

Work

Force If an object is moving in a straight line with position function $s(t)$, then the force F on the object at time t is the product of the mass of the object times its acceleration.

$$F = m \frac{d^2s}{dt^2}.$$

Note if velocity and acceleration have opposite signs, the force is acting in the direction opposite to the movement of the object. In the **SI metric system**, mass is measured in Kilograms (kg), displacement in meters (m) and the time is measured in seconds (s). The **force** is measured in **Newtons**; $N = kg \cdot m/s^2$. A force of one Newton acting on a mass of 1 kg produces an acceleration of $1 m/s^2$. In the **British system** the unit of force is a **pound (lb)**.

Work done by a constant force When a body moves a distance d along a straight line as a result of being acted on by a force of constant magnitude F , we define the work done by the force on the body with the formula; work = force \times distance,

$$W = Fd.$$

Note Forces applied in the direction of motion have positive values in this calculation and forces applied in a direction opposite to the direction of motion have negative values.

In the **SI system**, the unit of **work** is a Newton-meter ($N \cdot m$) or a **Joule J** . In the **British system** work is measured in **foot pounds (ft-lb)**. $1 \text{ ft-lb} \approx 1.36J$.

Example How much work is done in lifting a 3kg weight vertically upwards a distance of 0.5 meters? (recall that acceleration due to gravity is $9.8 m/s^2$.)

Note The work done by gravity in this example is $-14.7 J$.

Note Work done in the sense described above is not the same as our everyday notion of work. If I lift the weight as described above and then lower it, the work done by me on the way down is $-14.7 J$ and the sum total of the work I do is 0.

Example How much work is necessary to lift a 10 lb cat from the ground to a vet's examination table which is 3ft 6 in off the ground?

Variable Force

Here we consider an object moving along a straight line (the x axis) from the point a to the point b acted upon by a variable force $f(x)$. We can estimate the amount of work done with a Riemann sum.

Suppose a particle is moved along the x axis from $x = a$ to $x = b$ where the force acting on the particle at x is given by $f(x)$, a continuous function. We partition the interval $[a, b]$ into n equal subintervals $[x_{i-1}, x_i]$ each with length Δx . Since the function $f(x)$ is continuous, the variation in the force $f(x)$ on the subinterval $[x_{i-1}, x_i]$ should be small if Δx is small (we are of course free to make the value of Δx as small as we want.) Therefore on the interval $[x_{i-1}, x_i]$, the work done is approximately

$$W_i \approx f(x_i)\Delta x,$$

that done by a constant force of $f(x_i)$ on the subinterval. The work done over the course of the interval $[a, b]$ is the sum of the work done on each subinterval. Therefore our approximation for the work done over the interval $[a, b]$ is


$$W \approx \sum_{i=1}^n W_i \approx \sum_{i=1}^n f(x_i)\Delta x.$$

Now taking limits as n tends to infinity, we get

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x = \int_a^b f(x)dx.$$

Example A particle is moved along the x -axis by a force that measures $x^2 + 2$ lbs. at a distance x from the origin. Find the work done in moving the particle from the origin to a distance of 10 feet.

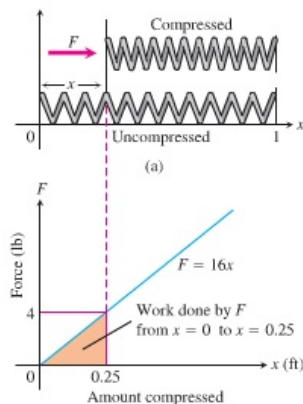
Springs

Hookes Law  states that the force required to maintain a stretched or compressed spring x units from its natural (unstressed) length is proportional to x ,

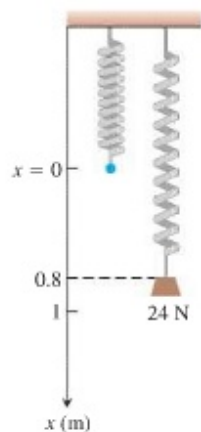
$$f(x) = kx.$$

where k is called the spring constant. (Not true if x is too large.)

Example Find the work required to compress a spring from its natural length of 1 ft. to a length of 0.75 ft. if the force constant is $k = 16$ (lb/ft).



Example A spring has a natural length of 1m. A force of 24 N holds the spring stretched to a total length of 1.8m.



- (a) Find the constant k .

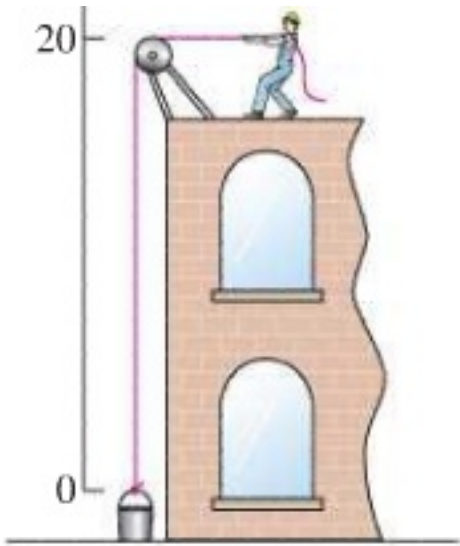
- (b) How much work would it take to stretch the spring 2m beyond its natural length?

- (c) How far will a 45 N force stretch the spring?

Lifting Objects With Varying weights

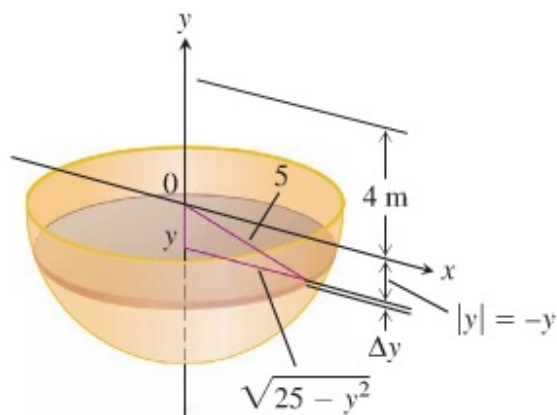
In the following problems we consider lifting objects with varying weights. For each problem, we strategically place the object to be lifted along an interval $[a, b]$ of the x axis. We then break that interval up into subintervals and estimate the work done on each subinterval by calculating the force for each subinterval and the distance for which that force will be applied (usually distance travelled). We then take limits as $\Delta x \rightarrow 0$ to get an integral.

Example A 20 ft. rope weighing 0.08 lbs per foot is used to haul a bucket weighing 5 lb. to the top of a building. How much work is done in lifting the rope and bucket to the top of the building ?



We use Riemann sums to solve this problem, by breaking the interval $[0, 20]$ into thin slices and adding the work done on each slice to approximate the total work done in lifting the rope and bucket. We then take the limit of the sum as the width of the slices tends to 0 to find the work done in lifting the rope and bucket.

Example How much work is necessary to empty a full hemispherical water reservoir of radius 5m by pumping the water to a height of 4m above the top of the reservoir. (The density of water is $1000\text{kg}/\text{m}^3$ and the force of gravity is approximately $9.80665\text{ m}/\text{s}^2$.)



Again we use limits of Riemann sums to find the appropriate integral for the work done in pumping the water from the reservoir. Using the vertical y -axis shown in the picture, we see that the water to be pumped lies along the interval $[-5, 0]$ of this axis. We divide this interval into thin slices and find an approximation for the work done on each slice.

We divide the interval $[-5, 0]$ into n subintervals $[y_{i-1}, y_i]$ of equal length $\Delta y = 5/n$. Let W_i denote the work done in pumping the water from the horizontal cross-section corresponding to the interval $[y_{i-1}, y_i]$.

$W_i = F_i \cdot D_i$ where F_i denotes the force exerted on the water in that cross section and D_i denotes the distance for which that force is applied.

$$F_i \approx 1000 \cdot 9.807 \cdot \text{volume of the slice} \approx 9807 \cdot \pi \left(\sqrt{25 - y_i^2} \right)^2 \cdot \Delta y = 9807 \cdot \pi (25 - y_i^2) \cdot \Delta y, \quad D_i \approx 4 - y_i.$$

Therefore

$$W_i \approx 9807\pi(25 - y_i^2)(4 - y_i)\Delta y$$

and the total amount of work done in pumping the water from the tank is

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=1}^n W_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n 9807\pi(25 - y_i^2)(4 - y_i)\Delta y = \int_{-5}^0 9807\pi(25 - y^2)(4 - y)dy \\ &= 9807\pi \int_{-5}^0 100 - 4y^2 - 25y - y^3 dy = 9807\pi \left[100y - \frac{4}{3}y^3 - \frac{25}{2}y^2 - \frac{1}{4}y^4 \right]_{-5}^0 \\ &= -9807\pi \left[-500 + \frac{4}{3}125 - \frac{25}{2}25 - \frac{1}{4}625 \right] = -9807\pi \left[-802.08 \right] = 7,865,998.56\pi \text{ J} \end{aligned}$$

Because we have approximated the force of gravity here, there is still an error, we could use more decimal places in our approximation for the force of gravity to get the desired level of accuracy in our answer.