Lecture 3 : Algebraic expressions, Polynomials

Algebra of Polynomials

A **variable** is a letter that can represent any number from a given set of numbers. If we start with the variables x, y and z (representing any real number) and some real numbers and combine them using addition, subtraction, multiplication, division, powers and roots, we obtain an **algebraic expression**.

Example here are some algebraic expressions:

$$2x^3 - 3\sqrt{x} + 1/2 \qquad \frac{x^2 - 3}{x^2 + 3} \qquad \frac{x + y + \sqrt{z}}{\sqrt{2}z - 1}$$

Definition A **Polynomial** in the variable x is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers and n is a positive integer. If $a_n \neq 0$, then we say the polynomial has **degree** n. (Note n is the highest power of x appearing in the expression. Each $a_k x^k$ is called a term of the polynomial and a_k is called the coefficient of x^k

To add polynomials, one should combine like terms and to multiply one should use the distributive law.

Example Calculate

$$(2x^3 + x + 5) + (x^5 + x^3 + 10) \qquad (x^2 - 1)(x + 1)$$

One should always keep in mind the following product formulas:

1. $(A + B)(A - B) = A^2 - B^2$
2. $(A + B)^2 = A^2 + 2AB + B^2$
3. $(A - B)^2 = A^2 - 2AB + B^2$

Example Calculate the following

$$(x^2 - 1)(x^2 + 1) \qquad (\sqrt{x + 1} - 1)^2 \qquad (\sqrt{x + 1} + 1)^2$$

Factoring

Sometimes in calculations, we want to simplify expressions by pulling out common factors:

Example Factor the following polynomials as much as you can:

$$x^4 - 1, \qquad x^2 - 9$$

Example Pull out the common factors in each expression:

$$4(x-3)x(x+2) + (x-5)(x-3) + (2x-6) \qquad\qquad x(x+2) + \sqrt{x}(x+2)$$

Often, we need to factor polynomials into smaller polynomials. We should keep in mind the basic identities 1, 2 and 3 above to help us recognize factors of this form.

Guessing The Factors
Sometimes we can guess the factors of a polynomial of degree 2 in the following way: To find the factors of

$$x^2 + bx + c$$

(if they exist) we must find factors of the form $(x+a_1)$ and $(x+a_2)$ with $(x+a_1)(x+a_2) = x^2+bx+c$ i.e. $x^2 + (a_1 + a_2)x + a_1a_2 = x^2 + bx + c$. Two polynomials are equal only if their corresponding coefficients are equal, so we must find two numbers a_1 and a_2 for which $a_1 + a_2 = b$ and $a_1a_2 = c$. (sometimes two such real numbers do not exist - see below).

Example Factor the following degree 2 polynomials:

$$x^2 + 6x + 9 \qquad x^2 - x - 6 \qquad\qquad x^2 - 2x - 8$$

Later you will be able to use the quadratic formula to factor degree 2 polynomials, when it is possible.

Later you will also see the fundamental theorem of Algebra which says that

Fact Every polynomial breaks down into factors of the form $ax + b$ or (irreducible) factors of the form $cx^2 + dx + e$ where $d^2 - 4ac < 0$.