Lecture 6 : Modeling with equations

Many problems in concrete applications of mathematics (and in calculus) require a translation of the problem to a problem in algebra where two variables are related by one or more equations. The problems given below are modifications of the models required in our lectures on related rates and optimization in calculus. They require knowledge of formulas for areas of rectangles, circles and triangles, similar triangles, Pythagoras' theorem, volumes of a cylinder and a box. We remind you of the relevant results at the end of this set of notes.

Problem solving. In complex problem solving, it is highly likely that one will not see the solution immediately. The process may require a number of steps. In general the following approach helps:

1. Read the problem carefully.
2. Draw a picture if possible.
3. Identify the variable(s) that you are required to find and introduce notation $(x, y, h....)$ for such unknown variables.
4. Identify the information you are given.
5. Write down any equations relating the unknown variables. Some such equations may be derived from the statement of the problem and some may follow from geometric properties of your picture.
6. Solve for the unknown variables.

Tip For Success In addition to writing complete mathematical sentences you should develop the habit of **showing all of the steps** in your calculations. This habit will really boost your performance in more complex calculations in Calculus.

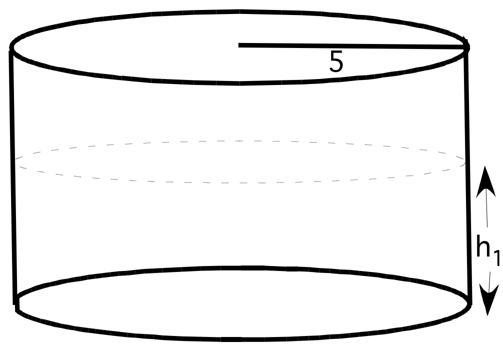
Tip For Success You should always **reflect** on your solution and make sure you understand the general principles and methods that helped you to derive your answer. This will help you to develop a big picture which is essential for problem solving. It is also important to reflect on the main ideas from each section or lecture as you progress through a course. It helps (especially for review at exam time) to make a short summary of the main formulas and ideas as you go.

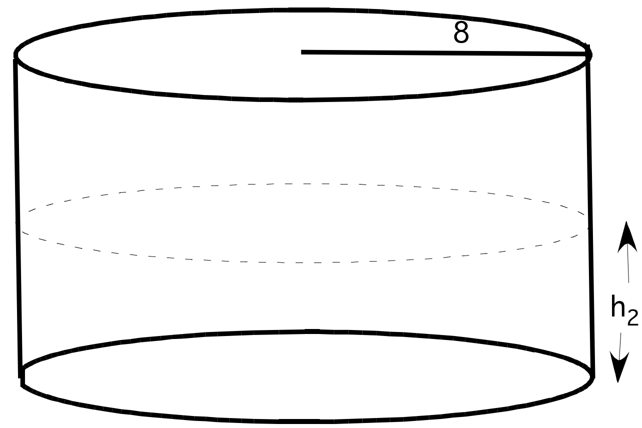
Example A twelve foot ladder is leaning against a wall. Jack pulls the base of the ladder away from the wall and stops when the base of the ladder is 5 ft. from the wall. What is the height of the top of the ladder above the ground when the base is 5ft. from the wall?

Example A person who is 5 ft. tall walks towards a street light that is 20 ft. above the ground. What is the length of her shadow when she is 10 ft. from the streetlight.

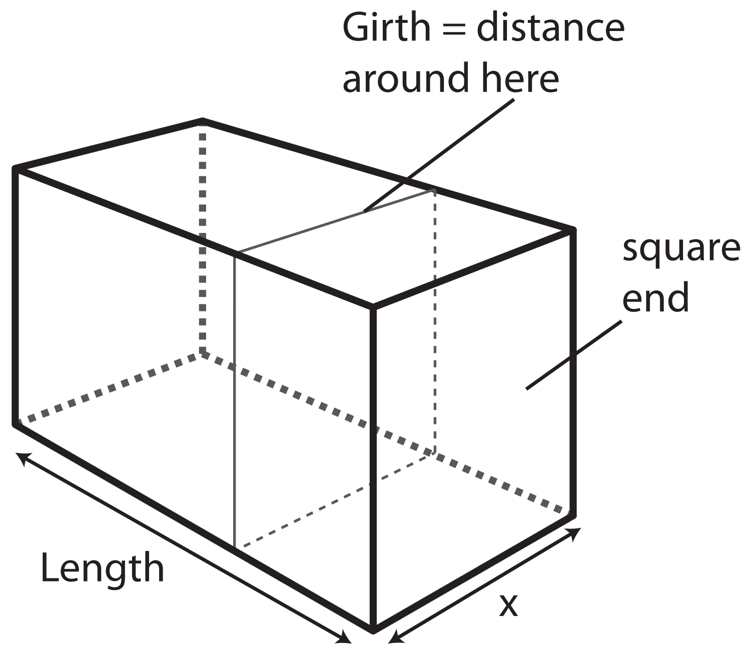
Example Two small planes approach an airport, one flying due west at 100 mi/hr and the other flying due north at 120 mi / hr. Assume that they are flying at the same constant elevation, what is the distance between the planes when the westbound plane is 180 mi. from the airport and the northbound plane is 200 mi. from the airport?

Example Two cylindrical swimming pools are being filled simultaneously with water at the same rate of $10m^3/min$. (Both pools were empty to start with and the water started pumping into both pools at the same time.) The smaller pool has a radius of 5 m. and the larger pool has a radius of 8 m. What is the height of the water in the larger pool when the height of the water in the smaller pool is 1 meter?



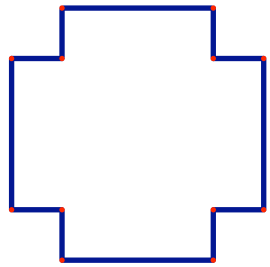


Example A package delivery service will accept a box for shipment only if the sum of its length and girth (distance around) does not exceed 60 inches. What are the dimensions of a box with a square end which has a girth of 40 inches and a volume of 500 in^2?
Will the box be accepted for shipment?



Note When we study derivatives, we will be able to solve for the dimensions of the box, which give the maximum volume under the given constraints.

Example An open top box is to be made by cutting small congruent squares from the corners of a 12-in by 12-in sheet of cardboard and bending up the sides.



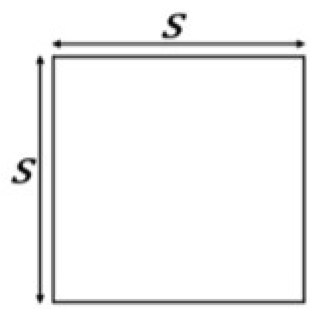
(A) Find the volume of the box when the squares removed are 1 in $\times$ 1in.

(B) Find the volume of the box when the squares removed are 2 in $\times$ 2in.

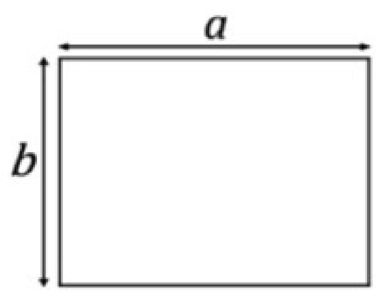
Note When we study derivatives, we will be able to solve for the size of the squares we should remove in order to create the box with the maximum volume from this piece of cardboard.

Formulas

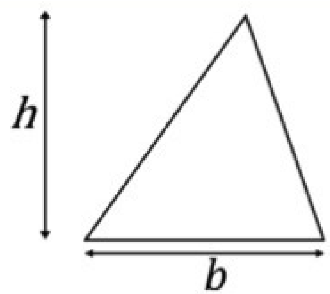
Area of square $= S^2$



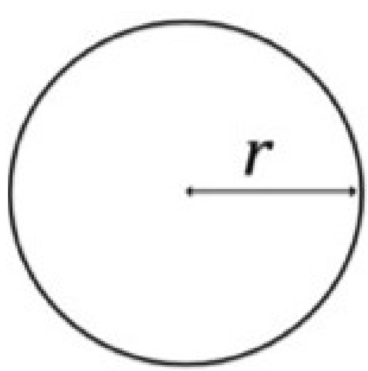
Area of rectangle $= ab$



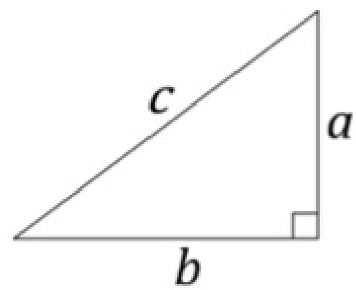
Area of triangle $= h\frac{b}{2}$



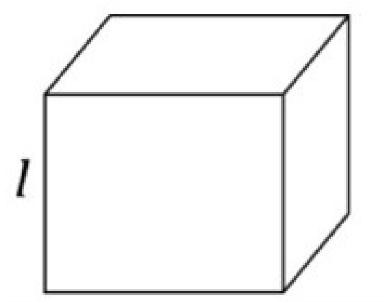
Area of circle (disk) $= \pi r^2$
Circumference of circle (disk) $= 2\pi r$



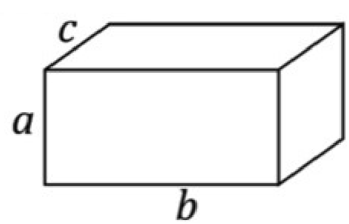
Theorem of Pythagoras $c^2 = a^2 + b^2$ or $c = \sqrt{a^2 + b^2}$



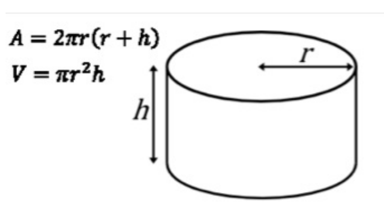
Volume of cube $= l^3$



Volume of box $= abc$



Volume (V) and surface area (A) of cylinder



Similar triangles

Similar triangles

Two triangles $\triangle ABC$ and $\triangle DEF$ are said to be similar if either of the following equivalent conditions holds:

1. They have two identical angles, which implies that their angles are all identical. For instance:

   $\angle BAC$ is equal in measure to $\angle EDF$, and $\angle ABC$ is equal in measure to $\angle DEF$. This also implies that $\angle ACB$ is equal in measure to $\angle DFE$.

2. Corresponding sides have lengths in the same ratio:

   $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$. This is equivalent to saying that one triangle (or its mirror image) is an enlargement of the other.

3. Two sides have lengths in the same ratio, and the angles included between these sides have the same measure. For instance:

   $\dfrac{AB}{DE} = \dfrac{BC}{EF}$ and $\angle ABC$ is equal in measure to $\angle DEF$.

When two triangles $\triangle ABC$ and $\triangle DEF$ are similar, one writes

$$\triangle ABC \sim \triangle DEF$$

or

$$\triangle ABC \,|||\, \triangle DEF$$

If any one of the conditions 1-3 above is true, then the other two are automatically true.