**Lecture 5 : Solving Equations, Completing the Square, Quadratic Formula**

An equation is a mathematical statement that two mathematical expressions are equal. For example the statement

$$1 + 2 = 3$$

is read as "one plus two equals three" and means that the quantity on the left hand side is equal to the quantity on the right hand side When we have an equation of the form

$$4x + 1 = 3$$

where $x$ is a variable, there are a limited number of possibilities for the value of $x$, in fact in this case, there is just one possible value for $x$. A **solution** to this equation is a value of $x$ that fits the equation or a value of $x$ which makes the statement true.

**Tip for Success.** It is important for students to write mathematical statements in such complete sentences and students who do not develop such habits often have difficulty in solving complex problems later, simply because they cannot keep track of their own calculations.

**Definition** A **linear equation** in one variable is an equation equivalent to one of the form

$$ax + b = 0$$

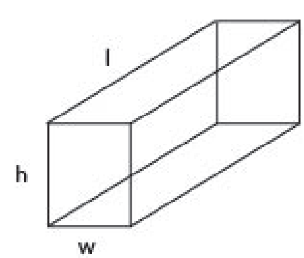
where $a$ and $b$ are real numbers and $x$ is the variable.

When we wish to solve for the feasible values of $x$ in such an equation, we bring all terms involving $x$ to one side of the equals sign and all other terms to the other side, and divide by the coefficient of $x$ to solve for $x$.

**Example** Solve for $x$ in the following linear equations:

$$4x + 1 = 3 \qquad\qquad 3x + 2 = x + 1$$

**Example** In related rates problems in Calculus I one frequently has to express a variable in terms of another variable. Here is an example:
Express the surface area of the box below in terms of its width (w), length (l) and height (h). Then solve for the width in terms of the other variables.



## Quadratic Equations

A **Quadratic Equation** is an equation of the form (or equivalent to)

$$ax^2 + bx + c = 0$$

where $a, b$ and $c$ are real numbers and $a \neq 0$.

The (real) solutions of a quadratic equation are the real numbers $x$ which satisfy the equation or make the statement true. There are three possible scenarios

1. There is exactly one real solution.
2. There are two distinct (not the same) real solutions.
3. There are no real solutions (in this case there are two solutions in the complex numbers).

There are a number of ways to approach finding the solutions to such an equation, all of which will be useful in calculus.

**Solving an equation by Factoring** The following basic property of real numbers is important for solving all equations and will be used frequently throughout calculus and higher mathematics courses:

**Zero-Product Property**

$$\boxed{AB = 0 \quad \text{if and only if} \quad A = 0 \ \text{ or } \ B = 0.}$$

**Note** The term "if and only if" is a frequently used logical statement and it means that if the statement on the left hand side is true then the statement on the right hand side is guaranteed to be true also and vice-versa (if the statement on the right is true, then the statement on the left is guaranteed to be true.)

We can use this and our factorization techniques to solve (some) quadratic equations.

**Example** Solve the equation $x^2 + 10x = 24$.

**Solving Simple Quadratic Equations** The solutions to the equation

$$x^2 = c, \quad \text{where} \quad c > 0$$

are $x = \sqrt{c}$ and $x = -\sqrt{c}$.

**Note** For a quadratic of the form $x^2 = c$ where $c < 0$, there are no solutions among the real numbers, because the square of any real number must be $\geq 0$.

**Example** Find the solutions to the following quadratic equations

$$x^2 = 9, \qquad (x-2)^2 = 16$$

**Completing The Square** This technique helps us to solve quadratic equations but is also very useful in its own right especially in graphing functions. It is important to master it before studying calculus.

- To make $x^2 + bx$ into a perfect square, we must add $\left(\frac{b}{2}\right)^2$ to it (i.e. we add the square of half of the coefficient of $x$ ). We get
$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

- To convert a quadratic equation $x^2 + bx + c = 0$ into the form $(x+d)^2 + e = 0$, we add $\left(\frac{b}{2}\right)^2$ to both sides of the equation and then bring all terms to the left. We get
$$x^2 + bx + \left(\frac{b}{2}\right)^2 + c = \left(\frac{b}{2}\right)^2 \quad \text{or} \quad \left(x + \frac{b}{2}\right)^2 + \left[c - \left(\frac{b}{2}\right)^2\right] = 0$$

**Example** Solve the following quadratic equations by completing the square:

$$x^2 - 6x + 12 = 0 \qquad 2x^2 - 8x - 20 = 0$$

**The Quadratic Formula** The above technique of completing the square allows us to derive a general formula for the solutions of a quadratic called the quadratic formula. Below we give both the formula and the proof.

**The Quadratic Formula** The roots (solutions) of the quadratic equation $ax^2+bx+c=0$ where $a \neq 0$ are

$$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}.$$

**Proof** First we divide each side of the equation by $a$ to get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

We then complete the square for $x^2 + \frac{b}{a}x$ by adding $\left(\frac{b}{2a}\right)^2$ to both sides:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = \left(\frac{b}{2a}\right)^2.$$

It follows that

$$\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} = \left(\frac{b}{2a}\right)^2.$$

Now subtracting $\frac{c}{a}$ from both sides, we get

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}.$$

Adding the fractions on the right, we get

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2-4ac}{4a^2}.$$

Now we have a simple quadratic equation

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

with solutions

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2-4ac}}{2a}.$$

Therefore our solutions are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

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**Tip For Success** If you intend to be a math major, it is a good idea to start reading proofs and following the logic involved. This is in fact a good idea for all students.

**Example** Use the quadratic formula to find the solutions of the following quadratic equations.

$$x^2+3x-1=0 \qquad 2x^2+10x+10=0 \qquad x^2+5x-4=0.$$

**The Dicriminant** Note that there are no real solutions to the quadratic above when $b^2 - 4ac < 0$ because we cannot take the square root of a negative number. Below we explore all possible scenarios.

The **discriminant** of a general quadratic equation $ax^2+bx+c=0$, where $a \neq 0$, is given by $D = b^2-4ac$. We have the following possibilities:

1. If $D > 0$, then the equation has two distinct real solutions.
2. If $D = 0$, the equation has exactly one real solution.
3. If $D < 0$, the equation has no real solutions (it has complex solutions in this case).

**Example** Use the discriminant to determine how many real solutions each equation has:

$$x^2 + 2x + 1 = 0 \qquad x^2 - 3x - 3 = 0 \qquad 2x^2 - 10x + 20 = 0$$

**Path of a projectile** A commonly used example in Calculus I and II is that of the path of an object thrown or fired straight upwards from an initial height $h_0$ with an initial speed of $v_0$ feet per second (ft/s). Letting $h(t)$ denote the height of the object after $t$ seconds, you will see in Calculus I that if the Earth's gravity is the only force acting on the object then

$$h(t) = -16t^2 + v_0 t + h_0.$$

Since this is degree two polynomial in the variable $t$, we can use our knowledge of quadratic equations to retrieve information about the path of the object.

**Example** A ball is thrown vertically upwards from a height of 32 feet with an initial velocity of 80 feet per second.

1. When does the ball fall back to ground level?
2. When does the ball reach a height of 20 feet?

## Other Types of Equations

We may have to solve equations involving fractional algebraic expressions. There are two rules that we should keep in mind when solving such expressions:
**Rule 1, Rational Expressions**: If $\frac{A}{B} = 0$, then we must have $A = 0$. Technical issue This is not an "if and only if" statement, the equation $A = 0$ may have more solutions than the original equation, since some of these solutions may satisfy $B = 0$ and thus may not be in the domain of this fractional expression. This technical issue becomes important when finding the roots of a rational function. So when we use this rule to find the roots of a rational function later, we should look over the solutions and discard any that do not belong to the domain of the original rational expression.

**Example** Solve for $x$ in the following equations

$$\frac{x^2 + 2x + 1}{x^2 - x - 2} = 0, \qquad \frac{1}{x-1} + \frac{x}{x+1} = 0, \qquad \frac{(x+1)(x-3)(x+2)}{x^2 - 9} = 0.$$

**Rule 2: Equations involving radicals** : If $\sqrt{A} = B$, then (squaring both sides) we must have $A = B^2$.
Again this is not an "if and only if" statement, since we may have a solution to $A = B^2$, which does not satisfy $\sqrt{A} = B$ (e.g. A = 1, B = -1). So when we use this rule to solve an equation, we should look over the solutions and discard any that do not satisfy the original equation.

**Example** Solve the following equations

$$\sqrt{x-1} = x - 7, \qquad \sqrt{(x-1)^2 + (2x-4)^2} = 5$$

**Equations involving absolute values** When dealing with absolute values, we must always keep in mind the definition of the absolute value:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases}$$

To solve an equation of the form $|A| = B$ we must consider the solutions to two equations $A = B$ and $-A = B$. We have $|A| = B$ if and only if either $A = B$ or $-A = B$.

**Example** Solve for $x$ in the following equations

$$|3x + 1| = 5, \qquad |x^2 + 2x| = 1$$

**Tip For Success** In addition to writing complete mathematical sentences you should develop the habit of **showing all of the steps** in your calculations. This habit will really boost your performance in more complex calculations in Calculus.

**Tip For Success** You should always **reflect** on your solution and make sure you understand the general principles and methods that helped you to derive your answer. This will help you to develop a big picture which is essential for problem solving. It is also important to reflect on the main ideas from each section or lecture as you progress through a course. It helps (especially for review at exam time) to make a short summary of the main formulas and ideas as you go.