Small generating sets for the Torelli group

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Abstract

Proving a conjecture of Dennis Johnson, we show that the Torelli subgroup $\mathcal{I}_g$ of the genus g mapping class group has a finite generating set whose size grows cubically with respect to g. Our main tool is a new space called the handle graph on which $\mathcal{I}_g$ acts cocompactly.

1 Introduction

Let $\Sigma_{g,n}$ be a compact connected oriented genus g surface with n boundary components. The *mapping class group* of $\Sigma_{g,n}$, denoted $\text{Mod}_{g,n}$, is the group of orientation-preserving homeomorphisms of $\Sigma_{g,n}$ that fix the boundary pointwise modulo isotopies that fix the boundary pointwise. We will often omit the n if it vanishes. For $n \leq 1$, the *Torelli group*, denoted $\mathcal{I}_{g,n}$, is the kernel of the action of $\text{Mod}_{g,n}$ on $\text{H}_1(\Sigma_{g,n}; \mathbb{Z})$. The Torelli group has been the object of intensive study ever since the seminal work of Dennis Johnson in the early '80's. See [10] for a survey of Johnson's work.

Finite generation of Torelli. One of Johnson's most celebrated theorems says that $\mathcal{I}_{g,n}$ is finitely generated for $g \geq 3$ and $n \leq 1$ (see [11]). This is a surprising result – though $\text{Mod}_{g,n}$ is finitely presentable, $\mathcal{I}_{g,n}$ is an infinite-index normal subgroup of $\text{Mod}_{g,n}$, so there is no reason to hope that $\mathcal{I}_{g,n}$ has any finiteness properties. Moreover, McCullough and Miller [13] proved that $\mathcal{I}_{2,n}$ is *not* finitely generated for $n \leq 1$, and later Mess [14] proved that $\mathcal{I}_2$ is an infinite rank free group.

Johnson's generating set. Johnson's generating set for $\mathcal{I}_{g,n}$ when $g \geq 3$ and $n \leq 1$ is enormous. Indeed, for $\mathcal{I}_g$ (resp. $\mathcal{I}_{g,1}$), it contains $9 \cdot 2^{2g-3} - 4g^2 + 2g - 6$ (resp. $9 \cdot 2^{2g-3} - 4g^2 + 4g - 5$) elements. In [12], Johnson proved that the abelianization of $\mathcal{I}_g$ (resp. $\mathcal{I}_{g,1}$) has rank $\frac{1}{3}(4g^3 + 5g + 3)$ (resp. $\frac{1}{3}(4g^3 - g)$). These give large lower bounds on the size of generating sets for $\mathcal{I}_{g,n}$; however, there is

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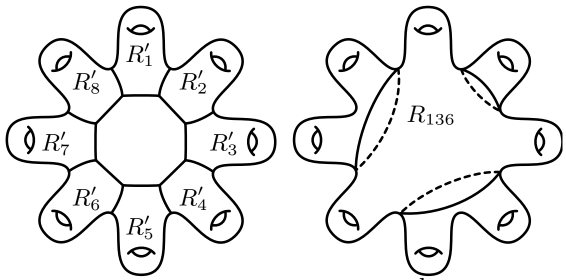

Figure 1: a. The subsurfaces $R'_i \cong \Sigma_{1,1}$. To avoid cluttering the picture, the portion of the boundaries of the R'_i which lie on the back side the figure are not drawn. b. A subsurface isotopic to R_{136}.

a huge gap between this cubic lower bound and Johnson's exponentially growing generating set. At the end of [11] and in [10, p. 168], Johnson conjectures that there should be a generating set for $\mathcal{I}_{g,n}$ whose size grows cubically with respect to the genus. Later, in [4, Problem 5.7] Farb asked whether there at least exists a generating set whose size grows polynomially.

Main theorem. In this paper, we prove Johnson's conjecture. Our main theorem is as follows.

Theorem A. *For $g \geq 3$, the group $\mathcal{I}_g$ has a generating set of size at most $57\binom{g}{3}$ and the group $\mathcal{I}_{g,1}$ has a generating set of size at most $57\binom{g}{3} + 2g + 1$.*

The generating set we construct was conjectured to generate $\mathcal{I}_{g,n}$ by Brendle and Farb [2]. To describe it, we must introduce some notation. As in Figure 1.a, let $R'_1, \ldots, R'_g$ be g subsurfaces of Σ_g each homeomorphic to $\Sigma_{1,1}$ such that the following hold. Interpret all indices modulo g.

- If $1 \leq i < j \leq g$ satisfy $i \notin \{j-1, j+1\}$, then $R'_i \cap R'_j = \emptyset$.
- For all $1 \leq i \leq g$, the intersection $R'_i \cap R'_{i+1}$ is homeomorphic to an interval.

For $1 \leq i < j < k \leq g$, define a subsurface R_{ijk} of Σ_g by $R_{ijk} = \overline{\Sigma_g \setminus \bigcup_{l \neq i,j,k} R'_l}$. Thus R_{ijk} is a genus 3 surface with at most 3 boundary components such that $R'_i, R'_j, R'_k \subset R_{i,j,k}$ (see Figure 1.b).

If S is a subsurface of Σ_g, define $\text{Mod}(\Sigma_g, S)$ to be the subgroup of Mod_g consisting of mapping classes that can be realized by homeomorphisms supported on S and $\mathcal{I}(\Sigma_g, S)$ to equal $\mathcal{I}_g \cap \text{Mod}(\Sigma_g, S)$. The key result for the proof of Theorem A is the following theorem.

Theorem B. *For $g \geq 3$, the group $\mathcal{I}_g$ is generated by $\bigcup_{1 \leq i<j<k \leq g} \mathcal{I}(\Sigma_g, R_{ijk})$.*

Using Johnson's work, it is easy to see that $\mathcal{I}(\Sigma_g, R_{ijk})$ is finitely generated by a generating set with at most 57 generators (see Lemma 2.2). Also, standard techniques (see Lemma 2.1) show that if $\mathcal{I}_g$ has a generating set with k elements, then $\mathcal{I}_{g,1}$ has a generating set with $k + 2g + 1$ elements. Since there are $\binom{g}{3}$ subsurfaces R_{ijk}, Theorem A follows from Theorem B.

Remark. To illustrate the relative sizes of our generating sets, Johnson's generating set for $\mathcal{I}_{20}$ contains more than one trillion elements while our generating set for $\mathcal{I}_{20}$ has 64980 elements.

New proof of Johnson's theorem. Our deduction of Theorem A from Theorem B depends on Johnson's theorem that $\mathcal{I}_3$ is finitely generated. However, Hain [6] has recently announced a direct conceptual proof that $\mathcal{I}_3$ is finitely generated. Hain's proof uses special properties of the moduli space of genus 3 Riemann surfaces and cannot be easily generalized to $g > 3$. Combining this with our paper, we obtain a new proof that $\mathcal{I}_{g,n}$ is finitely generated for $g \geq 3$ and $n \leq 1$.

Our new proof is more conceptual than Johnson's original one. To illustrate this, we will sketch Johnson's proof. He starts by writing down an enormous finite subset $S \subset \mathcal{I}_{g,n}$ which is known (from work of Powell [15]) to normally generate $\mathcal{I}_{g,n}$ as a subgroup of $\text{Mod}_{g,n}$. Letting T be a standard generating set for $\text{Mod}_{g,n}$, Johnson then proves via a laborious computation that for $t \in T$ and $s \in S$, the element $tst^{-1} \in \mathcal{I}_{g,n}$ can be written as a word in S. This implies that the subgroup Γ of $\mathcal{I}_{g,n}$ generated by S is a normal subgroup of $\text{Mod}_{g,n}$, and thus that $\Gamma = \mathcal{I}_{g,n}$.

Remark. Our proof of Theorem B appeals to a theorem of [17] whose proof depends on Johnson's theorem. However, Hatcher and Margalit [7] have recently given a new proof of this result that is independent of Johnson's work.

Nature of generators. Some basic elements of $\mathcal{I}_{g,n}$ are as follows (see, e.g., [16]). If x is a simple closed curve on $\Sigma_{g,n}$, then denote by $T_x \in \text{Mod}_{g,n}$ the Dehn twist about x. If x is a separating simple closed curve, then $T_x \in \mathcal{I}_{g,n}$; these are called *separating twists.* If x and y are disjoint homologous nonseparating simple closed curves, then $T_x T_y^{-1} \in \mathcal{I}_{g,n}$; these are called *bounding pair maps.* Following work of Birman [1], Powell [15] proved that $\mathcal{I}_{g,n}$ is generated by bounding pair maps and separating twists for $g \geq 1$ and $n \leq 1$ (see [16] and [7] for alternate proofs). Johnson's finite generating set for $\mathcal{I}_{g,n}$ for $g \geq 3$ and $n \leq 1$ consists entirely of bounding pair maps. It follows easily from our proofs of Lemma 2.1 and 2.2 that our generating set consists of bounding pair maps and separating twists; see the remark after Lemma 2.2.

The handle graph. Our proof of Theorem B is topological. To prove that a group G is finitely generated, it is enough to find a connected simplicial complex upon which G acts cocompactly with finitely generated stabilizers. We use a variant on the curve complex. If γ is an oriented simple closed curve on Σ_g, then denote by $[\gamma] \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ its homology class. Also, if γ_1 and γ_2 are isotopy classes of simple closed curves on Σ_g, then denote by $i_g(\gamma_1,\gamma_2)$ their *geometric intersection number*, i.e. the minimal possible number of intersections between two curves in the isotopy classes of γ_1 and γ_2. Finally, denote by $i_a(\cdot,\cdot)$ the algebraic intersection pairing on $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$.

Definition. Let $a, b \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ satisfy $i_a(a,b) = 1$. The *handle graph* associated to a and b, denoted $\mathcal{H}_{a,b}$, is the graph whose vertices are isotopy classes of oriented simple closed curves on Σ_g that are homologous to either a or b and where two vertices γ_1 and γ_2 are joined by an edge exactly when $i_g(\gamma_1,\gamma_2) = 1$.

We will show that $\mathcal{H}_{a,b}/\mathcal{I}_g$ consists of a single edge (see Lemma 5.2) and that $\mathcal{H}_{a,b}$ is connected for $g \geq 3$ (see Lemma 3.1).

A complication. It would appear that we have all the ingredients in place to use the space $\mathcal{H}_{a,b}$ to prove that $\mathcal{I}_g$ is finitely generated. However, there is one remaining complication. Namely, we do not know the answer to the following question.

Question 1.1. *For some $g \geq 4$, let γ be the isotopy class of a nonseparating simple closed curve on Σ_g. Is the stabilizer subgroup $(\mathcal{I}_g)_\gamma$ of γ finitely generated?*

In other words, we do not know if the vertex stabilizer subgroups of the action of $\mathcal{I}_g$ on $\mathcal{H}_{a,b}$ are finitely generated. Nonetheless, in §4 we will prove a weaker statement that suffices to prove Theorem B. The proof of Theorem B is in §5.

Smaller generating sets. A positive answer to Question 1.1 would likely lead to a smaller generating set for $\mathcal{I}_g$, though of course this depends on the nature of the finite generating sets for the stabilizer subgroups. Let us describe one way this could work. For $g \geq 3$, let σ_g be the smallest cardinality of a generating set for $\mathcal{I}_g$. Consider $g \geq 4$, and fix an edge $\{\alpha, \beta\}$ of $\mathcal{H}_{a,b}$. The proof of Theorem B shows that $\mathcal{I}_g$ is generated by $(\mathcal{I}_g)_\alpha \cup (\mathcal{I}_g)_\beta$. Let S be a subsurface of Σ_g such that $S \cong \Sigma_{g-1,1}$ and $\alpha \cup \beta \subset \Sigma_g \setminus S$. We have $\mathcal{I}(\Sigma_g, S) \cong \mathcal{I}_{g-1,1}$ (see §2) and $\mathcal{I}(\Sigma_g, S) \subset (\mathcal{I}_g)_\alpha$ and $\mathcal{I}(\Sigma_g, S) \subset (\mathcal{I}_g)_\beta$. Assume that there exists a finite set V_α (resp. V_β) such that $(\mathcal{I}_g)_\alpha$ (resp. $(\mathcal{I}_g)_\beta$) is generated by $\mathcal{I}(\Sigma_g, S) \cup V_\alpha$ (resp. $\mathcal{I}(\Sigma_g, S) \cup V_\beta$). The group $\mathcal{I}_g$ is then generated by $\mathcal{I}(\Sigma_g, S) \cup V_\alpha \cup V_\beta$. Lemma 2.1 says that $\mathcal{I}(\Sigma_g, S) \cong \mathcal{I}_{g-1,1}$ can be generated by $\sigma_{g-1} + 2g + 1$ elements. Moreover, it seems likely that there exists some relatively small K such that $|V_\alpha|, |V_\beta| \leq Kg^2$. This would imply that

$$\sigma_g \leq \sigma_{g-1} + 2g + 1 + 2Kg^2.$$

Iterating this, we would get that

$$\sigma_g \leq \sigma_3 + \sum_{i=4}^{g} (2i + 1 + 2Ki^2)$$

for $g \geq 4$. This bound is cubic in g (as it needs to be), but as long as K is not too large it is much smaller than $57\binom{g}{3}$.

Finite presentability. Perhaps the most important open question about the combinatorial group theory of $\mathcal{I}_g$ is whether or not it is finitely presentable for $g \geq 3$. One way of proving that a group G is finitely presentable is to construct a simply-connected simplicial complex X upon which G acts cocompactly with finitely presentable stabilizer subgroups (see, e.g., [3]). For example, Hatcher and Thurston use this technique in [8] to prove that the mapping class group is finitely presentable.

The handle graph $\mathcal{H}_{a,b}$ appears to be the first example of a useful space upon which $\mathcal{I}_g$ acts cocompactly (of course, there are trivial non-useful examples of such spaces; for example, the Cayley graph of $\mathcal{I}_g$ or a 1-point space). Unfortunately, while $\mathcal{H}_{a,b}$ is connected for $g \geq 3$, it is not simply connected. Indeed, it does not even have any 2-cells (and is not a tree). However, one could probably attach 2-cells to $\mathcal{H}_{a,b}$ to obtain a simply connected complex upon which $\mathcal{I}_g$ acts cocompactly. This would not be enough, however – one would also have to prove that the simplex stabilizer subgroups were finitely presentable. In other words, this complex would provide the inductive step in a proof that $\mathcal{I}_g$ was finitely presentable, but one would still need a base case.

A complex that does not work. We close this introduction by discussing an approach to Theorem B that does not work. One might think of trying to prove Theorem B using the following complex. Let $a \in \mathrm{H}_1(\Sigma_g; \mathbb{Z})$ be a primitive vector. Define $\mathcal{C}_a$ to be the graph whose vertices are isotopy classes of oriented simple closed curves γ on Σ_g such that $[\gamma] = a$ and where two vertices γ and γ' are joined by an edge if $i_g(\gamma, \gamma') = 0$. It is known ([17, Theorem 1.9]; see [7] for an alternate proof) that $\mathcal{C}_a$ is connected for $g \geq 3$. Moreover, $\mathcal{I}_g$ acts transitively on the vertices of $\mathcal{C}_a$. However, it does *not* act cocompactly; indeed, there are infinitely many edge orbits. To see this, consider edges $e_1 = \{\gamma_1, \gamma_1'\}$ and $e_2 = \{\gamma_2, \gamma_2'\}$ of $\mathcal{C}_a$. Assume that there exists some $f \in \mathcal{I}_g$ such that $f(e_1) = e_2$. Since γ_1 is homologous to γ_1', the multicurve $\gamma_1 \cup \gamma_1'$ divides Σ_g into two subsurfaces S_1 and S_1'. Similarly, $\gamma_2 \cup \gamma_2'$ divides Σ_g into two subsurfaces S_2 and S_2'. Relabeling if necessary, we have $f(S_1)$ isotopic to S_2 and $f(S_1')$ isotopic to S_2'. Since $f \in \mathcal{I}_g$, the images of $\mathrm{H}_1(S_1; \mathbb{Z})$ and $\mathrm{H}_1(S_2; \mathbb{Z})$ in $\mathrm{H}_1(\Sigma_g; \mathbb{Z})$ must be the same, and similarly for $\mathrm{H}_1(S_1'; \mathbb{Z})$ and $\mathrm{H}_1(S_2'; \mathbb{Z})$. It is easy to see that infinitely many such images occur for different edges of $\mathcal{C}_a$, so there must be infinitely many edges orbits. We remark that Johnson proved in [9, Corollary to Lemma 9 on p. 250] that the images of $\mathrm{H}_1(S_1;\mathbb{Z})$ and $\mathrm{H}_1(S_1';\mathbb{Z})$ in $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$ are a complete invariant for the edge orbits.

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2 The Torelli group on subsurfaces

We will need to understand how the Torelli group restricts to subsurfaces. For a general discussion of this, see [16]. In this section, we will extract from [16] results on two kinds of subsurfaces. In §2.1, we will show how to analyze subsurfaces like the subsurfaces R_{ijk} from §1. In §2.2, we will show how to analyze stabilizers of nonseparating simple closed curves (which are supported on the subsurface obtained by taking the complement of a regular neighborhood of the curve).

2.1 Analyzing the subsurfaces R_{ijk}

We begin by defining groups $\mathcal{I}_{g,n}$ for $n \geq 2$. There is a map $\mathrm{Mod}_{g,n} \to \mathrm{Mod}_g$ induced by gluing discs to the boundary components of $\Sigma_{g,n}$ and extending homeomorphisms by the identity. Define $\mathcal{I}_{g,n}$ to be the kernel of the resulting action of $\mathrm{Mod}_{g,n}$ on $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$. For the case $n = 1$, the map $\mathrm{H}_1(\Sigma_{g,1};\mathbb{Z}) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ is an isomorphism, so this agrees with our previous definition of $\mathcal{I}_{g,1}$.

Remark. In [16], the different definitions of the Torelli group on a surface with boundary are parametrized by partitions of the boundary components. The above definition of $\mathcal{I}_{g,n}$ corresponds to the discrete partition $\{\{\beta_1\},\ldots,\{\beta_n\}\}$ of the set $\{\beta_1,\ldots,\beta_n\}$ of boundary components of $\Sigma_{g,n}$.

In [16, Theorem 1.2], a version of the Birman exact sequence is proven for the Torelli group. For $\mathcal{I}_{g,n}$ with $g \geq 2$, it takes the form

$$1 \longrightarrow \pi_1(U\Sigma_{g,n}) \longrightarrow \mathcal{I}_{g,n+1} \longrightarrow \mathcal{I}_{g,n} \longrightarrow 1. \tag{1}$$

Here $U\Sigma_{g,n}$ is the unit tangent bundle of $\Sigma_{g,n}$. The subgroup $\pi_1(U\Sigma_{g,n})$ of $\mathcal{I}_{g,n+1}$ is often called the "disc-pushing subgroup" – the mapping class associated to $\gamma \in \pi_1(U\Sigma_{g,n})$ "pushes" a fixed boundary component around γ while allowing it to rotate. The following is an immediate consequence of (1) and the fact that $\pi_1(U\Sigma_g)$ can be generated by $2g+1$ elements.

Lemma 2.1. *$\mathcal{I}_{g,1}$ can be generated by $k+2g+1$ elements if $\mathcal{I}_g$ can be generated by k elements.*

Now assume that $S \cong \Sigma_{h,n}$ is an embedded subsurface of Σ_g and that all the boundary components of S are non-nullhomotopic separating curves in Σ_g. For example, S could be one of the surfaces R_{ijk} from §1. Letting $\mathrm{Mod}(S)$ be the mapping class group of S, the induced map $\mathrm{Mod}(S) \to \mathrm{Mod}_g$ is an injection. This gives a natural identification of $\mathrm{Mod}(S)$ with $\mathrm{Mod}(\Sigma_g, S)$. The group $\mathcal{I}(\Sigma_g, S)$ is thus naturally a subgroup of $\mathrm{Mod}(S) \cong \mathrm{Mod}_{h,n}$, and in [16, Theorem 1.1] it is proven that $\mathcal{I}(\Sigma_g, S) = \mathcal{I}_{h,n}$. Johnson [11] proved that $\mathcal{I}_3$ can be generated by 35 elements. Applying (1) repeatedly, we see that $\mathcal{I}_{3,1}$ can be generated by 42 elements, $\mathcal{I}_{3,2}$ by 49 elements, and $\mathcal{I}_{3,3}$ by 57 elements. Since $R_{ijk} \cong \Sigma_{3,k}$ with $k \leq 3$, we obtain the following.

Lemma 2.2. *For all $1 \leq i < j < k \leq g$, the group $\mathcal{I}(\Sigma_g, R_{ijk})$ can be generated by 57 elements.*

Remark. It is well-known (see, e.g., [16, §2.1]) that the mapping classes corresponding to the generators of $\pi_1(U\Sigma_{g,n})$ used to prove Lemmas 2.1 and 2.2 can be chosen to be bounding pair maps and separating twists. Additionally, Johnson's minimal-size generating set for $\mathcal{I}_3$ consists entirely of bounding pair maps, so the generating set for $\mathcal{I}(\Sigma_g, R_{ijk})$ in Lemma 2.2 can be taken to consist of bounding pair maps and separating twists.

2.2 Stabilizers of nonseparating simple closed curves

Let γ be a nonseparating simple closed curve on Σ_g. Define $\Sigma_{g,\gamma}$ to be the result of cutting Σ_g along γ, so $\Sigma_{g,\gamma} \cong \Sigma_{g-1,2}$. Letting $\mathrm{Mod}_{g,\gamma}$ be the mapping class group of $\Sigma_{g,\gamma}$, the natural map $\Sigma_{g,\gamma} \to \Sigma_g$ induces a map $i : \mathrm{Mod}_{g,\gamma} \to \mathrm{Mod}_g$. Define $\mathcal{I}_{g,\gamma} = i^{-1}(\mathcal{I}_g)$. The map i restricts to a surjection $\mathcal{I}_{g,\gamma} \to (\mathcal{I}_g)_\gamma$, where $(\mathcal{I}_g)_\gamma$ is the stabilizer subgroup of γ.

Remark. In the notation of [16], the group $\mathcal{I}_{g,\gamma}$ corresponds to the Torelli group of $\Sigma_{g-1,2}$ with respect to the "indiscrete partition" $\{\{\beta, \beta'\}\}$ of the boundary components β and β' of $\Sigma_{g,\gamma}$. Also, the kernel of the map $\mathcal{I}_{g,\gamma} \to (\mathcal{I}_g)_\gamma$ is isomorphic to $\mathbb{Z}$ and is generated by $T_\beta T_{\beta'}^{-1}$, where T_β and $T_{\beta'}$ are the Dehn twists about β and β', respectively.

In [16, Theorem 1.2], it is proven that for $g \geq 2$ there is a short exact sequence

$$1 \longrightarrow K_{g,\gamma} \longrightarrow \mathcal{I}_{g,\gamma} \longrightarrow \mathcal{I}_{g-1,1} \longrightarrow 1. \tag{2}$$

Here $K_{g,\gamma} \cong [\pi_1(\Sigma_{g-1,1}), \pi_1(\Sigma_{g-1,1})]$. This exact sequence splits via the inclusion $\mathcal{I}_{g-1,1} \hookrightarrow \mathcal{I}_{g,\gamma}$ induced by the inclusion $\Sigma_{g-1,1} \hookrightarrow \Sigma_{g,\gamma}$ indicated in Figure 2.a. In other words, the following holds.

Lemma 2.3. *$\mathcal{I}_{g,\gamma} = K_{g,\gamma} \ltimes \mathcal{I}_{g-1,1}$ for $g \geq 3$ and γ a simple closed nonseparating curve on Σ_g.*

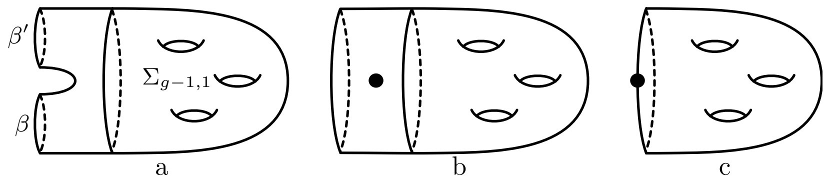

Figure 2: a. The surface $\Sigma_{g,\gamma}$ and and the subsurface $\Sigma_{g-1,1}$ of $\Sigma_{g,\gamma}$ such that the induced map $\mathcal{I}_{g-1,1} \to \mathcal{I}_{g,\gamma}$ splits the exact sequence (2). b. The basepoint for $\pi_1(\Sigma_{g-1,1})$ is obtained from $\Sigma_{g,\gamma}$ by collapsing the boundary component β to a point. c. The surface in b deformation retracts to $\Sigma_{g-1,1}$ such that the basepoint ends up on the boundary component.

The group $\mathcal{I}_{g-1,1}$ acts on $K_{g,\gamma} < \pi_1(\Sigma_{g-1,1})$ as follows. As is clear from [16, Theorem 1.2], the basepoint for $\pi_1(\Sigma_{g-1,1})$ is as indicated in Figure 2.b. As shown in Figure 2.c, the surface $\Sigma_{g-1,1}$ deformation retracts onto the surface $\Sigma_{g-1,1}$ on which $\mathcal{I}_{g-1,1}$ is supported. After this deformation retract, the basepoint ends up on $\partial\Sigma_{g-1,1}$. Summing up, $\mathcal{I}_{g-1,1}$ acts on $K_{g,\gamma} < \pi_1(\Sigma_{g-1,1})$ via the action of $\mathrm{Mod}_{g-1,1}$ on $\pi_1(\Sigma_{g-1,1})$, where the basepoint for $\pi_1(\Sigma_{g-1,1})$ is on $\partial\Sigma_{g-1,1}$.

3 The handle graph is connected

In this section, we prove the following.

Lemma 3.1. *Fix $g \geq 3$. Let $a, b \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ satisfy $i_a(a,b) = 1$. Then $\mathcal{H}_{a,b}$ is connected.*

We will need two lemmas. In the first, if ϵ is an oriented arc in a surface, then ϵ^{-1} denotes the arc obtained by reversing the orientation of ϵ.

Lemma 3.2. *Let the boundary components of $\Sigma_{g,2}$ be δ_0 and δ_1. Choose points $v_i \in \delta_i$ for $i = 0, 1$ and let ϵ be an oriented properly embedded arc in $\Sigma_{g,2}$ whose initial point is v_0 and whose terminal point is v_1. Then for any $h \in \mathrm{H}_1(\Sigma_{g,2};\mathbb{Z})$, there exists an oriented properly embedded arc ϵ' in $\Sigma_{g,2}$ whose initial point is v_0 and whose terminal point is v_1 such that the homology class of the loop $\epsilon' \cdot \epsilon^{-1}$ is h.*

Proof. Gluing (δ_0, v_0) to (δ_1, v_1), we obtain a surface $S \cong \Sigma_{g+1}$. Let α and $*$ be the images of δ_0 and v_0 in S, respectively. The image of ϵ in S is an oriented simple closed curve β with $i_g(\alpha, \beta) = 1$. There is a natural isomorphism $\mathrm{H}_1(\Sigma_{g,2};\mathbb{Z}) \cong [\alpha]^\perp$, where the orthogonal complement is taken with respect to $i_a(\cdot,\cdot)$. Under this identification, we can apply [16, Lemma A.3] to find an oriented simple closed curve β' on S such that $[\beta'] = [\beta] + h$ and such that $\alpha \cap \beta' = \{*\}$. Cutting S open along α, the curve β' becomes the desired arc ϵ'. $\square$

Lemma 3.3. *Let $a, b \in \mathrm{H}_1(\Sigma_g; \mathbb{Z})$ satisfy $i_a(a, b) = 1$. Let α_1 and α_2 be disjoint oriented simple closed curves on Σ_g such that $[\alpha_i] = a$ for $i = 1, 2$. There then exists some oriented simple closed curve β on Σ_g such that $[\beta] = b$ and $i_g(\alpha_i, \beta) = 1$ for $i = 1, 2$.*

Proof. Let β' be any simple closed curve on Σ_g such that $i(\alpha_i, \beta') = 1$ for $i = 1, 2$. Orient β' so that its intersections with α_1 and α_2 are positive. Let X_1 and X_2 be the two subsurfaces of Σ_g that result from cutting Σ_g along $\alpha_1 \cup \alpha_2$. For $i = 1, 2$, the surface X_i has 2 boundary components and the intersection of β' with X_i is an oriented properly embedded arc ϵ_i running between these boundary components. Also, the induced map $\mathrm{H}_1(X_i; \mathbb{Z}) \to \mathrm{H}_1(\Sigma_g; \mathbb{Z})$ is an injection, and we will identify $\mathrm{H}_1(X_i; \mathbb{Z})$ with its image in $\mathrm{H}_1(\Sigma_g; \mathbb{Z})$. The orthogonal complement to a with respect to the algebraic intersection pairing is spanned by $\mathrm{H}_1(X_1; \mathbb{Z}) \cup \mathrm{H}_1(X_2; \mathbb{Z})$. Since $i_a(a, b) = i_a(a, [\beta'])$, the homology class $b - [\beta']$ is orthogonal to a. There thus exist $h_i \in \mathrm{H}_1(X_i; \mathbb{Z})$ for $i = 1, 2$ such that $b = [\beta'] + h_1 + h_2$. Lemma 3.2 says that for $i = 1, 2$ there exists an oriented properly embedded arc ϵ_i' in X_i with the same endpoints as ϵ_i such that the homology class of the loop $\epsilon_i' \cdot \epsilon_i^{-1}$ equals h_i. Letting β be the loop $\epsilon_1' \cdot \epsilon_2'$, it follows that $[\beta] = [\beta'] + h_1 + h_2 = b$, as desired. □

Proof of Lemma 3.1. Let δ and δ' be vertices of $\mathcal{H}_{a,b}$. We will construct a path in $\mathcal{H}_{a,b}$ from δ to δ'. Without loss of generality, $[\delta] = [\delta'] = a$. By [17, Theorem 1.9] (see [7] for an alternate proof), we can find a sequence

$$\delta = \alpha_1, \alpha_2, \ldots, \alpha_n = \delta'$$

of isotopy classes of oriented simple closed curves on Σ_g such that $[\alpha_i] = a$ for $1 \leq i \leq n$ and $i_g(\alpha_i, \alpha_{i+1}) = 0$ for $1 \leq i < n$ (this is where we use the condition $g \geq 3$). Lemma 3.3 implies that there exist isotopy classes $\beta_1, \ldots, \beta_{n-1}$ of oriented simple closed curves on Σ_g such that $[\beta_i] = b$ and $i_g(\alpha_i, \beta_i) = i_g(\alpha_{i+1}, \beta_i) = 1$ for $1 \leq i < n$. Since β_i is adjacent to both α_i and α_{i+1} in $\mathcal{H}_{a,b}$, the desired path from δ to δ' is thus

$$\delta = \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \beta_{n-1}, \alpha_n = \delta'.$$ □

4 Generating the stabilizer of a nonseparating simple closed curve

Let the subsurfaces R_i' of Σ_g be as in the introduction. Define $S_i = \overline{\Sigma_g \setminus R_i'}$. The goal of this section is to prove the following lemma.

Lemma 4.1. *Assume that $g \geq 4$. Let γ be the isotopy class of a simple closed nonseparating curve on Σ_g that is contained in R_1'. Then the subgroup $(\mathcal{I}_g)_\gamma$ of $\mathcal{I}_g$ stabilizing γ is contained in the subgroup of $\mathcal{I}_g$ generated by $\cup_{i=1}^{g} \mathcal{I}(\Sigma_g, S_i)$.*

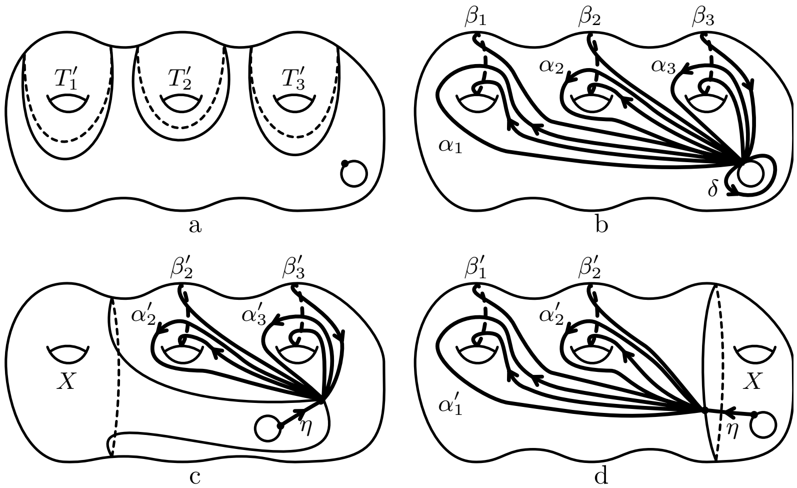

Figure 3: a. The subsurfaces T_i' b. The standard basis for π c. The surface X when $i = 1$ d. The surface X when $i = g$

Before proving this, we need a technical lemma. Set $\pi = \pi_1(\Sigma_{g,1}, *)$, where $* \in \partial\Sigma_{g,1}$. Let $T_1', \ldots, T_g'$ be disjoint subsurfaces of $\Sigma_{g,1}$ such that $T_i' \cong \Sigma_{1,1}$ and $T_i' \cap \partial\Sigma_{g,1} = \emptyset$ for $1 \leq i \leq g$ (see Figure 3.a). Define $T_i = \overline{\Sigma_{g,1} \setminus T_i'}$. We have $T_i \cong \Sigma_{g-1,2}$ and $* \in T_i$ for $1 \leq i \leq g$. The maps $\pi_1(T_i, *) \to \pi_1(\Sigma_{g,1}, *)$ and $\mathrm{H}_1(T_i'; \mathbb{Z}) \to \mathrm{H}_1(\Sigma_{g,1}; \mathbb{Z})$ are injective; we will identify $\pi_1(T_i, *)$ and $\mathrm{H}_1(T_i'; \mathbb{Z})$ with their images in $\pi_1(\Sigma_{g,1}, *)$ and $\mathrm{H}_1(\Sigma_g; \mathbb{Z})$, respectively. Define $K_i = [\pi, \pi] \cap \pi_1(T_i, *)$. We then have the following.

Lemma 4.2. *For $g \geq 3$, the group $[\pi, \pi]$ is generated by the $\mathcal{I}_{g,1}$-orbits of the set $\cup_{i=1}^g K_i$.*

The proof of this will have two ingredients. The first is the following theorem of Tomaszewski. As notation, if G is a group and $a, b \in G$, then $[a, b] := a^{-1}b^{-1}ab$ and $a^b := b^{-1}ab$.

Theorem 4.3 (Tomaszewski, [19])**.** *Let F_n be the free group on $\{x_1, \ldots, x_n\}$. Then the set*

$$\{[x_i, x_j]^{x_i^{k_i} x_{i+1}^{k_{i+1}} \cdots x_n^{k_n}} \mid 1 \leq i < j \leq n \text{ and } k_m \in \mathbb{Z} \text{ for all } i \leq m \leq n\}$$

is a free basis for $[F_n, F_n]$.

The second is the following lemma about the action of $\mathcal{I}_{g,1}$ on π. Choose a standard basis $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ for π (as in Figure 3.b) such that α_i and β_i are freely homotopic into T_i' for $1 \leq i \leq g$. Our proof of Lemma 4.2 would be much simpler if the image of $\mathrm{Mod}_{g,1}$ in $\mathrm{Aut}(\pi)$ contained the inner automorphisms – since inner automorphisms act trivially on homology, this would imply that the

$\mathcal{I}_g$-orbits of $\{[x,y] \mid x,y \in \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}\}$ generate $[\pi,\pi]$. However, the image of $\mathrm{Mod}_{g,1}$ in $\mathrm{Aut}(\pi)$ does not contain the inner automorphisms since $\mathrm{Mod}_{g,1}$ fixes the loop $\delta = [\alpha_1,\beta_1]\cdots[\alpha_g,\beta_g]$ depicted in Figure 3.b. The following lemma is a weak replacement for this.

Lemma 4.4. *Let i be either 1 or g. Consider $h \in \mathrm{H}_1(T_i';\mathbb{Z})$. There then exists some $w \in \langle \alpha_i, \beta_i, \delta\rangle$ and $f \in \mathcal{I}_{g,1}$ such that $[w] = h$ and such that $f(a_j) = a_j^w$ and $f(b_j) = b_j^w$ for $1 \le j \le g$ with $j \neq i$.*

Proof. Let X be a regular neighborhood of the curves $\alpha_i \cup \beta_i \cup \partial\Sigma_{g,1}$ depicted in Figure 3.b. Thus $X \cong \Sigma_{1,2}$, the surface T_i' is homotopic into X, and the image of $\pi_1(X,*)$ in π is $\langle \alpha_i, \beta_i, \delta\rangle$. Let $Y = \overline{\Sigma_{g,1} \setminus X}$, so $Y \cong \Sigma_{g-1,1}$ and $X \cap Y \cong S^1$. The key property of X is as follows (this is where we use the assumption that i is either 1 or g). There exists some $*' \in X \cap Y$, a properly embedded arc η in X from $*$ to $*'$, and elements

$$\{\alpha_j', \beta_j' \mid 1 \le j \le g,\, j \neq i\} \subset \pi_1(Y, *')$$

such that $\alpha_j = \eta \cdot \alpha_j' \cdot \eta^{-1}$ and $\beta_j = \eta \cdot \beta_j' \cdot \eta^{-1}$ for $1 \le j \le g$ with $j \neq i$. See Figure 3.c for the case $i = 1$ and Figure 3.d for the case $i = g$.

By Lemma 3.2, there exists an oriented properly embedded arc η' in X whose endpoints are the same as those of η such that the homology class of $w := \eta \cdot (\eta')^{-1} \in \pi$ in $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$ is h. Observe that $w \in \langle \alpha_i, \beta_i, \delta\rangle$. Also,

$$\eta' \cdot \alpha_j' \cdot (\eta')^{-1} = w^{-1} \cdot \eta \cdot \alpha_j' \cdot \eta^{-1} \cdot w = \alpha_j^w$$

for $j \neq i$, and similarly for β_j. It is thus enough find some $f \in \mathcal{I}(\Sigma_g, X)$ such that $f(\eta) = \eta'$.

The "change of coordinates principle" from [5, §1.3] implies that there exists some $f' \in \mathrm{Mod}(\Sigma_g, X)$ such that $f'(\eta) = \eta'$. Briefly, an Euler characteristic calculation shows that cutting X open along either η or η' results in a surface homeomorphic to $\Sigma_{1,1}$. Choosing an orientation-preserving homeomorphism between these two cut-open surfaces and gluing the boundary components back together in an appropriate way, we obtain some $f' \in \mathrm{Mod}(\Sigma_g, X)$ such that $f'(\eta) = \eta'$. See [5, §1.3] for more details and many other examples of arguments of this form.

The mapping class f' need not lie in Torelli; however, it satisfies $f'([\alpha_j]) = [\alpha_j]$ and $f'([\beta_j]) = [\beta_j]$ for $j \neq i$ and $f'(\mathrm{H}_1(T_i';\mathbb{Z})) = \mathrm{H}_1(T_i';\mathbb{Z})$. Since the image of $\mathrm{Mod}(T_i')$ in $\mathrm{Aut}(\mathrm{H}_1(T_i';\mathbb{Z})) = \mathrm{Aut}(\mathbb{Z}^2)$ is $\mathrm{SL}_2(\mathbb{Z})$, we can choose some $f'' \in \mathrm{Mod}(\Sigma_g, T_i')$ such that $f'([\alpha_i]) = f''([\alpha_i])$ and $f'([\beta_i]) = f''([\beta_i])$. It follows that $f := f' \cdot (f'')^{-1}$ lies in $\mathcal{I}(\Sigma_g, X)$ and satisfies $f(\eta) = \eta'$, as desired. □

Proof of Lemma 4.2. The generating set for $[F_n, F_n]$ in Theorem 4.3 depends on an ordering of the generators for F_n. It seems hard to prove the lemma using the generating set corresponding to the standard ordering

$$(x_1, x_2, \ldots, x_{2g}) = (\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$$

of the generators for $\pi \cong F_{2g}$. However, consider the following nonstandard ordering on the generators for π:

$$(x_1, x_2, \ldots, x_{2g}) = (\alpha_2, \beta_2, \alpha_1, \beta_1, \alpha_3, \beta_3, \alpha_4, \beta_4, \ldots, \alpha_g, \beta_g).$$

Let S be the generating set for $[\pi, \pi]$ given by Theorem 4.3 using this ordering of the generators. All the elements of S lie in K_2 except for

$$[\alpha_2, \zeta]^{\alpha_2^{n_2}\beta_2^{m_2}\alpha_1^{n_1}\beta_1^{m_1}\alpha_3^{n_3}\cdots\beta_g^{m_g}} \quad \text{and} \quad [\beta_2, \zeta']^{\beta_2^{m_2}\alpha_1^{n_1}\beta_1^{m_1}\alpha_3^{n_3}\cdots\beta_g^{m_g}}; \tag{3}$$

here $\zeta \in \{\beta_2, \alpha_1, \beta_1, \alpha_3, \ldots, \beta_g\}$ and $\zeta' \in \{\alpha_1, \beta_1, \alpha_3, \ldots, \beta_g\}$ and $n_i, m_i \in \mathbb{Z}$. Letting $T \subset S$ be the elements in (3), we must show that every $t \in T$ can be expressed as a product of elements in the $\mathcal{I}_{g,1}$-orbit of the set $\cup_{i=1}^{g} K_i$. Consider $t \in T$, so either $t = [\alpha_2, \zeta]^{\alpha_2^{n_2}\beta_2^{m_2}\alpha_1^{n_1}\beta_1^{m_1}\alpha_3^{n_3}\cdots\beta_g^{m_g}}$ or $t = [\beta_2, \zeta]^{\beta_2^{m_2}\alpha_1^{n_1}\beta_1^{m_1}\alpha_3^{n_3}\cdots\beta_g^{m_g}}$. There are two cases.

Case 1. $\zeta \notin \{\alpha_1, \beta_1\}$.

We will do the case where $t = [\alpha_2, \zeta]^{\alpha_2^{n_2}\beta_2^{m_2}\alpha_1^{n_1}\beta_1^{m_1}\alpha_3^{n_3}\cdots\beta_g^{m_g}}$; the other case is treated in a similar way. Set $t' = [\alpha_2, \zeta]^{\alpha_2^{n_2}\beta_2^{m_2}\alpha_3^{n_3}\cdots\beta_g^{m_g}}$, so $t' \in K_1$. By Lemma 4.4, there exists some $w \in \{\alpha_1, \beta_1, \delta\}$ and $f \in \mathcal{I}_{g,1}$ such that $[w] = [\alpha_1^{n_1}\beta_1^{m_1}]$ and such that $f(a_j) = a_j^w$ and $f(b_j) = b_j^w$ for $j > 1$. This implies that $f(t') = [\alpha_2, \zeta]^{\alpha_2^{n_2}\beta_2^{m_2}\alpha_3^{n_3}\cdots\beta_g^{m_g}w}$. Now, $\alpha_3^{n_3}\cdots\beta_g^{m_g}w$ and $\alpha_1^{n_1}\beta_1^{m_1}\alpha_3^{n_3}\cdots\beta_g^{m_g}$ are homologous, so there exists some $\theta \in [\pi, \pi]$ such that $\alpha_3^{n_3}\cdots\beta_g^{m_g}w\theta = \alpha_1^{n_1}\beta_1^{m_1}\alpha_3^{n_3}\cdots\beta_g^{m_g}$. Moreover, since $w \in \langle a_1, b_1, \delta\rangle$ we have $\theta \in K_2$. Observe now that

$$\theta^{-1} \cdot f(t') \cdot \theta = [\alpha_2, \zeta]^{\alpha_2^{n_2}\beta_2^{m_2}\alpha_3^{n_3}\cdots\beta_g^{m_g}w\theta} = [\alpha_2, \zeta]^{\alpha_2^{n_2}\beta_2^{m_2}\alpha_1^{n_1}\beta_1^{m_1}\alpha_3^{n_3}\cdots\beta_g^{m_g}} = t.$$

We have thus found the desired expression for t.

Case 2. $\zeta' \in \{\alpha_1, \beta_1\}$.

This case is similar to Case 1. The only difference is that the $\alpha_g^{n_g}\beta_g^{m_g}$ term of t is deleted to form t' instead of the $\alpha_1^{n_1}\beta_1^{m_1}$ term. □

Proof of Lemma 4.1. Let I be the subgroup of $\mathcal{I}_g$ generated by $\cup_{i=1}^{g}\mathcal{I}(\Sigma_g, S_i)$. Using the notation of §2, there is a surjection $\rho : \mathcal{I}_{g,\gamma} \to (\mathcal{I}_g)_\gamma$ induced by a continuous map $\phi : \Sigma_{g,\gamma} \to \Sigma_g$. Define $X = \phi^{-1}(S_1)$, so $X \cong \Sigma_{g-1,1}$. Letting $\mathcal{I}(X)$ be the Torelli group of X, Lemma 2.3 gives a decomposition $\mathcal{I}_{g,\gamma} = K_{g,\gamma} \ltimes \mathcal{I}(X)$. Clearly $\rho(\mathcal{I}(X)) = \mathcal{I}(\Sigma_g, S_1) \subset I$. Also, Lemma 4.2 implies that $K_{g,\gamma}$ is generated by the $\mathcal{I}(X)$-conjugates of a set $S \subset K_{g,\gamma}$ such that $\rho(S) \subset I$. We conclude that $\rho(\mathcal{I}_{g,\gamma}) \subset I$, as desired. □

5 Proof of main theorem

We finally prove our main theorem. The key is the following standard lemma, whose proof is similar to that given in [20, (1) of Appendix to §3] and is thus omitted.

Lemma 5.1. *Consider a group G acting without inversions on a connected graph X. Assume that X/G consists of a single edge $\overline{e}$. Let e be a lift of $\overline{e}$ to X and let v and v' be the endpoints of e. Then G is generated by $G_v \cup G_{v'}$.*

To apply this, we will need the following lemma.

Lemma 5.2. *Let $a, b \in \mathrm{H}_1(\Sigma_g; \mathbb{Z})$ satisfy $i_a(a, b) = 1$. Then $\mathcal{H}_{a,b}/\mathcal{I}_g$ is isomorphic to a graph with a single edge.*

The proof is similar to the proofs of [16, Lemma 6.2] and [18, Lemma 6.9], and is thus omitted.

Proof of Theorem B. Let $R_1', \ldots, R_g'$ and R_{ijk} be the subsurfaces of Σ_g from the introduction. Let Γ be the subgroup of $\mathcal{I}_g$ generated by $\bigcup_{1 \leq i<j<k \leq g} \mathcal{I}(\Sigma_g, R_{ijk})$. Our goal is to prove that $\Gamma = \mathcal{I}_g$.

The proof will be by induction on g. The base case $g = 3$ is trivial, so assume that $g \geq 4$ and that the theorem is true for all smaller g such that $g \geq 3$. Choose simple closed curves α and β in R_1' such that $i_g(\alpha, \beta) = 1$. Observe that R_1' is a closed regular neighborhood of $\alpha \cup \beta$. Set $a = [\alpha]$ and $b = [\beta]$. Clearly $\mathcal{I}_g$ acts on $\mathcal{H}_{a,b}$ without inversions. Lemmas 3.1 and 5.2 show that the action of $\mathcal{I}_g$ on $\mathcal{H}_{a,b}$ satisfies the other conditions of Lemma 5.1. We deduce that $\mathcal{I}_g$ is generated by the union $(\mathcal{I}_g)_\alpha \cup (\mathcal{I}_g)_\beta$ of the stabilizer subgroups of α and β.

Recall that $S_i = \overline{\Sigma_g \setminus R_i'}$ for $1 \leq i \leq g$. By Lemma 4.1, both $(\mathcal{I}_g)_\alpha$ and $(\mathcal{I}_g)_\beta$ are contained in the subgroup generated by $\cup_{i=1}^{g} \mathcal{I}(\Sigma_g, S_i)$. We must prove that $\mathcal{I}(\Sigma_g, S_i) \subset \Gamma$ for $1 \leq i \leq g$. We will do the case $i = g$; the other cases are similar. We have a Birman exact sequence

$$1 \longrightarrow \pi_1(U\Sigma_{g-1}) \longrightarrow \mathcal{I}(\Sigma_g, S_g) \longrightarrow \mathcal{I}_{g-1} \longrightarrow 1.$$

By induction, the subset $\bigcup_{1 \leq i<j<k \leq g-1} \mathcal{I}(\Sigma_g, R_{ijk})$ of $\mathcal{I}(\Sigma_g, S_g)$ projects to a generating set for $\mathcal{I}_{g-1}$. Also, it is clear that the disc-pushing subgroup $\pi_1(U\Sigma_{g-1})$ of $\mathcal{I}(\Sigma_g, S_g)$ is generated by elements that lie in $\bigcup_{1 \leq i<j<g} \mathcal{I}(\Sigma_g, R_{ijg})$. We conclude that $\mathcal{I}(\Sigma_g, S_g) \subset \Gamma$, as desired. $\square$

References

[1] J. S. Birman, On Siegel's modular group, Math. Ann. **191** (1971), 59–68.

[2] T. Brendle and B. Farb, personal communication.

[3] K. S. Brown, Presentations for groups acting on simply-connected complexes, J. Pure Appl. Algebra **32** (1984), no. 1, 1–10.

[4] B. Farb, Some problems on mapping class groups and moduli space, in *Problems on mapping class groups and related topics*, 11–55, Proc. Sympos. Pure Math., 74, Amer. Math. Soc., Providence, RI, 2006.

[5] B. Farb and D. Margalit, A Primer on Mapping Class Groups, to be published by Princeton University Press.

[6] R. Hain, Fundamental groups of branched coverings and the Torelli group in genus 3, in preparation.

[7] D. Margalit and A. Hatcher, Generating the Torelli group, in preparation.

[8] A. Hatcher and W. Thurston, A presentation for the mapping class group of a closed orientable surface, Topology **19** (1980), no. 3, 221–237.

[9] D. Johnson, Conjugacy relations in subgroups of the mapping class group and a group-theoretic description of the Rochlin invariant, Math. Ann. **249** (1980), no. 3, 243–263.

[10] D. Johnson, A survey of the Torelli group, in *Low-dimensional topology (San Francisco, Calif., 1981)*, 165–179, Contemp. Math., 20, Amer. Math. Soc., Providence, RI, 1983.

[11] D. Johnson, The structure of the Torelli group. I. A finite set of generators for $\mathcal{I}$, Ann. of Math. (2) **118** (1983), no. 3, 423–442.

[12] D. Johnson, The structure of the Torelli group. III. The abelianization of $\mathcal{T}$, Topology **24** (1985), no. 2, 127–144.

[13] D. McCullough and A. Miller, The genus 2 Torelli group is not finitely generated, Topology Appl. **22** (1986), no. 1, 43–49.

[14] G. Mess, The Torelli groups for genus 2 and 3 surfaces, Topology **31** (1992), no. 4, 775–790.

[15] J. Powell, Two theorems on the mapping class group of a surface, Proc. Amer. Math. Soc. **68** (1978), no. 3, 347–350.

[16] A. Putman, Cutting and pasting in the Torelli group, Geom. Topol. **11** (2007), 829–865.

[17] A. Putman, A note on the connectivity of certain complexes associated to surfaces, Enseign. Math. (2) **54** (2008), no. 3-4, 287–301.

[18] A. Putman, An infinite presentation of the Torelli group, Geom. Funct. Anal. **19** (2009), no. 2, 591–643.

[19] W. Tomaszewski, A basis of Bachmuth type in the commutator subgroup of a free group, Canad. Math. Bull. **46** (2003), no. 2, 299–303.

[20] J.-P. Serre, *Trees*, translated from the French original by John Stillwell, corrected 2nd printing of the 1980 English translation, Springer Monographs in Mathematics, Springer, Berlin, 2003.

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