{Res}_{S_n}^{S_{n+k}} S^\mu(\mathbb{F}) \cong \bigoplus_{\mathfrak{s} \in \mathfrak{S}} S^{\mu_{\mathfrak{s}}}(\mathbb{F})$ .*

## 6.2 A filtration on representations of the symmetric group

We now discuss a type of filtration on a representation of the symmetric group which will play a key role in the rest of this paper. First, some definitions concerning filtrations.

**Definition.**

- A *filtered vector space* of length  $N$  is a vector space  $V$  equipped with a descending filtration

$$V = \mathcal{F}_N V \supseteq \mathcal{F}_{N-1} V \supseteq \cdots \supseteq \mathcal{F}_0 V = 0.$$

We will use the convention that  $\mathcal{F}_i V = V$  for  $i > N$  and  $\mathcal{F}_i V = 0$  for  $i \leq 0$ .

- If  $f : V \rightarrow W$  is a linear map between filtered vector spaces, then  $f$  is a *filtered map* of degree  $k \geq 0$  if  $f(\mathcal{F}_i V) \subset \mathcal{F}_{i+k} W$  for all  $i$ .
- If  $f : V \rightarrow W$  is a filtered map of degree  $k$ , then we get induced maps

$$f_i : \mathcal{F}_i V / \mathcal{F}_{i+1} V \rightarrow \mathcal{F}_{i+k} W / \mathcal{F}_{i+k+1} W$$

for all  $i$ . We will call  $f_i$  the  $i^{\text{th}}$  *graded map* associated to  $f$ .

We now give some motivation for our filtration. Assume for the moment that  $\text{char}(\mathbb{F}) = 0$ , and let  $V$  and  $W$  be representations over  $\mathbb{F}$  of  $S_n$  and  $S_{n+1}$ , respectively. Consider an  $S_n$ -equivariant map  $f : V \rightarrow W$ . Decompose  $V$  and  $W$  as direct sums

$$V = \bigoplus_{i \in I} S^{\mu_i}(\mathbb{F}) \quad \text{and} \quad W = \bigoplus_{j \in J} S^{\nu_j}(\mathbb{F})$$

of Specht modules. What can we say about  $f(S^{\mu_i}(\mathbb{F})) \subset W$ ?

A hint is provided by Theorem 6.2 (the restriction rule). Let the first row of  $\mu_i$  have  $r$  boxes, and let

$$J' = \{j \in J \mid \nu_j \text{ has } r \text{ or } r + 1 \text{ boxes in its first row}\}.$$

Theorem 6.2 implies that

$$f(S^{\mu_i}(\mathbb{F})) \subset \bigoplus_{j \in J'} S^{\nu_j}(\mathbb{F}) \subset W.$$

This suggests that it might be worthwhile to filter a representation of  $S_n$  by the “length of the top rows of its Specht modules”.

We now return to considering general fields  $\mathbb{F}$ . We make the above type of filtration precise as follows.



**Definition.** Let  $V$  be an  $S_n$ -representation over  $\mathbb{F}$ . A *top-indexed Specht filtration* for  $V$  is an  $S_n$ -invariant filtration

$$V = \mathcal{F}_n V \supset \mathcal{F}_{n-1} V \supset \cdots \supset \mathcal{F}_0 V = 0$$

together with a decomposition

$$\mathcal{F}_i V / \mathcal{F}_{i-1} V = \bigoplus_{j \in I_i} S^{\mu(i,j)}(\mathbb{F})$$

for each  $i$  such that the first row of  $\mu(i, j)$  has  $i$  boxes for  $j \in I_i$ .

**Definition.** Let  $V$  and  $W$  be representations of  $S_n$  and  $S_{n+1}$ , respectively, which are equipped with top-indexed Specht filtrations. A *Specht filtration map*  $f : V \rightarrow W$  is an  $S_n$ -equivariant filtered map of degree 1.

*Remark.* It follows from what we said above that if  $\text{char}(\mathbb{F}) = 0$ , then all representations  $V$  of  $S_n$  over  $\mathbb{F}$  can be uniquely equipped with top-indexed Specht filtrations, and if  $W$  is a representation of  $S_{n+1}$  over  $\mathbb{F}$  and  $f : V \rightarrow W$  is  $S_n$ -equivariant, then  $f$  is a Specht filtration map. Neither of these need to hold if  $\text{char}(\mathbb{F}) > 0$ .

### 6.3 Definition of Specht stability

In this section, we define Specht stability. We begin by describing how to stabilize a single Specht module. This notion of stability was first introduced by Church and Farb; see their paper [9] for many examples of situations “in nature” in which it occurs.

**Definition.** If  $\mu = (\mu_1, \dots, \mu_\ell) \vdash n$ , then  $\text{st}(\mu) = (\mu_1 + 1, \mu_2, \dots, \mu_\ell) \vdash n + 1$ . There is an  $S_n$ -equivariant map  $M^\mu(\mathbb{F}) \hookrightarrow M^{\text{st}(\mu)}(\mathbb{F})$  which appends an  $n + 1$  to the first row of a tabloid in  $M^\mu(\mathbb{F})$ . Restricting this to  $S^\mu(\mathbb{F})$ , we get an  $S_n$ -equivariant map  $S^\mu(\mathbb{F}) \hookrightarrow S^{\text{st}(\mu)}(\mathbb{F})$  that we will call the *stabilization map*.

We now extend this to representations equipped with top-indexed Specht filtrations.

**Definition.** Let  $V$  and  $W$  be representations over  $\mathbb{F}$  of  $S_n$  and  $S_{n+1}$ , respectively. Assume that  $V$  and  $W$  are equipped with top-indexed Specht filtrations and that  $f : V \rightarrow W$  is a Specht filtration map. The map  $f$  is a *stabilization map* if the following holds for all  $i \in \mathbb{Z}$ . Let  $f_i : \mathcal{F}_i V / \mathcal{F}_{i-1} V \rightarrow \mathcal{F}_{i+1} W / \mathcal{F}_i W$  be the graded map and let

$$\mathcal{F}_i V / \mathcal{F}_{i-1} V = \bigoplus_{j \in I_i} S^{\mu(i,j)}(\mathbb{F}) \quad \text{and} \quad \mathcal{F}_{i+1} W / \mathcal{F}_i W = \bigoplus_{j \in I'_i} S^{\nu(i,j)}(\mathbb{F})$$

be the decompositions. There then exists a bijection  $\sigma : I_i \rightarrow I'_i$  such that  $f_i$  restricts to the stabilization map  $S^{\mu(i,j)}(\mathbb{F}) \rightarrow S^{\nu(i,\sigma(i))}(\mathbb{F})$  for all  $j \in I_i$ .

*Remark.* If  $f : V \rightarrow W$  is a stabilization map as in the previous definition, then since  $\mathcal{F}_0 V = 0$  we must have  $\mathcal{F}_1 W = 0$ .

We can now define Specht stability.

**Definition.** Let

$$V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow V_4 \longrightarrow \cdots$$

be a coherent sequence of representations of the symmetric group. This sequence is *Specht stable* with stability starting at  $N$  if for all  $n \geq N$ , the  $S_n$ -representation  $V_n$  can be equipped with a top-indexed Specht filtration  $\mathcal{F}_\bullet V_n$  such that the maps  $V_n \rightarrow V_{n+1}$  are stabilization maps.

*Remark.* If

$$V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow V_4 \longrightarrow \cdots$$

is a coherent sequence of representations of the symmetric group which is Specht stable starting at  $N$ , then by a reasoning similar to the remark after the definition of the stabilization map we must have  $\mathcal{F}_i(V_n) = 0$  for  $n \geq N$  and  $i \leq n - N$ .

*Remark.* For coherent sequences of finite-dimensional representations over a field of characteristic 0, Specht stability is easily seen to imply both representation stability in the sense of Church-Farb [9] and monotonicity in the sense of Church [10].

## 6.4 Reduction of Proposition 6.1 to a special case

The following is a special case of Proposition 6.1.

**Proposition 6.3.** *Fix  $\mu \vdash n$  and  $k \geq 1$ . Let*

$$IA_k(S^\mu(\mathbb{F})) \rightarrow IA_{k-1}(S^{\text{st}(\mu)}(\mathbb{F})) \rightarrow \cdots \rightarrow S^{\text{st}^k(\mu)}(\mathbb{F}) \rightarrow 0 \quad (6)$$

be the  $(n+k)$ -central stability chain complex associated to the potentially stable sequence

$$S^\mu(\mathbb{F}) \rightarrow S^{\text{st}(\mu)}(\mathbb{F}) \rightarrow \cdots \rightarrow S^{\text{st}^k(\mu)}(\mathbb{F})$$

of representations of the symmetric group. Then (6) is exact.

The proof of Proposition 6.3 is in §6.5.

The following corollary follows from Lemma 4.3 and the case  $k = 2$  of Proposition 6.3.

**Corollary 6.4.** *For  $\mu \vdash n$ , we have  $\mathcal{C}(S^\mu(\mathbb{F}) \rightarrow S^{\text{st}(\mu)}(\mathbb{F})) \cong S^{\text{st}^2(\mu)}(\mathbb{F})$ . Moreover, the map  $S^{\text{st}(\mu)}(\mathbb{F}) \rightarrow S^{\text{st}^2(\mu)}(\mathbb{F})$  obtained by composing the map  $S^{\text{st}(\mu)}(\mathbb{F}) \hookrightarrow IA_1(S^{\text{st}(\mu)}(\mathbb{F}))$  with the projection  $IA_1(S^{\text{st}(\mu)}(\mathbb{F})) \rightarrow \mathcal{C}(S^\mu(\mathbb{F}) \rightarrow S^{\text{st}(\mu)}(\mathbb{F}))$  is the stabilization map.*

We now show how to derive Proposition 6.1 from Proposition 6.3.

*Proof of Proposition 6.1.* Let us recall the setup. Let

$$V_1 \rightarrow V_2 \rightarrow \cdots$$

be a coherent sequence of representations of the symmetric group which is Specht stable starting at  $N$ . Consider  $n$  and  $m$  and  $M$  such that  $N \leq n \leq m \leq M$ . Then the claim is that the  $M$ -central stability chain complex

$$IA_{M-n}(V_n) \rightarrow IA_{M-n-1}(V_{n+1}) \rightarrow \cdots \rightarrow IA_{M-m}(V_m) \quad (7)$$

associated to the potentially centrally stable sequence

$$V_n \rightarrow V_{n+1} \rightarrow \cdots \rightarrow V_m$$

is exact. The filtrations on the  $V_i$  given by Specht stability induce filtrations on the terms  $IA_{M-i}(V_i)$ . These are compatible with the differentials in (7), so (7) is a filtered chain complex. The associated graded pieces of this filtered chain complex are direct sums of chain complexes like in Proposition 6.3, and thus by Proposition 6.3 the homology of the associated graded pieces of (7) vanish. Standard homological algebra (for instance, the spectral sequence of a filtered chain complex) then shows that the homology of (7) vanishes.  $\square$

## 6.5 Specht stability and the central stability chain complex

Our goal is to prove Proposition 6.3. It turns out that this was essentially proven by G. James in [15, §17], though his formulation is quite different and it takes some effort to extract Proposition 6.3 from James's work. We begin by going over some necessary representation-theoretic background material.

**Duality.** If  $\nu \vdash n$ , then the *conjugate partition* of  $\nu$ , denoted  $\nu'$ , is the partition whose Young diagram is obtained by converting each row of  $\nu$  into a column. For instance,

$$\nu = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \quad \text{and} \quad \nu' = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array}$$

Recalling that  $\mathcal{A}_n$  is the sign representation of  $S_n$ , define  $S_{\mathcal{A}}^\nu(\mathbb{F}) = S^\nu(\mathbb{F}) \otimes \mathcal{A}_n$ . It is then classical that  $(S_{\mathcal{A}}^\nu(\mathbb{F}))^* \cong S^{\nu'}(\mathbb{F})$  (see, e.g., [11, §4]). For a representation  $V$  of  $S_n$  and  $j \geq 0$ , define

$$I_j(V) = \text{Ind}_{S_n \times S_j}^{S_{n+j}} V \boxtimes \mathcal{T}_j,$$

where  $\mathcal{T}_k$  is the trivial representation of  $S_j$ . Recall that if  $H$  is a subgroup of  $G$  and  $W$  is an  $H$ -representation, then  $(\text{Ind}_H^G W)^* \cong \text{Ind}_H^G W^*$  (see, e.g., [2, §3.3]). This implies that  $(I_j(S_{\mathcal{A}}^\nu(\mathbb{F})))^* \cong I_j(S^{\nu'}(\mathbb{F}))$ .

**Weak partitions.** A *weak partition*  $\eta$  of an integer  $n$  is an ordered sequence  $(\eta_1, \dots, \eta_k)$  of nonnegative integers whose sum is  $n$ . Young diagrams, tableau and tabloids of shape  $\eta$  are defined in the obvious way, and given a tableau  $t$  of shape  $\eta$ , we will let  $\{t\}$  denote the associated tabloid. Also,  $M^\eta(\mathbb{F})$  will still denote the set of  $\mathbb{F}$ -linear combinations of tabloids of shape  $\eta$ . If  $\eta = (\eta_1, \dots, \eta_k)$  and  $\nu = (\nu_1, \dots, \nu_\ell)$  are weak partitions with  $\ell \leq k$ , then we will write  $\nu \subset \eta$  if  $\nu_i \leq \eta_i$  for all  $1 \leq i \leq k$ , where by convention  $\nu_i = 0$  for  $\ell < i \leq k$ . If  $\nu \subset \eta$ , then we will regard the Young diagram of  $\nu$  as being contained in the Young diagram for  $\eta$ . For example, if  $\nu = (2, 1)$  and  $\eta = (3, 1, 3)$ , then the Young diagrams are as follows.



The  $\bullet$ 's indicate the location of  $\nu$ . Given a tableau  $t$  of shape  $\eta$ , this allows us to refer to the portion of  $t$  lying inside/outside  $\nu$ .

**Specht modules for weak partitions.** Assume now that  $\eta$  is a weak partition of  $n$  and  $\nu$  is a partition (not just a weak partition) satisfying  $\nu \subset \eta$ . If  $t$  is a tableau of shape  $\eta$ , then let  $\text{ColStab}(t, \nu)$  be the subgroup of  $S_n$  that acts as the identity on the portion of  $t$  lying outside of  $\nu$  and preserves the columns of the portion of  $t$  lying inside  $\nu$ . The *polytabloid*  $e_t^\nu$  associated to  $t$  is then

$$e_t^\nu = \sum_{\sigma \in \text{ColStab}(t, \nu)} (-1)^{|\sigma|} \{\sigma \cdot t\} \in M^\eta(\mathbb{F}). \quad (8)$$

The *generalized Specht module* associated to the pair  $(\nu, \eta)$ , denoted  $S^{\nu, \eta}(\mathbb{F})$ , is the span in  $M^\eta(\mathbb{F})$  of the set  $\{e_t^\nu \mid t \text{ tableau of shape } \eta\}$ . The group  $S_n$  clearly acts on  $S^{\nu, \eta}(\mathbb{F})$ .

**Adding tails and stabilizing.** Generalized Specht modules are closely related to certain kinds of induced representations. For this, we need some notation. Consider a partition  $\nu = (\nu_1, \dots, \nu_\ell)$ . For  $k \geq 1$ , define

$$\nu[k] = (\nu_1, \dots, \nu_\ell + k) \quad \text{and} \quad \widehat{\text{st}}^k(\nu) = (\nu_1, \dots, \nu_\ell, k).$$

We will omit the  $k$  in  $\widehat{\text{st}}^k(\nu)$  if  $k = 1$ . Also, we will use the conventions  $\widehat{\text{st}}^0(\nu) = \nu$  and  $\nu[0] = \nu$ . We then have the following.

**Theorem 6.5** ([15, Theorem 17.13]). *Let  $\nu$  be a partition of  $n$ . The for  $m > n$ , there exists a short exact sequence*

$$0 \longrightarrow S^{\widehat{\text{st}}(\nu), (\widehat{\text{st}}(\nu))[m-n-1]}(\mathbb{F}) \longrightarrow I_{m-n}(S^\nu(\mathbb{F})) \longrightarrow S^{\nu, \nu[m-n]}(\mathbb{F}) \longrightarrow 0$$

of  $S_m$ -representations.

*Remark.* To relate Theorem 6.5 to the statement in [15, Theorem 17.13], we make the following two remarks.

- In the notation of [15, Theorem 17.13], we are taking  $\mu^\# = \nu$  and  $\mu = (\nu_1, \dots, \nu_\ell, m-n)$ .
- Instead of  $I_{m-n}(S^\nu(\mathbb{F}))$ , the statement of [15, Theorem 17.13] has  $S^{\mu^\#, \mu}(\mathbb{F})$  (though actually, the field  $\mathbb{F}$  is not specified in the notation in [15]). The isomorphism  $S^{\mu^\#, \mu}(\mathbb{F}) \cong I_{m-n}(S^\nu(\mathbb{F}))$  is discussed in the proof of [15, Corollary 17.14].

**An exact sequence.** Let  $\nu$  be a partition of  $n$ . Consider  $k \geq 0$ . Theorem 6.5 gives the following short exact sequences.

$$\begin{aligned} 0 &\longrightarrow S^{\widehat{\text{st}}(\nu), (\widehat{\text{st}}(\nu))[k-1]}(\mathbb{F}) \longrightarrow I_k(S^\nu(\mathbb{F})) \longrightarrow S^{\nu, \nu[k]}(\mathbb{F}) \longrightarrow 0 \\ 0 &\longrightarrow S^{\widehat{\text{st}}^2(\nu), (\widehat{\text{st}}^2(\nu))[k-2]}(\mathbb{F}) \longrightarrow I_{k-1}(S^{\widehat{\text{st}}(\nu)}(\mathbb{F})) \longrightarrow S^{\widehat{\text{st}}(\nu), (\widehat{\text{st}}(\nu))[k-1]}(\mathbb{F}) \longrightarrow 0 \\ &\vdots \\ 0 &\longrightarrow S^{\widehat{\text{st}}^{k-1}(\nu), (\widehat{\text{st}}^{k-1}(\nu))[1]}(\mathbb{F}) \longrightarrow I_2(S^{\widehat{\text{st}}^{k-2}(\nu)}(\mathbb{F})) \longrightarrow S^{\widehat{\text{st}}^{k-2}(\nu), (\widehat{\text{st}}^{k-2}(\nu))[2]}(\mathbb{F}) \longrightarrow 0 \\ 0 &\longrightarrow S^{\widehat{\text{st}}^k(\nu), (\widehat{\text{st}}^k(\nu))[0]}(\mathbb{F}) \longrightarrow I_1(S^{\widehat{\text{st}}^{k-1}(\nu)}(\mathbb{F})) \longrightarrow S^{\widehat{\text{st}}^{k-1}(\nu), (\widehat{\text{st}}^{k-1}(\nu))[1]}(\mathbb{F}) \longrightarrow 0 \end{aligned}$$

Stringing these short exact sequences together and using the obvious isomorphism  $S^{\widehat{\text{st}}^k(\nu), (\text{st}^k(\nu)) [0]}(\mathbb{F}) \cong S^{\widehat{\text{st}}^k(\nu)}(\mathbb{F})$ , we obtain the following.

**Corollary 6.6.** *Let  $\nu$  be a partition of  $n$  and let  $k \geq 0$ . There is then an exact sequence*

$$0 \longrightarrow S^{\widehat{\text{st}}^k(\nu)}(\mathbb{F}) \longrightarrow I_1(S^{\widehat{\text{st}}^{k-1}(\nu)}(\mathbb{F})) \longrightarrow \cdots \longrightarrow I_k(S^\nu(\mathbb{F}))$$

of representations of  $S_{n+k}$ .

**The proof.** We are finally in a position to prove Proposition 6.3.

*Proof of Proposition 6.3.* Let us recall the setup. Fix  $\mu \vdash n$  and  $k \geq 1$ . Let

$$\text{IA}_k(S^\mu(\mathbb{F})) \longrightarrow \text{IA}_{k-1}(S^{\text{st}(\mu)}(\mathbb{F})) \longrightarrow \cdots \longrightarrow S^{\text{st}^k(\mu)}(\mathbb{F}) \longrightarrow 0 \quad (9)$$

be the  $(n+k)$ -central stability chain complex associated to the potentially stable sequence

$$S^\mu(\mathbb{F}) \longrightarrow S^{\text{st}(\mu)}(\mathbb{F}) \longrightarrow \cdots \longrightarrow S^{\text{st}^k(\mu)}(\mathbb{F})$$

of representations of the symmetric group. Then we must prove that (9) is exact.

Recall that if  $H$  is a subgroup of  $G$  and  $V$  is an  $H$ -representation and  $W$  is a  $G$ -representation, then  $W \otimes \text{Ind}_H^G V \cong \text{Ind}_H^G (V \otimes \text{Res}_H^G W)$  (see, e.g., [2, Proposition 3.3.31]). This implies that  $\mathcal{A}_{n+k} \otimes \text{IA}_{k-j}(S^{\text{st}^j(\mu)}(\mathbb{F})) \cong \text{IA}_{k-j}(S_{\mathcal{A}}^{\text{st}^j(\mu)}(\mathbb{F}))$  for all  $0 \leq j \leq k$ . Tensoring (9) with  $\mathcal{A}_{n+k}$ , it is therefore enough to prove that the resulting chain complex

$$I_k(S_{\mathcal{A}}^\mu(\mathbb{F})) \longrightarrow I_{k-1}(S_{\mathcal{A}}^{\text{st}(\mu)}(\mathbb{F})) \longrightarrow \cdots \longrightarrow S_{\mathcal{A}}^{\text{st}^k(\mu)}(\mathbb{F}) \longrightarrow 0 \quad (10)$$

is exact.

Recall from above that  $I_j(S_{\mathcal{A}}^{\text{st}^{k-j}(\mu)}(\mathbb{F}))$  is dual to  $I_j(S^{(\text{st}^{k-j}(\mu))'}(\mathbb{F}))$  for all  $0 \leq j \leq k$ . Since  $(\text{st}^{k-j}(\mu))' = \widehat{\text{st}}^{k-j}(\mu')$ , the dual chain complex of (10) is

$$0 \longrightarrow S^{\widehat{\text{st}}^k(\mu')}(\mathbb{F}) \longrightarrow I_1(S^{\widehat{\text{st}}^{k-1}(\mu')}(\mathbb{F})) \longrightarrow \cdots \longrightarrow I_k(S^{\mu'}(\mathbb{F})). \quad (11)$$

It is enough to prove that (11) is exact. In fact, letting  $\nu = \mu'$ , this is exactly the chain complex that Corollary 6.6 asserts is exact (modulo the signs of the boundary maps, which depend on the noncanonical choice of a duality pairing). This follows easily from the formulas for the various maps involved given in [15, §17] combined with the explicit duality isomorphism given in [11, §4].  $\square$

## 7 Central stability implies Specht stability

In this section, we will prove Theorem E, which asserts that a centrally stable sequence of representations of the symmetric group is also Specht stable. We start with some definitions.

**Definition.** The *width* of a Specht module  $S^\mu(\mathbb{F})$  is the number of boxes in the first row of  $\mu$ .

**Definition.** Let  $\phi_{N-1} : V_{N-1} \rightarrow V_N$  be an  $S_{N-1}$ -equivariant map from a representation of  $S_{N-1}$  to a representation of  $S_N$  and let  $Q_{N+1}$  be an  $S_{N+1}$ -subrepresentation of  $\mathcal{C}(V_{N-1} \xrightarrow{\phi_{N-1}} V_N)$ . The *quotiented central stabilization sequence* associated to  $\phi_{N-1}$  and  $Q_{N+1}$  is the sequence

$$V_{N-1} \xrightarrow{\phi_{N-1}} V_N \xrightarrow{\phi_N} V_{N+1} \xrightarrow{\phi_{N+1}} V_{N+2} \xrightarrow{\phi_{N+2}} \cdots$$

which is inductively defined as follows. First,  $V_{N+1} = \mathcal{C}(V_{N-1} \xrightarrow{\phi_{N-1}} V_N) / Q_{N+1}$  and  $\phi_N$  is the natural map. Next, assume that  $V_{n-1}$  and  $V_n$  and  $\phi_{n-1} : V_{n-1} \rightarrow V_n$  are defined for some  $n \geq N+1$ . Then  $V_{n+1} = \mathcal{C}(V_{n-1} \xrightarrow{\phi_{n-1}} V_n)$  and  $\phi_n : V_n \rightarrow V_{n+1}$  is the natural map. If  $Q_{N+1} = 0$ , then we will simply call this the *central stabilization sequence* associated to  $\phi_{N-1}$ .

Our first lemma restricts the Specht modules that can appear in a central stabilization sequence. Recall that if either  $\text{char}(\mathbb{F}) = 0$  or  $\text{char}(\mathbb{F}) \geq n + 1$ , then every representation of  $S_n$  can be decomposed into a direct sum of Specht modules and the isotypic components of this decomposition are unique.

**Lemma 7.1.** *Let  $\phi_{N-1} : V_{N-1} \rightarrow V_N$  be an  $S_{N-1}$ -equivariant map from a representation of  $S_{N-1}$  to a representation of  $S_N$  and let*

$$V_{N-1} \xrightarrow{\phi_{N-1}} V_N \xrightarrow{\phi_N} V_{N+1} \xrightarrow{\phi_{N+1}} V_{N+2} \xrightarrow{\phi_{N+2}} \dots$$

*be the associated central stabilization sequence. Consider  $n \geq N$ . Assume that either  $\text{char}(\mathbb{F}) = 0$  or  $\text{char}(\mathbb{F}) \geq n + 1$ . Then every Specht module that occurs in  $V_n$  has width at least  $n - N$ .*

*Proof.* Let  $S^\mu(\mathbb{F})$  be a Specht module that occurs in  $V_n$ . Recall that  $\mathcal{T}_k \cong \mathbb{F}$  is the trivial representation of  $S_k$ . By construction,  $V_n$  is a quotient of  $V'_n := \text{Ind}_{S_N \times S_{n-N}}^{S_n} V_N \boxtimes \mathcal{T}_{n-N}$ , so  $S^\mu(\mathbb{F})$  appears in  $V'_n$ . The Littlewood-Richardson rule says that there exists some  $\nu \vdash N$  such that  $S^\nu(\mathbb{F})$  appears in  $V_N$  and such that  $\mu$  is obtained by adding  $n - N$  boxes to  $\nu$  with no two boxes added to the same column (see ([15, §16]; the special case we are using is often called Pieri's formula). Thus  $\mu$  has at least  $n - N$  columns, and hence at least  $n - N$  boxes in its first row.  $\square$

We now need three more definitions.

**Definition.** Let  $V_n$  be a representation of  $S_n$ . Assume that either  $\text{char}(\mathbb{F}) = 0$  or  $\text{char}(\mathbb{F}) \geq n + 1$ . Then the *width* of  $V_n$  is the maximum width of a Specht module that occurs in  $V_n$ .

In the following definition, observe that there is no assumption on  $\text{char}(\mathbb{F})$ .

**Definition.** Let  $V_n$  be a representation of  $S_n$ . We say that  $V_n$  has *constant width*  $k$  if it can be decomposed into a direct sum of Specht modules of width  $k$ . We say that an  $S_n$ -subrepresentation  $W_n$  of  $V_n$  has *constant cowidth*  $k$  if  $V_n/W_n$  has constant width  $k$ .

*Remark.* A theorem of Hemmer-Nakano [13] says that if a representation of  $S_n$  can be decomposed into a direct sum of Specht modules, then the Specht modules that occur are independent of the decomposition (as long as  $\text{char}(\mathbb{F}) \geq 5$ ).

Recall that if  $\mu \vdash n$ , then there is a natural stabilization map  $S^\mu(\mathbb{F}) \hookrightarrow S^{\text{st}(\mu)}(\mathbb{F})$ .

**Definition.** Let  $\phi_n : V_n \rightarrow V_{n+1}$  be an  $S_n$ -equivariant map from a representation of  $S_n$  to a representation of  $S_{n+1}$ . Assume that  $V_n$  has constant width  $k$ , and let

$$V_n = \bigoplus_{i \in I} S^{\mu_i}(\mathbb{F})$$

be the associated decomposition. Then  $\phi_n$  is a *stabilization map* if we can write

$$V_{n+1} = \bigoplus_{i \in I} S^{\text{st}(\mu_i)}(\mathbb{F})$$

such that the restriction of  $\phi_n$  to  $S^{\mu_i}(\mathbb{F})$  is the stabilization map  $S^{\mu_i}(\mathbb{F}) \hookrightarrow S^{\text{st}(\mu_i)}(\mathbb{F})$ .

**Lemma 7.2.** *Let  $V_n$  be a representation of  $S_n$ , let  $V_{n+1}$  be an  $S_{n+1}$  representation obtained as a quotient of  $IA_1(V_n)$ , and let  $\phi_n : V_n \rightarrow V_{n+1}$  be the natural  $S_n$ -equivariant map. Assume that either  $\text{char}(\mathbb{F}) = 0$  or  $\text{char}(\mathbb{F}) \geq n + 2$ . Let  $k$  be the width of  $V_n$ , let  $W_n$  be the subspace of  $V_n$  spanned by Specht modules of width strictly less than  $k$ , and let  $W_{n+1}$  be the subspace of  $V_{n+1}$  spanned by Specht modules of width strictly less than  $k + 1$ . The following then hold.*

- $\phi_n(W_n) \subset W_{n+1}$ , so there is an induced map  $\hat{\phi}_n : V_n/W_n \rightarrow V_{n+1}/W_{n+1}$ .
- We can factor  $\hat{\phi}_n$  as  $V_n/W_n \xrightarrow{\hat{\phi}'_n} \hat{V}'_n \xrightarrow{\hat{\phi}''_n} V_{n+1}/W_{n+1}$ , where  $\hat{\phi}'_n$  is a surjection and  $\hat{\phi}''_n$  is a stabilization map.

*Proof.* The restriction rule (Theorem 6.2) implies that  $\phi_n(W_n) \subset W_{n+1}$ , so we concentrate on the second claim. To simplify our notation, write  $\hat{V}_n = V_n/W_n$  and  $\hat{V}_{n+1} = V_{n+1}/W_{n+1}$

Assume first that  $V_n = S^\mu(\mathbb{F})$  and  $V_{n+1} = \text{IA}_1(V_n)$ . The branching rule [15, §9.2] implies that  $\hat{V}_{n+1} = S^{\text{st}(\mu)}(\mathbb{F})$ . The universal property  $\text{Hom}_{S_n}(V_n, \text{Res}_{S_n}^{S_{n+1}} \hat{V}_{n+1}) = \text{Hom}_{S_{n+1}}(\text{IA}_1(V_n), \hat{V}_{n+1})$  implies that  $\hat{\phi}_n \neq 0$ . Since  $V_n$  is irreducible,  $\hat{\phi}_n$  must be injective. The restriction rule (Theorem 6.2) then implies that  $\hat{\phi}_n$  is the stabilization map.

We now consider general  $V_n$ . Let  $\hat{V}'_{n+1}$  be the quotient of  $\text{IA}_1(V_n)$  by the subspace spanned by all Specht modules of width strictly less than  $k+1$ . There then exists an  $S_{n+1}$ -subrepresentation  $Q_{n+1}$  of  $\hat{V}'_{n+1}$  such that  $\hat{V}_{n+1} = \hat{V}'_{n+1}/Q_{n+1}$ . We can write

$$\hat{V}'_{n+1} = \bigoplus_{j \in J} S^{v_j}(\mathbb{F}) \quad \text{and} \quad Q_{n+1} = \bigoplus_{j \in J'} S^{v_j}(\mathbb{F})$$

with  $J' \subset J$ . It is an easy exercise using the results in the previous paragraph together with Schur's lemma to show that we can write

$$\hat{V}_n = \bigoplus_{j \in J} S^{\mu_j}(\mathbb{F}),$$

where for all  $j \in J$  we have  $v_j = \text{st}(\mu_j)$  and the restriction of the natural map  $\hat{V}_n \rightarrow \hat{V}'_{n+1}$  to  $S^{\mu_j}(\mathbb{F})$  is the stabilization map  $S^{\mu_j}(\mathbb{F}) \hookrightarrow S^{v_j}(\mathbb{F})$ . We can then let  $\hat{V}'_n = \hat{V}_n/Q_n$ , where

$$Q_n = \bigoplus_{j \in J'} S^{\mu_j}(\mathbb{F}) \subset \hat{V}_n,$$

and let  $\hat{\phi}'_n : \hat{V}_n \rightarrow \hat{V}'_n$  and  $\hat{\phi}''_n : \hat{V}'_n \rightarrow \hat{V}_{n+1}$  be the natural maps. □

**Lemma 7.3.** *Let  $\phi_{N-1} : V_{N-1} \rightarrow V_N$  be an  $S_{N-1}$ -equivariant map from a representation of  $S_{N-1}$  to a representation of  $S_N$  and let  $Q_{N+1}$  be an  $S_{N+1}$ -subrepresentation of  $\mathcal{C}(V_{N-1} \xrightarrow{\phi_{N-1}} V_N)$ . Let*

$$V_{N-1} \xrightarrow{\phi_{N-1}} V_N \xrightarrow{\phi_N} V_{N+1} \xrightarrow{\phi_{N+1}} V_{N+2} \xrightarrow{\phi_{N+2}} \dots,$$

*be the associated quotiented central stabilization sequence. Assume that either  $\text{char}(\mathbb{F}) = 0$  or  $\text{char}(\mathbb{F}) \geq N+2$ , and let  $k$  be the width of  $V_N$ . For all  $n \geq N$ , there then exist constant cowidth  $k + (n - N)$  subrepresentations  $W_n$  of  $V_n$  such that the following hold.*

1. *The representations  $W_N$  and  $W_{N+1}$  have width at most  $k-1$  and  $k$ , respectively.*
2. *For  $n \geq N$ , we have  $\phi_n(W_n) \subset W_{n+1}$ . Moreover, for  $n \geq N+1$  the induced map  $\hat{\phi}_n : V_n/W_n \rightarrow V_{n+1}/W_{n+1}$  is a stabilization map.*
3. *For  $n \geq N$ , let  $\phi'_n : W_n \rightarrow W_{n+1}$  be the restriction of  $\phi_n$ . There then exists some  $S_{N+2}$ -subrepresentation  $Q'_{N+2}$  of  $\mathcal{C}(W_N \xrightarrow{\phi'_N} W_{N+1})$  such that the sequence*

$$W_N \xrightarrow{\phi'_N} W_{N+1} \xrightarrow{\phi'_{N+1}} W_{N+2} \xrightarrow{\phi'_{N+2}} W_{N+3} \xrightarrow{\phi'_{N+3}} \dots$$

*is the quotiented central stabilization sequence associated to  $\phi'_N$  and  $Q'_{N+2}$ .*

*Proof.* Let  $W_N$  (resp.  $W_{N+1}$ ) be the subspace of  $V_N$  (resp.  $V_{N+1}$ ) spanned by Specht modules of width strictly less than  $k$  (resp.  $k+1$ ). Condition 1 is clearly satisfied, and Lemma 7.2 says that  $\phi_N(W_N) \subset W_{N+1}$ . Assume now that  $n \geq N+1$  and that we have constructed  $W_N, \dots, W_n$  satisfying the conclusions of the lemma.

**Step 1.** *We construct  $W_{n+1}$ .*

We need some notation.

- Let  $\phi'_{n-1} : W_{n-1} \rightarrow W_n$  be the restriction of  $\phi_{n-1}$  and let  $\hat{\phi}_{n-1} : V_{n-1}/W_{n-1} \rightarrow V_n/W_n$  be the induced map.

- Let  $\partial_{n-1} : \text{IA}_2(V_{n-1}) \rightarrow \text{IA}_1(V_n)$  and  $\partial'_{n-1} : \text{IA}_2(W_{n-1}) \rightarrow \text{IA}_1(W_n)$  and  $\hat{\partial}_{n-1} : \text{IA}_2(V_{n-1}/W_{n-1}) \rightarrow \text{IA}_1(V_n/W_n)$  be the  $(n+1)$ -boundary maps associated to  $\phi_{n-1}$  and  $\phi'_{n-1}$  and  $\hat{\phi}_{n-1}$ , respectively.
- Let  $\tilde{W}_{n+1} = \mathcal{C}(W_{n-1} \xrightarrow{\phi'_{n-1}} W_n)$ .

Lemma 4.3 together with our assumptions implies that  $V_{n+1} = \text{coker}(\partial_{n-1})$  and  $\tilde{W}_{n+1} = \text{coker}(\partial'_{n-1})$ . For all  $k \geq 0$ , the functor  $\text{IA}_k(\bullet)$  is exact. We thus have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{IA}_2(W_{n-1}) & \longrightarrow & \text{IA}_2(V_{n-1}) & \longrightarrow & \text{IA}_2(V_{n-1}/W_{n-1}) \longrightarrow 0 \\
& & \downarrow \partial'_{n-1} & & \downarrow \partial_{n-1} & & \downarrow \hat{\partial}_{n-1} \\
0 & \longrightarrow & \text{IA}_1(W_n) & \longrightarrow & \text{IA}_1(V_n) & \longrightarrow & \text{IA}_1(V_n/W_n) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & \tilde{W}_{n+1} & & V_{n+1} & & 
\end{array}$$

whose rows and columns are exact. We can form  $V_{n+1} = \text{coker}(\partial_{n-1})$  in two steps. First, let  $\tilde{V}_{n+1} = \text{IA}_1(V_n)/\partial_{n-1}(\text{IA}_2(W_{n-1}))$ . Chasing the above diagram, we see that there is an exact sequence

$$0 \longrightarrow \tilde{W}_{n+1} \longrightarrow \tilde{V}_{n+1} \longrightarrow \text{IA}_1(V_n/W_n) \longrightarrow 0.$$

Also, the map  $\text{IA}_2(V_{n-1}) \rightarrow \tilde{V}_{n+1}$  factors through a map  $\bar{\partial}_{n-1} : \text{IA}_2(V_{n-1}/W_{n-1}) \rightarrow \tilde{V}_{n+1}$  satisfying  $V_{n+1} = \text{coker}(\bar{\partial}_{n-1})$ . Let  $W_{n+1}$  be the image of  $\tilde{W}_{n+1}$  in  $V_{n+1}$ .

**Step 2.**  $W_{n+1}$  satisfies the second conclusion of the lemma.

We have  $\phi_n(W_n) \subset W_{n+1}$  by construction, so we must show that the induced map  $\hat{\phi}_n : V_n/W_n \rightarrow V_{n+1}/W_{n+1}$  is a stabilization map. Observe that we have a commutative diagram

$$\begin{array}{ccccccc}
& & & & \text{IA}_2(V_{n-1}/W_{n-1}) & & \\
& & & & \downarrow \bar{\partial}_{n-1} & \searrow \hat{\partial}_{n-1} & \\
0 & \longrightarrow & \tilde{W}_{n+1} & \longrightarrow & \tilde{V}_{n+1} & \longrightarrow & \text{IA}_1(V_n/W_n) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_{n+1} & \longrightarrow & V_{n+1} & \longrightarrow & V_{n+1}/W_{n+1} \longrightarrow 0
\end{array}$$

whose rows and columns are exact. Chasing this diagram, we deduce that there is a short exact sequence

$$\text{IA}_2(V_{n-1}/W_{n-1}) \xrightarrow{\hat{\partial}_{n-1}} \text{IA}_1(V_n/W_n) \longrightarrow V_{n+1}/W_{n+1} \longrightarrow 0. \quad (12)$$

Lemma 4.3 then implies that  $V_{n+1}/W_{n+1} = \mathcal{C}(V_{n-1}/W_{n-1} \rightarrow V_n/W_n)$ .

There are now two cases. If  $n \geq N+2$ , then the map  $V_{n-1}/W_{n-1} \rightarrow V_n/W_n$  is a stabilization map by induction, so Corollary 6.4 implies that  $\hat{\phi}_n$  is a stabilization map. If instead  $n = N+1$ , then Lemma 7.2 says that we can factor  $\hat{\phi}_{n-1}$  as a composition

$$V_{n-1}/W_{n-1} \xrightarrow{\hat{\phi}'_{n-1}} \hat{V}'_{n-1} \xrightarrow{\hat{\phi}''_{n-1}} V_n/W_n,$$

where  $\hat{\phi}'_{n-1}$  is a surjection and  $\hat{\phi}''_{n-1}$  is a stabilization map. Letting  $\hat{\partial}''_{n-1}$  be the  $(n+1)$ -boundary map associated to  $\hat{\phi}''_{n-1}$ , we can therefore factor  $\hat{\partial}_{n-1}$  as a composition

$$\text{IA}_2(V_{n-1}/W_{n-1}) \longrightarrow \text{IA}_2(\hat{V}'_{n-1}) \xrightarrow{\hat{\partial}''_{n-1}} \text{IA}_1(V_n/W_n).$$

Combining this with (12), we obtain an exact sequence

$$\mathrm{IA}_2(\hat{V}'_{n-1}) \xrightarrow{\hat{\partial}''_{n-1}} \mathrm{IA}_1(V_n/W_n) \longrightarrow V_{n+1}/W_{n+1} \longrightarrow 0,$$

so we can apply Lemma 4.3 to deduce that  $V_{n+1}/W_{n+1} = \mathcal{C}(\hat{V}'_{n-1} \rightarrow V_n/W_n)$  and then apply Corollary 6.4 to deduce that  $\hat{\phi}_n$  is a stabilization map.

**Step 3.**  $W_{n+1}$  satisfies the third conclusion of the lemma.

Letting  $\mathcal{Q}'_{n+1} = \tilde{W}_{n+1} \cap \mathrm{Im}(\bar{\partial}_{n-1})$ , we have a short exact sequence

$$0 \longrightarrow \mathcal{Q}'_{n+1} \longrightarrow \tilde{W}_{n+1} \longrightarrow W_{n+1} \longrightarrow 0.$$

Since  $\tilde{W}_{n+1} = \mathcal{C}(W_{n-1} \rightarrow W_n)$ , the desired conclusion follows if  $n = N + 1$ . Assume now that  $n \geq N + 2$ . To prove the desired conclusion, we must show that  $\mathcal{Q}'_{n+1} = 0$ . Let  $\partial_{n-2} : \mathrm{IA}_3(V_{n-2}) \rightarrow \mathrm{IA}_2(V_{n-1})$  and  $\hat{\partial}_{n-2} : \mathrm{IA}_3(V_{n-2}/W_{n-2}) \rightarrow \mathrm{IA}_2(V_{n-1}/W_{n-1})$  be the  $(n+1)$ -boundary maps associated to  $\phi_{n-2}$  and  $\hat{\phi}_{n-2}$ , respectively. Lemma 4.4 implies that  $\partial_{n-1} \circ \partial_{n-2} = 0$ . Observe that we have the following commutative diagram.

$$\begin{array}{ccccc} \mathrm{IA}_3(V_{n-2}) & \xrightarrow{\partial_{n-2}} & \mathrm{IA}_2(V_{n-1}) & \xrightarrow{\partial_{n-1}} & \mathrm{IA}_1(V_n) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{IA}_3(V_{n-2}/W_{n-2}) & \xrightarrow{\hat{\partial}_{n-2}} & \mathrm{IA}_2(V_{n-1}/W_{n-1}) & \xrightarrow{\hat{\partial}_{n-1}} & \mathrm{IA}_1(V_n/W_n) \\ & & \nearrow \bar{\partial}_{n-1} & \nwarrow & \\ & & \tilde{V}_{n+1} & & \end{array} \quad (13)$$

Since  $\partial_{n-1} \circ \partial_{n-2} = 0$ , we can chase this diagram to see that  $\bar{\partial}_{n-1} \circ \hat{\partial}_{n-2} = 0$ , i.e. that  $\mathrm{Im}(\hat{\partial}_{n-2}) \subset \ker(\bar{\partial}_{n-1})$ . Let  $\bar{\partial}'_{n-1} : \mathrm{IA}_2(V_{n-1}/W_{n-1})/\mathrm{Im}(\hat{\partial}_{n-2}) \rightarrow \tilde{V}_{n+1}$  be the induced map.

There are now two cases. If  $n \geq N + 3$ , then by induction the maps  $\hat{\phi}_{n-2}$  and  $\hat{\phi}_{n-1}$  are stabilization maps, so Proposition 6.1 implies that the bottom row of (13) is exact. This implies that the composition

$$\mathrm{IA}_2(V_{n-1}/W_{n-1})/\mathrm{Im}(\hat{\partial}_{n-2}) \xrightarrow{\bar{\partial}'_{n-1}} \tilde{V}_{n+1} \longrightarrow \mathrm{IA}_1(V_n/W_n)$$

is injective. Since  $\tilde{W}_{n+1} = \ker(\tilde{V}_{n+1} \rightarrow \mathrm{IA}_1(V_n/W_n))$ , we conclude that

$$\mathcal{Q}'_{n+1} = \mathrm{Im}(\bar{\partial}_{n-1}) \cap \tilde{W}_{n+1} = 0.$$

If  $n = N + 2$ , then  $\hat{\phi}_{n-1}$  is a stabilization map but  $\hat{\phi}_{n-2}$  need not be. However, Lemma 7.2 says that we can factor  $\hat{\phi}_{n-1}$  as a composition of a surjection with a stabilization map, and just like in Step 2 we can use this to run the above argument and get that  $\mathcal{Q}'_{n+1} = 0$ , as desired.  $\square$

*Proof of Theorem E.* Let us first recall the setup. We have a coherent sequence

$$V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} V_4 \xrightarrow{\phi_4} \dots \quad (14)$$

of representations of the symmetric group over  $\mathbb{F}$  which is centrally stable starting at  $N$ . Also, we have either  $\mathrm{char}(\mathbb{F}) = 0$  or  $\mathrm{char}(\mathbb{F}) \geq 2N + 2$ . Our goal is to prove that (14) is Specht stable starting at  $2N + 1$ .

By assumption, the sequence

$$V_{N-1} \xrightarrow{\phi_{N-1}} V_N \xrightarrow{\phi_N} V_{N+1} \xrightarrow{\phi_{N+1}} V_{N+2} \xrightarrow{\phi_{N+2}} \dots$$

is the central stabilization sequence associated to  $\phi_{N-1}$ . Let  $k_1$  be the maximal width of  $V_N$ , which is well-defined by our assumptions on  $\mathrm{char}(\mathbb{F})$ . Clearly  $k_1 \leq N$ . For  $n \geq N$ , let  $W_n^1 < V_n$  be the constant cowidth



$$\begin{array}{cccccccc}
V_2 & \longrightarrow & V_3 & \rightrightarrows & V_4 & \rightrightarrows & V_5 & \rightrightarrows & V_6 & \rightrightarrows & V_7 & \rightrightarrows & \dots \\
& & \uparrow^{k_1} & & \uparrow^{k_1+1} & & \uparrow^{k_1+2} & & \uparrow^{k_1+3} & & \uparrow^{k_1+4} & & \\
& & W_3^1 & \longrightarrow & W_4^1 & \rightrightarrows & W_5^1 & \rightrightarrows & W_6^1 & \rightrightarrows & W_7^1 & \rightrightarrows & \dots \\
& & & & \uparrow^{k_2} & & \uparrow^{k_2+1} & & \uparrow^{k_2+2} & & \uparrow^{k_2+3} & & \\
& & & & W_4^2 & \longrightarrow & W_5^2 & \rightrightarrows & W_6^2 & \rightrightarrows & W_7^2 & \rightrightarrows & \dots \\
& & & & & & \uparrow^{k_3} & & \uparrow^{k_3+1} & & \uparrow^{k_3+2} & & \\
& & & & & & W_5^3 & \longrightarrow & W_6^3 & \rightrightarrows & W_7^3 & \rightrightarrows & \dots \\
& & & & & & & & \uparrow^{k_4} & & \uparrow^{k_4+1} & & \\
& & & & & & & & W_6^4 & \longrightarrow & W_7^4 & \rightrightarrows & \dots
\end{array}$$

**Table 1:** The  $V_i$  and  $W_i^j$  for  $N = 3$ . Triple horizontal arrows are central stabilizations, double horizontal arrows are quotients of central stabilizations, and numbers on the vertical arrows are the cowidths of the constant cowidth subrepresentations

$k_1 + (n - N) \leq n$  subrepresentation given by Lemma 7.3. Let  $k_2$  be the maximal width of  $W_{N+1}^1$ , which again is well-defined. Since  $W_{N+1}^1$  has width at most  $k_1$  by assumption, we see that  $k_2 \leq N$ . The sequence

$$W_N^1 \longrightarrow W_{N+1}^1 \longrightarrow W_{N+2}^1 \longrightarrow W_{N+3}^1 \longrightarrow \dots$$

is a quotiented central stabilization sequence, so we can apply Lemma 7.3 again and obtain constant cowidth  $k_2 + (n - N - 1)$  subrepresentations  $W_n^2 < W_n^1$  for  $n \geq N + 1$ .

By our assumptions on  $\text{char}(\mathbb{F})$ , this process can be repeated several times to obtain  $W_n^i$  for  $n \geq N - 1 + i$  and  $1 \leq i \leq N + 1$ . Here  $W_n^{i+1}$  is a constant cowidth  $k_{i+1} + (n - N - i)$  subrepresentation of  $W_n^i$ , where  $k_{i+1} \leq N$ . To help keep all of this straight, see Table 1. Now, by assumption  $W_{2N}^{N+1}$  (resp.  $W_{2N+1}^{N+1}$ ) has width at most  $k_{N+1} - 1 \leq N - 1$  (resp.  $k_{N+1} \leq N$ ). Since  $W_{2N}^{N+1}$  (resp.  $W_{2N+1}^{N+1}$ ) is a subrepresentation of  $V_{2N}$  (resp.  $V_{2N+1}$ ), Lemma 7.1 implies that  $W_{2N}^{N+1} = 0$  and  $W_{2N+1}^{N+1} = 0$ . But this implies that  $W_i^{N+1} = 0$  for all  $i \geq 2N$ . It follows that for  $n \geq 2N$  we have a filtration

$$V_n \supset W_n^1 \supset W_n^2 \supset \dots \supset W_n^{2N+1} = 0.$$

This might not quite be a top-indexed Specht filtration (for example, if  $k_1 < N$ ), but we can obtain one by adding repeated terms as necessary. Our assumptions then imply that with respect to these filtrations the maps  $V_n \rightarrow V_{n+1}$  are stabilization maps for  $n \geq 2N + 1$ , and we are done.  $\square$

## 8 Central stability implies polynomial dimensions

We now prove Theorem D. By Theorem E, it is enough to prove that if  $\mu \vdash n$ , then there is a polynomial  $\phi(k)$  such that  $\phi(k) = \dim S^{\text{st}^k(\mu)}(\mathbb{F})$  for  $k \geq 0$ . This follows easily from the results in [8], but we give a short direct proof. If  $\mathbf{v}$  is a partition, then let  $\text{ST}(\mathbf{v})$  be the set of standard tableau of shape  $\mathbf{v}$ , so  $|\text{ST}(\mathbf{v})| = \dim S^{\mathbf{v}}(\mathbb{F})$ . If  $t$  is a tableau, then denote by  $\text{UR}(t)$  the entry in the upper right hand corner of  $t$ . Finally, define

$$X_k = \{(t, \sigma) \mid t \in \text{ST}(\mu) \text{ and } \sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n+k\} \text{ is an order-preserving injection with } \sigma(i) = i \text{ for } 1 \leq i \leq \text{UR}(\mu)\}.$$

We will construct a bijection  $\psi_k : \text{ST}(\text{st}^k(\mu)) \rightarrow X_k$ . Since there are  $\binom{n-m+k}{k}$  order-preserving injections  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n+k\}$  such that  $\sigma(i) = i$  for  $1 \leq i \leq m$ , it will follow that

$$\dim S^{\text{st}^k(\mu)}(\mathbb{F}) = |\text{ST}(\text{st}^k(\mu))| = \sum_{t \in \text{ST}(\mu)} \binom{n - \text{UR}(t) + k}{k},$$

a polynomial in  $k$ .

For  $s \in \text{ST}(\text{st}^k(\mu))$ , define  $\psi_k(s) = (t, \sigma)$ , where  $t$  and  $\sigma$  are as follows. Let  $s_2$  be the result of deleting the last  $k$  boxes from the first row of  $s$ . Let  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n+k\}$  be the unique order-preserving injection whose image is the set of entries of  $s_2$ . Finally, let  $t$  be the result of replacing each entry  $i$  in  $s_2$  with  $\sigma^{-1}(i)$ . It is easy to see that  $(t, \sigma) \in X_k$ . To see that  $\psi_k$  is a bijection, define a map  $\phi_k : X_k \rightarrow \text{ST}(\text{st}^k(\mu))$  via  $\phi_k(t, \sigma) = s$ , where  $s$  is obtained by first replacing each entry  $i$  in  $t$  with  $\sigma(i)$  and then appending the numbers  $\{1, \dots, n+k\} \setminus \text{Im}(\sigma)$  (in order) to the end of the first row of the result. Clearly  $\phi_k$  is a 2-sided inverse to  $\psi_k$ .

## References

- [1] H. Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York, 1968.
- [2] D. J. Benson, *Representations and cohomology. I*, Cambridge Studies in Advanced Mathematics, 30, Cambridge Univ. Press, Cambridge, 1991.
- [3] A. Borel, Stable real cohomology of arithmetic groups, *Ann. Sci. École Norm. Sup. (4)* **7** (1974), 235–272 (1975).
- [4] A. Borel and J.-P. Serre, Corners and arithmetic groups, *Comment. Math. Helv.* **48** (1973), 436–491.
- [5] K. S. Brown, *Cohomology of groups*, Corrected reprint of the 1982 original, Springer, New York, 1994.
- [6] K. Bux, R. Gramlich, and S. Witzel, Higher finiteness properties of reductive arithmetic groups in positive characteristic: the rank theorem, preprint 2011.
- [7] R. Charney, On the problem of homology stability for congruence subgroups, *Comm. Algebra* **12** (1984), no. 17-18, 2081–2123.
- [8] T. Church, J. Ellenberg, and B. Farb, FI-modules: a new approach to stability for  $S_n$ -representations, preprint 2012.
- [9] T. Church and B. Farb, Representation theory and homological stability, preprint 2010.
- [10] T. Church, Homological stability for configuration spaces of manifolds, to appear in *Invent. Math.*
- [11] M. Fayers, On the structure of Specht modules, *J. London Math. Soc. (2)* **67** (2003), no. 1, 85–102.
- [12] A. Hatcher and N. Wahl, Stabilization for mapping class groups of 3-manifolds, *Duke Math. J.* **155** (2010), no. 2, 205–269.
- [13] D. J. Hemmer and D. K. Nakano, Specht filtrations for Hecke algebras of type A, *J. London Math. Soc. (2)* **69** (2004), no. 3, 623–638.
- [14] D. J. Hemmer, Stable decompositions for some symmetric group characters arising in braid group cohomology, *J. Combin. Theory Ser. A* **118** (2011), no. 3, 1136–1139.
- [15] G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics, 682, Springer, Berlin, 1978.
- [16] R. Lee and R. H. Szczarba, On the homology and cohomology of congruence subgroups, *Invent. Math.* **33** (1976), no. 1, 15–53.
- [17] H. Maazen, Homology Stability for the General Linear Group, thesis, University of Utrecht, 1979.
- [18] M. H. Peel, Specht modules and symmetric groups, *J. Algebra* **36** (1975), no. 1, 88–97.
- [19] A. Putman, Stability for the homology of mapping class groups of arbitrary manifolds with marked points, preprint 2012.
- [20] S. V. Sam and J. Weyman, Pieri resolutions for classical groups, *J. Algebra* **329** (2011), 222–259.
- [21] W. van der Kallen, Homology stability for linear groups, *Invent. Math.* **60** (1980), no. 3, 269–295.

Andrew Putman  
Department of Mathematics  
Rice University, MS 136  
6100 Main St.  
Houston, TX 77005  
E-mail: andyp@rice.edu