

# Interpretations and differential Galois extensions. DRAFT

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July 30, 2014

## Abstract

We give model-theoretic accounts and proofs of the following results: Suppose  $\partial y = Ay$  is a linear differential equation over a differential field  $K$  of characteristic 0, and the field  $C_K$  of constants of  $K$  is existentially closed in  $K$ . Then: (i) There exists a Picard-Vessiot extension  $L$  of  $K$ , namely a differential field extension  $L$  of  $K$  which is generated by a fundamental system of solutions of the equation, and has no new constants. (ii) There is a field  $C$  of constants which is an elementary extension of  $C_K$  such that  $K(C)$  has field of constants  $C$ , and has a Picard-Vessiot extension  $L$  such that  $C_K$  is existentially closed in  $L$ . (iii) Assume that the field  $C_K$  has finitely many extensions of degree  $n$  for all  $n$ . Then in (ii) we can choose  $C$  to be  $C_K$ , namely already  $K$  has a Picard Vessiot extension  $L$  such that  $C_K$  is existentially closed in  $L$ . (iv) If  $L_1$  and  $L_2$  are two Picard Vessiot extensions of  $K$  which (as fields) have a common embedding over  $K$  into an elementary extension of  $C_K$ , then  $L_1$  and  $L_2$  are isomorphic over  $K$  as differential fields.

Our results are proved in the more general context of logarithmic differential equations over  $K$  on not necessarily linear algebraic groups

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\*Partial support from MSRI and NSF

over  $C_K$ , and the corresponding *strongly normal* extensions of  $K$ . We make use of the general yoga of interpretations and definable automorphism groupoids from model theory, which is closely related to the Tannakian theory in say [2], but goes beyond the linear context.

## 1 Introduction

Let  $K$  be a differential field of characteristic 0 with field of constants  $C_K$ , and let  $\partial y = Ay$  be a (homogeneous) linear differential equation over  $K$  (in vector form). Namely  $y$  is a  $n \times 1$  column vector of indeterminates and  $A$  is an  $n \times n$  matrix over  $K$ . If  $L$  is a differential field extension of  $K$  then the solution set of the equation in  $L$  is a vector space over  $C_L$  of dimension at most  $n$ . A *fundamental system of solutions* of this equation, in a differential field  $L$  extending  $K$ , is by definition a set  $Y_1, \dots, Y_n$  of solutions in  $L$  which form a basis of the  $C_L$ -vector space of solutions (which thus has maximal dimension). It is well known that linear independence of  $Y_1, \dots, Y_n$  over  $C_L$  is equivalent to the  $n \times n$  matrix over  $L$  whose columns are  $Y_1, \dots, Y_n$  being nonsingular. In any case by a Picard-Vessiot (or PV) extension of  $K$  for the equation we mean a differential field extension  $L$  of  $K$  which is generated over  $K$  by such a fundamental system  $Y_1, \dots, Y_n$  of solutions, and has *no new constants*, i.e.  $C_L = C_K$ . When  $C_K$  is algebraically closed it is well-known that such a PV extension of  $K$  exists and is moreover unique up to isomorphism over  $K$  (and is generated over  $K$  by some/any fundamental system of solutions in the *differential closure*  $K^{diff}$  of  $K$ , bearing in mind that  $C_{K^{diff}} = C_K^{alg} = C_K$ ). In general ( $C_K$  algebraically closed or not) we can always find a fundamental system of solutions in  $K^{diff}$ , and the question is whether we can find such a fundamental system  $Z$  such that the constants of  $K(Z)$  coincide with  $C_K$ . Moreover one can ask for uniqueness: for another such  $Z_1$ ,  $K(Z_1)$  is isomorphic to  $K(Z_2)$  over  $K$ . In general one has neither existence nor uniqueness. Some recent papers [3], [1] give sufficient conditions for existence of PV-extensions possibly with additional properties, and also uniqueness under additional constraints: In [3] the existence is proved when  $C_K$  is existentially closed in  $K$  (as fields). In [1] it is shown that if  $C_K$  is a real closed field and  $K$  is a formally real field then a formally real PV-extension of  $K$  exists, and moreover two formally real extensions of  $K$  which are compatible (have common embeddings over  $K$  in a common real closed extension of  $K$ ) are isomorphic over  $K$ , as differential fields. Likewise in the  $p$ -

adic case. These results use the full strength of Deligne’s work on Tannakian categories [2]. The aim of the current paper is to give a model theoretic account of these results and more, at a somewhat greater level of generality, to which, at least on the surface, the Tannakian theory does not apply. The first author gives a model-theoretic account of the Tannakian formalism in [5]. The current paper is related to this, but can be read independently.

We now recall logarithmic differential equations on algebraic groups and “strongly normal extensions”. We start to assume familiarity with differential algebra in the style of Kolchin [6], and in particular with the notion of the differential closure  $K^{diff}$  of a differential field  $K$  and the fact that the field of constants of  $K^{diff}$  is the algebraic closure of the field of constants of  $K$ . The reader is also referred to [8] and [9] for basic model theory and the model theory of differential fields. Fix  $K$  as above (an arbitrary differential field of characteristic 0) and let  $G$  be an algebraic group over  $C_K$  which we will take to be connected. By a logarithmic differential equation on  $G$  over  $K$ , we mean something of the form  $\partial(y)y^{-1} = A$  for some  $A \in LG(K)$  (where  $LG$  denotes the Lie algebra of  $G$ ), where  $y$  ranges over  $G$  (i.e. over  $L$ -points of  $G$  for  $L$  any differential field containing  $K$ ). Here  $\partial(y)y^{-1}$  is Kolchin’s logarithmic derivative, a crossed homomorphism from  $G$  to its Lie algebra, which can be explained as follows: For  $g \in G(L)$ ,  $\partial(g)$  can be considered as a point in the tangent space to  $G$  at  $g$ . The tangent bundle  $TG$  of  $G$  is an algebraic group which splits as the semidirect product of  $G$  and  $LG$ . Identifying  $g$  with the point  $(g, 0)$  of  $TG$ , both  $\partial(g)$  and  $g$  are points of  $TG_g$  so their difference  $\partial(g)g^{-1}$  lies in  $LG$ .

We will write  $dlog_G(-)$  for this logarithmic derivative map from  $G$  to  $LG$ , where  $G$  is an algebraic group over  $C_K$  (namely for any differential field  $L$  containing  $K$ ,  $dlog_G$  takes  $G(L)$  to  $LG(L)$ ). Of course  $dlog_G(g) = A$  if and only if  $\partial(g) = Ag$  where the right hand side is multiplication in the sense of the algebraic group  $TG$ . We typically write the group operation on  $LG$  additively.  $dlog_G$  being a crossed homomorphism means that  $dlog_G(gh) = dlog_G(g) + (dlog_G(h))^g$ . The kernel of  $dlog_G$ ,  $\{g : dlog_G(g) = 0\}$  is a subgroup and its points in any differential field  $L > K$  coincide with  $G(C_L)$ . For any  $A \in LG$ ,  $\{g \in G : dlog_G(g) = A\}$  is a left coset of the kernel.

When  $G = GL_n$ , then  $dlog_G$  actually coincides with multiplication of matrices  $\partial(y)y^{-1}$  in  $gl_n$ . When  $G$  is an elliptic curve in Weierstrass form the logarithmic derivative coincides with  $\partial(y)/x$ . Returning to  $GL_n$  we note that if  $g \in GL_n(L)$  and  $A$  is an  $n \times n$  matrix over  $K$  and  $\partial(g)g^{-1} = A$  then the columns of  $g$  form a fundamental system of solutions for the linear differential

equation  $\partial(y) = Ay$ . Hence seeking a solution of a logarithmic differential equation on an algebraic group  $G$  is a generalization of seeking a fundamental system of solutions for a (homogeneous) linear differential equation. The generalization of the notion of Picard-Vessiot extension to this broader class of equations is what is called a “strongly normal” extension. We may as well present Kolchin’s original definition from [6] (so as to keep our notations consistent with the literature) although the ‘spirit of the matter’ is more important. We will work in a “universal differential field”  $\mathcal{U}$  (in the sense of Kolchin, or of model theory), and  $K, L, \dots$  will denote small differential subfields unless we say otherwise.  $\mathcal{C}$  denotes the field of constants of  $\mathcal{U}$  and  $C_K$  the field of constants of  $K$ .

**Definition 1.1.** *Let  $K, L$  be differential fields with  $L$  finitely generated over  $K$ .  $L$  is said to be a strongly normal extension of  $K$  if*

- (i) for each isomorphic copy  $L_1$  of  $L$  over  $K$  in  $\mathcal{U}$ ,  $L_1 \subseteq L(\mathcal{C})$  (the differential field generated over  $L$  by  $\mathcal{C}$ ), and*
- (ii)  $C_L = C_K$ .*

Including (ii) in the definition (as Kolchin does) makes for consistency with the notion of Picard-Vessiot extension. The following is well-known.

**Remark 1.2.** *Suppose  $d\log_G(y) = A$  is a logarithmic differential equation on  $G$  over  $K$  where  $G$  is over  $C_K$ . Let  $g$  be a solution of the equation in  $G(\mathcal{U})$  and let  $L = K(g)$  be the (differential) field generated over  $K$  by  $g$ . Then*

- (a) Condition (i) in Definition 1.1 is automatically satisfied,*
- (b) If  $L$  is a strongly normal extension of  $K$  (namely condition (ii) is also satisfied) then  $L$  is contained in some differential closure of  $K$ .*
- (iii) Conversely if  $L$  is contained in some differential closure of  $K$  and  $C_K$  is algebraically closed, then  $L$  is a strongly normal extension of  $K$ .*
- (iv) As we can always find a solution of  $d\log_G(y) = A$  in a differential closure of  $K$ , (iii) gives the existence of a strongly normal extension of  $K$  generated by a solution of the equation, when  $C_K$  is algebraically closed.*
- (v) When  $C_K$  is algebraically closed, there is a unique (as differential field, up to isomorphism over  $K$ ) strongly normal extension of  $K$  generated by a solution of  $d\log_G(y) = A$ .*

Remark 1.2 (a) justifies us calling  $L$  a *strongly normal extension of  $K$*  for  $d\log_G(y) = A$ , if  $L$  is generated over  $K$  by a solution  $g \in G(L)$  of the equation, and  $C_L = C_K$ . From the model-theoretic point of view the question

of the existence of such  $L$  has an “omitting types” flavour: we seek a solution  $y \in G(\mathcal{U})$  of the equation such that any constant in the definable closure of  $K, y$  is already in  $C_K$ .

We are focusing on strongly normal extensions generated by certain kinds of differential equations. But this is close to the general case, as will be discussed in Section 6.

There is a Galois theory of strongly normal extensions generalizing the linear case. But from our point of view the relevant Galois group intrinsic to the equation  $d\log_G(y) = A$  is the group of automorphisms of  $K(\mathcal{C})(\mathcal{Y})$  over  $K(\mathcal{C})$ , where  $\mathcal{Y}$  is the solution set of  $d\log_G(y) = A$  in  $G(\mathcal{U})$ , and where the automorphisms should respect the derivation. This is discussed more in section 2.

If  $M \subseteq N$  are structures for a common language  $L$ ,  $M$  is said to be existentially closed in  $N$  if any quantifier-free formula over  $M$  with a solution in  $N$  has a solution in  $M$ . When we are concerned with fields  $K \leq L$  (of characteristic 0 say) in the language of unitary rings, this is equivalent to asking that any algebraic variety  $V$  over  $K$  (not necessarily affine or irreducible) with an  $L$ -rational point, has a  $K$ -rational point.  $M$  being existentially closed in  $N$  is equivalent to each of the following

- (i)  $N$  is a model of the universal part of the complete diagram of  $M$ ,
- (ii)  $N$  embeds over  $M$  into an elementary extension of  $M$ .

We now state the main results. A logarithmic differential equation (\*)

$$d\log_G(y) = A$$

is fixed in advance, where  $G$  is a connected algebraic group over  $C_K$  and  $A \in LG(K)$ .

**Theorem 1.3.** *Suppose  $C_K$  is existentially closed in  $K$  (as a field). Then there is a strongly normal extension  $L$  of  $K$  for the equation (\*).*

The idea is straightforward. Work in an ambient differentially closed field  $\mathcal{U}$  with field of constants  $\mathcal{C}$ . A key point is to find a (quantifier-free) formula  $\psi$  in the language of fields with parameters from  $C_K$  such that (essentially)  $\psi$  defines the set of  $a \in \mathcal{C}$  such that  $K(a)$  has a strongly normal extension for the equation (\*). We want to find a solution of  $\psi$  in  $C_K$ . Now a certain interpretability result shows that  $\psi$  has a solution in  $K$ . Our assumptions imply that  $\psi$  has a solution in  $C_K$ , as required.

Before continuing, let us note that  $C_K \leq C \leq \mathcal{C}$ , then the field of constants of the differential field  $K(C)$  is precisely  $C$ . We want now to have more control over the strongly normal extension  $L$ . It would be natural to ask that  $C_K$  is also existentially closed in  $L$ . (So for example, if  $C_K$  is real closed and  $K$  is formally real, then one would like  $L$  to be formally real too.) The next result says that, without additional hypotheses, this can be accomplished, at the expense of replacing the field of constants  $C_K$  by an elementary extension. For  $C$  an extension of  $C_K$ , contained in  $\mathcal{C}$ , we say that  $L$  is a *strongly normal extension of  $K$  over  $C$  (for  $(*)$ )* if  $L$  is a strongly normal extension of  $K(C)$  for  $(*)$ . (So  $L$  is generated over  $K(C)$  by a solution of  $(*)$  and  $L$  has the same field of constants as  $K(C)$ , namely  $C$ ). Proofs of the remaining results depend on an associated Galois groupoid  $\mathcal{G}$  definable in  $\mathcal{C}$  with parameters from  $C_K$  as well as another family of interpretations.

**Theorem 1.4.** *Suppose again  $C_K$  is existentially closed in  $K$ . Then there is an elementary extension  $C$  of  $C_K$ , with  $C \leq \mathcal{C}$  and a strongly normal extension  $L$  of  $K$  over  $C$ , such that  $C_K$  is also existentially closed in  $L$  (as fields).*

Here is the uniqueness statement over  $K$ , with no further hypotheses.

**Theorem 1.5.** *Suppose again  $C_K$  is existentially closed in  $K$ . Suppose that  $L_1, L_2$  are strongly normal extensions of  $K$  for  $(*)$  such that there is a common embedding over  $K$  of  $L_1$  and  $L_2$  into an elementary extension of  $C_K$  (as fields). Then  $L_1$  and  $L_2$  are isomorphic over  $K$  as differential fields.*

The existence statement *over  $K$*  requires a further hypothesis.

**Theorem 1.6.** *Suppose again  $C_K$  is existentially closed in  $K$ . Suppose in addition that  $C_K$  has only finitely many extensions of degree  $n$  for all  $n$  (Serre's property (F)),  
Then  $K$  has a strongly normal extension  $L$  for the equation  $(*)$  such that  $C_K$  is existentially closed in  $L$ .*

Serre's property (F) is used, essentially as in [1], to preserve Galois cohomology when passing to elementary extensions, and thus to see that the restriction of the Galois groupoid to  $Th(C_K)$  has finitely many components. Note that real closed and  $p$ -adically closed fields have property (F). In any case the above theorems subsume the afore mentioned results from [3] and [1]. In fact in Theorems 1.4 and 1.6 we can fix in advance an embedding of

$K$  in an elementary extension  $L_1$  of  $C_K$ , define  $T_1$  to be the universal theory of  $L_1$  in a language with constants for elements of  $K$ , and require that the strongly normal extensions  $L$  are models of  $T_1$ .

These notions of Picard-Vessiot and strongly normal extensions of differential fields and their automorphism groups, are a special case of a phenomenon widely studied in model theory, namely “internality” and definable automorphism groups (also called “liason” or “binding” groups). This was initiated by Zilber, with subsequent refinements by Poizat, Hrushovski, and others, including the current authors. Poizat [11] in particular made clear the connection of the general model theoretic constructions with differential Galois theory. A kind of culmination of the model-theoretic perspective appears in [4] where not only a definable automorphism group but also a definable automorphism groupoid, is attached to an “internal cover”. This point of view will be present throughout the current paper.

As already mentioned we will assume familiarity with basic model theory from say [8], as well as some basic model theory of differentially closed fields which can be found in [9]. We nevertheless try to give proofs of the main theorems which will be accessible to a broad audience, including differential algebraists. So both in this introduction and in sections 2 to 6, the presentation may be a bit heavy-handed for some people’s taste. Parts of section 7 where we place some of the results in a general model-theoretic context may require a little more fluency in model theory.

$T, T', ..$  usually denote *complete* theories in languages  $L, L', ..$ . If  $M$  is a model of  $T$  and  $A$  a subset of (the universe of)  $M$ , by  $L_A$  we mean  $L$  together with new constants for elements of  $A$ , by  $M_A$  we mean the tautological expansion of  $M$  to an  $L_A$ -structure, and by  $T_A$  we mean the theory of  $M_A$  in  $L_A$ . (So  $T_A$  depends on  $M$ .)

Note that if  $T$  has quantifier elimination (which will often be the case) then  $T_A$  depends only on  $T$  and the isomorphism type of  $A$ .

If and when we discuss possibly incomplete theories we will mention any modifications to the above conventions. In a departure from the second author’s usual conventions, when we talk about “definability” with respect to a given structure or theory, we will mean  $\emptyset$ -definable, i.e. definable without parameters.

Important theories we will be dealing with are  $ACF_0$ , the theory of algebraically closed fields of characteristic 0 which is complete with quantifier elimination in the language of unitary rings  $(+, -, \times, 0, 1)$ , and  $DCF_0$ , the

theory of differentially closed fields of characteristic 0 which is complete with quantifier elimination in the language of unitary rings above together with a symbol  $\partial$  for the derivation. We often work in an ambient “saturated” differentially closed field  $\mathcal{U}$ , with field  $\mathcal{C}$  of constants. We will use repeatedly the fact that the field  $C_L$  of constants of a differentially closed field  $L$  is an algebraically closed field “without additional structure”. Actually we use a bit more:

**Fact 1.7.** *Suppose that  $L$  is a differentially closed field, and  $K$  is a differential subfield. Then the subsets of Cartesian powers of  $C_L$  which are definable with parameters from  $K$  in the structure  $(L, +, -, \times, 0, 1, \partial)$  are precisely the sets definable with parameters from  $C_K$  in the field  $(C_L, +, -, 0, 1, \times)$ .*

A “definability of types” restatement is:  $tp(K/C_L)$  is definable over  $C_K$ . Another consequence is that for any field  $C$  such that  $C_K \leq C \leq \mathcal{C}$ , the field of constants of the differential field  $K(C)$  is precisely  $C$ .

Sometimes we will mention constants in the sense of logic, namely constant symbols, which should not be confused with constants in a differential field.

## 2 Interpretations

To a logarithmic differential equation over  $K$  we can and will attach a two-sorted structure  $(\mathcal{C}, \mathcal{Y})$  consisting of the constants  $\mathcal{C}$ , and the solution set  $\mathcal{Y}$  of the equation, in an ambient differentially closed field, equipped with all  $K$ -definable relations. The main point of this section is to show that this structure (or rather its first order theory) is interpretable in a rather special way in  $ACF_K$  the theory of algebraically closed fields of characteristic 0 with constant symbols for elements of  $K$ . Our interpretation result is an easy version of so-called “quantifier-elimination for algebraic  $\partial$ -groups” from [7] which says that any finite-dimensional differential algebraic group is interpretable in the theory of algebraically closed fields. Nevertheless we will give a few details, for completeness.

Let us first be precise about interpretations: Let  $T_1, T_2$  be first order theories, possibly many sorted in languages  $L_1, L_2$ . As above we assume both  $T_1, T_2$  are complete. An interpretation  $\omega$  of  $T_1$  in  $T_2$  is an assignment, to each sort  $S$  of  $T_1$  a formula  $\omega(S)$  of  $L_2$ , and to each  $L_1$ -symbol  $R$ , an  $L_2$ -formula  $\omega(R)$  (appropriately sorted) such that, for some (any) model  $M$  of



$T_2$  the  $L_1$ -structure  $\omega^*(M)$  which has sorts  $\omega(S)(M)$  and relations  $\omega(R)(M)$ , is a model of  $T_1$  (in the obvious sense). Note that  $\omega$  associates to each  $L_1$ -formula (sentence)  $\psi$ , an  $L_2$ -formula (sentence)  $\omega(\psi)$ , and the condition to be an interpretation of  $T_1$  in  $T_2$  can be restated as: for each  $L_1$ -sentence  $\sigma$  which is in  $T$ , the  $L_2$ -sentence  $\omega(\sigma)$  is in  $T_2$ . Note that any model of  $T_1$  can be elementarily embedded in a model of the form  $\omega(M)$  for some  $M \models T_2$ , although may not itself be of that form.

We now fix, once and for all, the data consisting of a logarithmic differential equation

$$(*) \quad d\log_G(y) = A$$

where  $G$  is a connected algebraic group over  $C_K$ , and  $A \in LG(K)$ .

We will fix a differentially closed field  $\mathcal{U}$  containing  $K$ . In later sections we may want to apply compactness and so will choose  $\mathcal{U}$  to be saturated, but in this section there is no harm in taking  $\mathcal{U}$  to be the differential closure  $K^{diff}$  of  $K$ . As mentioned earlier  $\text{Ker}(d\log_G) = G(\mathcal{C})$ , and  $\mathcal{Y}$  is the left coset (so right torsor)  $bG(\mathcal{C})$  for some/any  $b \in \mathcal{Y}$ .

Let  $\mathcal{C}$  be the field of constants of  $\mathcal{U}$  and let  $\mathcal{Y}$  be the solution set of  $(*)$  in  $\mathcal{U}$ , namely  $\{y \in G(\mathcal{U}) : d\log_G(y) = A\}$ .

Let  $M$  be  $(\mathcal{C}, \mathcal{Y})$  equipped with all relations which are definable over  $K$  in  $\mathcal{U}$ , so a 2-sorted structure. Let  $L(M)$  be the language of  $M$ , namely with symbols for all these relations. We will show that we can make  $(\mathcal{U}, G(\mathcal{U}))$  into an  $L(M)$ -structure  $N$  say in such a way that

- (i)  $M \prec N$ , and
- (ii) All the relations on  $N$  are definable over  $K$  in the algebraically closed field  $(\mathcal{U}, +, \cdot)$ .

This will yield the interpretation we are looking for. By Fact 1.7:

**Remark 2.1.** *The subsets of  $\mathcal{C}^n$  which are  $\emptyset$ -definable in  $M$  are precisely the subsets definable over  $C_K$  in the algebraically closed field  $(\mathcal{C}, +, \cdot)$ .*

To facilitate the construction and proof we will choose some auxiliary languages:  $L_{\partial, \mathcal{U}}$  and  $L_{\partial, K}$ , the languages of  $\partial$ -varieties over  $\mathcal{U}$  and  $\partial$ -varieties over  $K$ , respectively (with respect to the “connections” or differential equations  $\partial(y) = Ay$  on  $G$ , and  $\partial(x) = 0$  on  $\mathcal{U}$ ). We refer to [7] for more details and/or background.

**Definition 2.2.** (i) *By a  $\partial$ -subvariety of  $\mathcal{U}^n \times G(\mathcal{U})^m$  (over  $\mathcal{U}$ ), we mean a subvariety  $X$  over  $\mathcal{U}$ , such that  $X^\partial =_{def} X \cap (\mathcal{C}^n \times \mathcal{Y}^m)$  is Zariski-dense in*

$X$ .

(ii)  $L_{\partial, \mathcal{U}}$  is the language with symbols  $R_X$  for each such  $\partial$ -subvariety, and  $L_{\partial, K}$  is the sublanguage consisting of the  $R_X$  where  $X$  is defined over  $K$ .

(iii)  $(\mathcal{U}, G)_{L_{\partial, \mathcal{U}}}$  is the  $L_{\partial, \mathcal{U}}$  structure with sorts  $\mathcal{U}$  and  $G$ , and with the tautological interpretations of the  $R_X$ . Similarly for  $(\mathcal{U}, G)_{L_{\partial, K}}$ .

(iv)  $(\mathcal{C}, \mathcal{Y})_{L_{\partial, \mathcal{U}}}$  is the  $L_{\partial, \mathcal{U}}$ -structure with sorts  $\mathcal{C}$  and  $\mathcal{Y}$  and with  $X^\partial$  as the interpretation of  $R_X$ . Similarly for  $(\mathcal{C}, \mathcal{Y})_{L_{\partial, K}}$ .

**Remark 2.3.** (i) The class of  $\partial$ -subvarieties of the various Cartesian powers  $\mathcal{U}^n \times G(\mathcal{U})^m$  is closed under finite unions, finite intersections, Cartesian products, and passing to irreducible components.

(ii) If  $A$  is a Boolean combination of  $\partial$ -subvarieties of  $\mathcal{U}^n \times G(\mathcal{U})^m$ , then  $A^\partial =_{\text{def}} A \cap (\mathcal{C}^n \times \mathcal{Y}^m)$  is Zariski-dense in  $A$ .

(iii) The  $\partial$ -subvarieties of  $\mathcal{U}^n$  are precisely the subvarieties defined over  $\mathcal{C}$ .

(iv) The  $\emptyset$ -definable sets in the structure  $M$  above are precisely the Boolean combinations of the  $X^\partial$  for  $X$  a  $\partial$ -subvariety defined over  $K$ , so from now on we identify  $M$  with the  $L_{\partial, K}$ -structure on  $(\mathcal{C}, \mathcal{Y})$ . In particular the structure  $(\mathcal{C}, \mathcal{Y})_{L_{\partial, K}}$  has quantifier-elimination.

*Proof.* (i) and (ii) are contained in Facts 2.2, Fact 2.3, and Lemma 2.5 of [7].

(iii) is obvious, and (iv) follows from quantifier elimination in  $DCF_0$ .  $\square$

Intrinsic description of  $\partial$ -subvarieties.

**Lemma 2.4.** *The structure  $(\mathcal{U}, G)_{L_{\partial, \mathcal{U}}}$  has quantifier elimination.*

*Proof.* Fix a point  $d \in \mathcal{Y}$ . Let  $f$  be translation by  $d^{-1}$ , taking  $G$  to  $G$ .  $f$  induces a bijection between  $\mathcal{Y}$  and  $G(\mathcal{C})$ . So for any subvariety  $X$  of  $G$ ,  $X \cap \mathcal{Y}$  is Zariski-dense in  $X$  if and only if  $G(\mathcal{C})$  is Zariski-dense in  $f(X)$  if and only if  $f(X)$  is defined over  $\mathcal{C}$ . More generally, fixing  $n$  and  $m$ ,

(\*) a subvariety  $Z$  of  $\mathcal{U}^n \times G^m$  is a  $\partial$ -subvariety if and only if  $(id^n, f^m)(Z)$  is a subvariety of  $\mathcal{U}^n \times G^m$  which is defined over  $\mathcal{C}$ .

Now let  $W \subseteq \mathcal{U}^n \times G^m$  be  $\partial$ -constructible, namely a Boolean combination of  $\partial$ -varieties over  $\mathcal{U}$ . Then  $(id^n, f^m)$  is a Boolean combination of subvarieties of  $\mathcal{U}^n \times G^m$  which are defined over  $\mathcal{C}$ . Let  $\pi$  be any projection of  $\mathcal{U}^n \times G^m$  onto some coordinate axes, say onto  $\mathcal{U}^{n'} \times G^{m'}$ . Then by quantifier elimination in algebraically fields,  $\pi \circ (id^n, f^m)(W)$  is also a Boolean combination of subvarieties of  $\mathcal{U}^{n'} \times G^{m'}$  defined over  $\mathcal{C}$ . Applying  $(id^{n'}, f^{m'})$  and using (\*), we see that  $\pi(W)$  is a Boolean combination of  $\partial$ -subvarieties of  $\mathcal{U}^{n'} \times G^{m'}$ , as required.  $\square$

**Corollary 2.5.** *The structure  $(\mathcal{U}, G)_{L_{\partial, K}}$  has quantifier elimination.*

*Proof.* Let  $X \subseteq \mathcal{U}^n \times G^m$  be a Boolean combination of  $\partial$ -varieties which are defined over  $K$ , and let  $\pi$  be some projection on coordinate axes. We have to show that  $\pi(X)$  is a Boolean combination of  $\partial$ -varieties defined over  $K$ . By Lemma 2.4,  $\pi(X)$  is a Boolean combination of  $\partial$ -varieties which are defined over  $\mathcal{U}$ . Now  $\pi(X)$  as a definable set in  $(\mathcal{U}, +\cdot)$  is invariant under  $K$ -automorphisms, as is its Zariski closure  $\overline{\pi(X)}$ . Clearly (using Remark 2.3)  $\overline{\pi(X)}$  is a  $\partial$  variety, so by  $K$ -invariance is a  $\partial$ -variety defined over  $K$ . Now  $\overline{\pi(X)} \setminus \pi(X)$  is a Boolean combination of  $\partial$ -varieties and is also definable over  $K$ . By induction (on dimension say)  $\overline{\pi(X)} \setminus \pi(X)$  is a Boolean combination of  $\partial$ -varieties which are defined over  $K$ .  $\square$

**Corollary 2.6.**  *$(\mathcal{C}, \mathcal{Y})_{L_{\partial, K}}$  is an elementary substructure of  $(\mathcal{U}, G)_{L_{\partial, K}}$*

*Proof.* First note that by definition  $(\mathcal{C}, \mathcal{Y})_{L_{\partial, K}}$  is a substructure of  $(\mathcal{U}, G)_{L_{\partial, K}}$ . Now suppose that  $Z$  is a nonempty definable set in  $(\mathcal{U}, G)_{L_{\partial, K}}$ , defined with parameters from  $(\mathcal{C}, \mathcal{Y})_{L_{\partial, K}}$ . By Corollary 2.5 above and Remark 2.3,  $Z$  is a Boolean combination of  $\partial$ -varieties. By Remark 2.3 (ii),  $Z$  has a point in  $(\mathcal{C}, \mathcal{Y})_{L_{\partial, K}}$ . The result follows (Tarski-Vaught).  $\square$

In particular we have:

**Theorem 2.7.** *The map assigning to each relation symbol  $R_X \in L_{\partial, K}$  the formula (over  $K$ ) defining  $X$  in  $(\mathcal{U}, +, -, \times, 0, 1)$  is an interpretation of  $Th((\mathcal{C}, \mathcal{Y})_{L_{\partial, K}})$  in  $ACF_K$ .*

### 3 The Galois group and the proof of Theorem 1.3

We remain in the general setup of the previous section, namely a connected algebraic group  $G$  over  $C_K$ , and a logarithmic differential equation  $dlog_G(-) = A$  over  $K$ , and again we take  $\mathcal{Y}$  to be the solution set in  $\mathcal{U}$ , an ambient differentially closed field extending  $K$ , which we will now take to be saturated for various reasons.  $\mathcal{C}$  denotes the constants of  $\mathcal{U}$  as before.

By  $Aut(\mathcal{Y}/K(\mathcal{C}))$  we mean the group of permutations of  $\mathcal{Y}$  induced by automorphisms of  $\mathcal{U}$  which fix  $K(\mathcal{C})$  pointwise. Equivalently, via quantifier-elimination it is simply the group of automorphisms of the differential field  $K(\mathcal{C})(\mathcal{Y})$  which fix the differential subfield  $K(\mathcal{C})$  pointwise.

We will work in the structure  $\mathcal{U}$ . The first aim is to obtain a  $K$ -definable function  $f$  from  $\mathcal{Y}$  to  $\mathcal{C}^m$  (some  $m$ ) such that for each  $b \in \mathcal{Y}$ ,  $b$  is “constrained over  $K(\mathcal{C})$ ” by some differential equations and inequations over  $K(f(b))$ . Or in model-theoretic notation  $tp(b/K(\mathcal{C}))$  is isolated by a formula over  $K(f(b))$ . It will follow that  $C_{K(b)} = C_{K(f(b))}$ .

**Lemma 3.1.**  *$\mathcal{Y}$  is contained in any differential closure of  $K(\mathcal{C})$  in  $\mathcal{U}$ . In particular for all  $b \in \mathcal{Y}$ ,  $tp(b/K(\mathcal{C}))$  is isolated.*

*Proof.* Fix a solution  $b_1$  in some differential closure  $K_1$  of  $K(\mathcal{C})$  in  $\mathcal{U}$ . Any other  $b \in \mathcal{Y}$  differs from  $b_1$  by an element of  $G(\mathcal{C})$  so is also in  $K_1$ .  $\square$

Let us now fix some  $b \in \mathcal{Y}$ . Let  $\phi(y, z)$  be a formula over  $K$  and  $a$  some tuple from  $\mathcal{C}$  such that  $\phi(y, a)$  isolates  $tp(b/K(\mathcal{C}))$ . By elimination of imaginaries in algebraically closed fields we may choose  $\phi$  and  $a$  such that  $a$  is a canonical parameter for  $\phi(y, a)$  over  $K$ . In particular we can write  $a = f(b)$  where  $f$  is a  $K$ -definable partial function, depending of course only on  $r = tp(b/K)$ , so written as  $f_r$ . Note that the formula  $\phi(y, z)$  also depends only on  $r$  so we write it as  $\phi_r(y, z)$ . The outcome is:

**Remark 3.2.** *For each complete type  $r(y)$  over  $K$  of the sort  $\mathcal{Y}$ , there is a formula  $\phi_r(y, z)$  over  $K$  and partial function  $f_r(y)$  defined over  $K$ , such that for each such  $r$  and realization  $b$  of  $r$ ,  $f_r(b)$  is a tuple  $a$  from  $\mathcal{C}$  and  $\phi_r(y, a)$  isolates  $tp(b/K(\mathcal{C}))$ . (Strictly speaking the variable  $z$  depends also on  $r$ .)*

The following relates isolation to strongly normal extensions.

**Lemma 3.3.** *Suppose  $a = f_r(b)$  (where  $b$  realizes  $r$ ). Then for any field  $C \leq \mathcal{C}$  containing  $C_K$ , the field of constants of  $K(\mathcal{C})(b)$  is precisely  $C(a)$ . In particular if  $a \in C$ , then  $K(\mathcal{C})(b)$  is a strongly normal extension of  $K$  over  $C$ .*

*Proof.* This is very basic, but we go through the routine details, as it is a key point of the paper. First as  $a \in K(b)$ ,  $a \in K(\mathcal{C})(b)$ . For the converse, we first point out that  $C_{K(b)} = C_K(a)$ : Let  $d \in C_{K(b)}$ , so  $d = g(b)$  for some definable function  $g$  over  $K$ . By 3.2, we have that  $\forall y(\psi_r(y, a) \rightarrow g(y) = d)$  is true in  $\mathcal{U}$ . So  $d \in dcl(K, a)$ , whereby, using Fact 1.7,  $d \in dcl(C_K(a))$  in the field  $\mathcal{C}$ , namely  $d \in C_K(a)$ . The same argument shows that if  $d$  is a constant of  $K(\mathcal{C})(b)$  then it is in  $C(a)$ .  $\square$

Now we aim towards describing the “intrinsic” Galois group of the equation (\*).

**Lemma 3.4.** *Let  $b \in \mathcal{Y}$  and  $\sigma \in \text{Aut}(\mathcal{Y}/K(\mathcal{C}))$ . Then  $\sigma(b)b^{-1}$  (multiplication in the group  $G$ ) does not depend on  $b$*

*Proof.* Let  $b_1$  be another element of  $\mathcal{Y}$ . As  $\mathcal{Y}$  is a left coset of  $G(\mathcal{C})$  there is  $d \in G(\mathcal{C})$  such that  $b = b_1d$ . Applying  $\sigma$  we get  $\sigma(b) = \sigma(b_1)d$ . So  $\sigma(b)b^{-1} = \sigma(b_1)dd^{-1}b_1^{-1} = \sigma(b_1)b_1^{-1}$ .  $\square$

**Lemma 3.5.** (i) *The map  $\rho$  from  $\text{Aut}(\mathcal{Y}/K(\mathcal{C}))$  to  $G$  taking  $\sigma$  to  $\sigma(b)b^{-1}$  (for some/any  $b \in \mathcal{Y}$ ) is an isomorphism between  $\text{Aut}(\mathcal{Y}/K(\mathcal{C}))$  and a definable over  $K$  subgroup  $H^+$  of  $G$ .*

(ii) *This is also an isomorphism of group actions where the action of  $H^+$  on  $\mathcal{Y}$  is by left multiplication in  $G$  (so  $\mathcal{Y}$  is a right coset of  $H^+$ ).*

*Proof.* By Lemma 3.4, this map is well-defined (does not depend on the choice of  $b$ ). Now for a given  $b$ , the set of  $\sigma(b)$  for  $\sigma \in \text{Aut}(\mathcal{Y}/K(\mathcal{C}))$  is precisely the set of solutions of  $\phi_r(y, f_r(b))$  from Remark 3.2, hence the image  $H^+$  is definable over  $K(f_r(b))$  as  $\{yb^1 : \models \phi_r(y, f_r(b))\}$ . As  $H^+$  does not depend on the choice of  $b$  it is defined over  $K$ . The rest is easy to check.  $\square$

Hence we see that the group  $\text{Aut}(\mathcal{Y}/K(\mathcal{C}))$  together with its action, is defined over  $K$  in the differentially closed field, namely is a differential algebraic subgroup of  $G$ , defined over  $K$ . In fact it is also  $\emptyset$ -definable (or rather interpretable) in the  $L_{\partial, K}$ -structure  $(\mathcal{C}, \mathcal{Y})$  from Section 2.

**Remark 3.6.** *Although we will not need this, let us remark that if  $H$  is the Zariski closure of  $H^+$  in  $G$ , then  $H$  is precisely the intersection of the stabilizers (under left multiplication in  $G$ ) of all  $\partial$ -subvarieties of  $G$  defined over  $K$ . It is an algebraic group over  $K$  which can be viewed as the automorphism group of the “forgetful” interpretation  $\omega$  from Section 2.*

We can now find a single  $f$  and  $\phi$  doing the job of the  $f_r$  and  $\phi_r$ .

**Corollary 3.7.** *There is a formula  $\phi(y, z)$  over  $K$  and  $K$ -definable function  $f : \mathcal{Y} \rightarrow \mathcal{C}^m$  some  $m$  such that for all  $b \in \mathcal{Y}$  the formula  $\phi(y, f(b))$  isolates  $tp(b/K(\mathcal{C}))$ .*

*Proof.* There are different ways of doing it. For example, let  $Z$  be the set  $\mathcal{Y}/H^+$  of right cosets of  $H^+$  in  $\mathcal{Y}$  equivalently orbits under the action of  $H^+$  on  $\mathcal{Y}$  by left multiplication in  $G$ . Then  $Z$  is pointwise fixed by  $\text{Aut}(\mathcal{U}/K(\mathcal{C}))$ , so by elimination of imaginaries in  $DCF_0$  can be considered as a definable (over  $K$ , so over  $C_K$ ) set in  $\mathcal{C}$ . Then  $f(b)$  is precisely the orbit of  $b$  under  $H^+$  as an element of  $Z$ .

Alternatively, fixing  $r$ , and realization  $b$  of  $r$ , we have that  $\phi(y, f(b))$  implies  $yb^{-1} \in H^+$ , so we can apply compactness to find a formula  $\psi_r(y)$  in  $r$  responsible for this. Apply compactness again to get that finitely many  $\psi_r(y)$  cover  $\mathcal{Y}$ . □

*Conclusion of proof of Theorem 1.3.*

Let  $f$  be as in Corollary 3.7. So  $f$  is  $\emptyset$ -definable in the structure  $M$  on  $(\mathcal{C}, \mathcal{Y})$  discussed in the previous section. Also the image of  $f$  is a  $\emptyset$ -definable set of tuples from  $\mathcal{C}$  in  $M$ . By Remark 2.1,  $\text{Im}(f)$  is (quantifier-free) definable over  $C_K$  in the algebraically closed field  $\mathcal{C}$ , by formula  $\chi(x)$  say. Now the sentence “ $f$  is a function from  $\mathcal{Y}$  to  $\mathcal{C}^m$ ” is true in  $(\mathcal{C}, \mathcal{Y})$ . So taking  $F$  to be the interpretation of the formula defining  $f$  in  $(\mathcal{U}, G)$ ,  $F$  is a function from  $G$  to  $\mathcal{U}^m$ . The interpretation result says that  $F$  is definable over  $K$  in the algebraically closed field  $\mathcal{U}$ . Let  $e$  be the identity element of  $G$ . So  $e \in G(C_K) \subseteq G(K)$ , whereby  $F(e) \in K^m$ . So  $\text{Im}(F)$  has a point in  $K$ , namely  $\chi(x)$  is realized in  $K$ . By the assumption that  $C_K$  is existentially closed in  $K$ ,  $\chi(x)$  is realized in  $C_K$ . What this means is that there is  $b \in \mathcal{Y}$  such that  $f(b) \in C_K$ . By Lemma 3.3,  $K(b)$  has constant field  $C_K$  so is a strongly normal extension of  $K$  for the given logarithmic differential equation.

## 4 The Galois groupoid and more interpretations

For now we continue working in the differentially closed field  $\mathcal{U}$ , and use  $f$  and  $\phi$  from Corollary 3.7 where appropriate. We may and will assume that for each  $a \in \text{Im}(f)$ ,  $f^{-1}(a)$  is the set defined by  $\phi(y, a)$  (which isolates  $\text{tp}(b/K(\mathcal{C}))$  for some/any realization  $b$  of  $\phi(y, a)$ ). But we also be referring to two other theories,  $\tilde{T} = \text{Th}(\mathcal{C}, \mathcal{Y})_{L_{\emptyset, K}}$  which has already been discussed, and its reduct  $T$  to the sort  $\mathcal{C}$  which we know to be bi-definable (without

parameters) with the theory of the algebraically closed field  $\mathcal{C}$  with constants for elements of  $C_K$ .

We feel free to identify  $Aut(\mathcal{Y}/K(\mathcal{C}))$  with the definable subgroup  $H^+$  of  $G$ , acting on  $\mathcal{Y}$  by multiplication in  $G$  on the left. First we describe the “everybody’s” Galois group of our logarithmic differential equation, namely as an algebraic group over the constants.

**Definition 4.1.** (i) For  $b \in \mathcal{Y}$  let  $h_b$  be the bijection between  $\mathcal{Y}$  and  $G(\mathcal{C})$  which takes  $y$  to  $b^{-1}y$ .  
(ii) For  $b \in \mathcal{Y}$  let  $\rho_b$  be the map from  $Aut(\mathcal{Y}/K(\mathcal{C}))$  to  $G(\mathcal{C})$  taking  $\sigma$  to  $b^{-1}\sigma(b)$ .

**Lemma 4.2.** (i)  $\rho_b$  is a group isomorphism between  $Aut(\mathcal{Y}/K(\mathcal{C}))$  and an algebraic subgroup of  $G(\mathcal{C})$  defined over  $C_K(a)$ , where  $a = f(b)$ , which we call  $H_a$ .  
(ii)  $(\rho_b, h_b)$  is an isomorphism of group actions between  $Aut(\mathcal{Y}/K(\mathcal{C}))$  acting naturally on  $\mathcal{Y}$ , and  $H_a$  acting on the set  $G(\mathcal{C})$  by left multiplication. In other words  $\rho_b$  is induced by  $h_b$ .  
(iii)  $\{h_{b_1}^{-1} \circ h_b : f(b_1) = f(b) = a\}$  is precisely  $Aut(\mathcal{Y}/K(\mathcal{C}))$ .  
(iv)  $\{h_{b_1} \circ h_b^{-1} : f(b_1) = f(b) = a\}$  is precisely  $H_a$  acting on  $G(\mathcal{C})$  by multiplication in  $G$  on the left.

*Proof.* (i)  $\rho_b(\sigma)\rho_b(\tau) = b^{-1}\sigma(b)b^{-1}\tau(b) = b^{-1}\sigma(bb^{-1}\tau(b))$  (as  $b^{-1}\tau(b) \in G(\mathcal{C})$  and  $\sigma$  fixes  $\mathcal{C}$ )  $= b^{-1}(\sigma\tau)(b)$ . So the image of  $\rho_b$  is a subgroup of  $G(\mathcal{C})$  definable in  $\mathcal{U}$ . By 3.3 and 1.7 this subgroup is defined in  $\mathcal{C}$  over  $C_K(a)$  (and so we call it  $H_a$ ).

(ii) Given  $\sigma \in Aut(\mathcal{Y}/K(\mathcal{C}))$ , the bijection  $h_b$  between  $\mathcal{Y}$  and  $G(\mathcal{C})$  induces the permutation  $h_b(\sigma)$  of  $G(\mathcal{C})$  which takes  $b^{-1}y$  to  $b^{-1}\sigma(y)$ . Using Lemma 3.4 this is precisely left multiplication in  $G(\mathcal{C})$  by  $b^{-1}\sigma(b)$ .

(iii) Note that  $f(b_1) = f(b)$  iff  $tp(b_1/K(\mathcal{C})) = tp(b/K(\mathcal{C}))$  iff there is  $\sigma \in Aut(\mathcal{Y}/K(\mathcal{C}))$  such that  $\sigma(b) = b_1$ . So apply Lemma 3.5.

(iv) Clear. □

So for any  $a \in Im(f)$ ,  $H_a$ , an algebraic subgroup of  $G(\mathcal{C})$ , is isomorphic to  $Aut(\mathcal{Y}/K(\mathcal{C}))$ . But it needs the parameter  $a$  to be defined, and moreover its action on  $\mathcal{Y}$  requires the choice of a parameter  $b \in f^{-1}(a)$ .

The “Galois groupoid” is a natural way of putting all this data together in an “invariant” fashion. A groupoid is a category in which all morphisms have inverses, and it is said to be connected if  $Mor(a, b)$  is nonempty for

any objects  $a, b$ . The notion of a groupoid definable in a theory or structure appears in [4], and is in itself straightforward: the set of objects should be definable, as well as the set OF morphisms, the domain and codomain maps from the set of morphisms to the set of objects, and the composition map. (Remember by definable we mean without parameters. So for example for objects  $a, b$  the set of morphisms from  $a$  to  $b$  should be definable over  $a, b$ , uniformly in  $a, b$ .) The reader is referred to [4] for the finer notion of a “concrete definable groupoid” which is really what we are concerned with.

We will define two groupoids definable in the differentially closed field  $\mathcal{U}$ :  $\mathcal{G}^+$ , definable with parameters from  $K$ , and a full subgroupoid  $\mathcal{G}$  definable with parameters  $C_K$  and living in the sort  $\mathcal{C}$ . (In fact  $\mathcal{G}^+$  and  $\mathcal{G}$  will be groupoids definable in  $\tilde{T}^{eq}$ , and  $T$  respectively, without parameters. )

The set  $O = O_{\mathcal{G}}$  of objects of  $\mathcal{G}$  will be the  $\emptyset$ -definable set  $Im(f)$  in  $\mathcal{C}$ , where  $f$  is from Corollary 3.7. The set  $O^+ = O_{\mathcal{G}^+}$  of objects of  $\mathcal{G}^+$  will be  $O$  together with a single  $\emptyset$ -definable object which we call  $+$ .

Fix  $a \in O$ , let  $Mor^+(+, a) = Mor_{\mathcal{G}^+}(+, a)$  between  $+$  and  $a$  in  $\mathcal{G}^+$  be the set  $\{h_b : f(b) = a\}$  of bijections between  $\mathcal{Y}$  and  $G(\mathcal{C})$ . The set of inverses of the  $h_b$ 's form the set of morphisms from  $a$  to  $+$ .  $Mor(+, +) (= Aut(+))$  will be  $Aut(\mathcal{Y}/K(\mathcal{C})) = H^+$  from the last section.

For  $a_1, a_2 \in O$ , the set of morphisms between  $a_1$  and  $a_2$  in both  $\mathcal{G}$  and  $\mathcal{G}^+$  is simply  $\{h_{b_2} \circ h_{b_1}^{-1} : f(b_1) = a_1 \text{ and } f(b_2) = a_2\}$ . Finally the set of morphisms between  $+$  and itself in  $\mathcal{G}^+$  is  $Aut(\mathcal{Y}/K(\mathcal{C}))$  as defined earlier.

Note that for  $a_1, a_2 \in O$ ,  $Mor(a_1, a_2)$  is the uniformly (in  $a_1, a_2$ )-definable set of permutations of  $G(\mathcal{C})$  given by left multiplication in  $G(\mathcal{C})$  by the set of  $b_2^{-1}b_1$  for  $b_1 \in f^{-1}(a_1)$  and  $b_2 \in f^{-1}(a_2)$ . It is a right coset (translate) of  $H_{a_1} = Mor(a_1, a_1)$  and a left coset (translate) of  $H_{a_2} = Mor(a_2, a_2)$ . Likewise  $Mor(+, a)$  is a right coset of  $H_+ = Mor(+, +)$  and left coset of  $H_{a_1}$  (working inside  $G$ ). So  $\mathcal{G}$  and  $\mathcal{G}^+$  are groupoids.  $\mathcal{G}$  is clearly (quantifier-free) definable (without parameters) in  $T$ . On the face of it  $\mathcal{G}^+$  is (quantifier-free) definable in  $DFC_K$ . Now  $H^+$  is a  $K$ -definable subgroup of  $G$  in  $\mathcal{U}$ , but  $G$  is not definable in  $\tilde{T}$ . On the other hand  $H^+$  (and the action) is interpretable without parameters in  $\tilde{T}$ , hence  $\mathcal{G}^+$  can be viewed as a definable (without parameters) groupoid in  $\tilde{T}$ . From the very general point of view of [4],  $\mathcal{G}$  represents a “generalized or groupoid imaginary” in  $T$ , and adjoining the canonical object  $+$ , eliminates this imaginary in the enveloping theory  $\tilde{T}$ . Both groupoids we have introduced are connected.

For  $b \in \mathcal{Y}$  the bijection  $h_b$  of  $\mathcal{Y}$  with  $G(\mathcal{C})$  induces an interpretation (with pa-



rameters) of  $\tilde{T}$  in  $T$  (which is the identity on the sort  $\mathcal{C}$ ): For any  $\emptyset$  definable set  $X$  in  $(\mathcal{Y}, \mathcal{C})$ , apply the identity to coordinates in  $\mathcal{C}$  and  $h_b$  to coordinates in  $\mathcal{Y}$  to obtain a definable set, say  $h_b(X)$ , a subset of  $\mathcal{C}^n \times G(\mathcal{C})^m$  which is definable in  $(\mathcal{Y}, \mathcal{C})$  over  $b$ . But then by 1.7 (and the fact that  $C_{K(b)} = C_K(f(b))$ ),  $h_b(X)$  is definable in  $\mathcal{C}$  over  $f(b) = a$  say. Hence the interpretation depends only on  $a = f(b)$  and we write it as  $\omega_a$  (so strictly speaking an interpretation of  $\tilde{T}$  in  $T_a$ ). As both  $\tilde{T}$  and  $T_a$  have quantifier elimination in their respective languages  $\omega_a$  is a “quantifier-free interpretation”: it takes quantifier-free formulas to quantifier-free formulas.

**Remark 4.3.** *The family of interpretations  $\omega_a$  for  $a \in O$  is uniformly definable, in the obvious sense.*

## 5 Substructures

We point out how various notions above such as groupoids and interpretations, behave under passing to substructures, and prove some key lemmas.

We start with the groupoid  $\mathcal{G}$ ,  $\emptyset$ -quantifier-free, definable in  $T$ . Let  $C$  be a substructure of a model of  $T$ , i.e. a field containing  $C_K$ . Then  $\mathcal{G}(C)$  denotes the interpretation in  $C$  (equivalently the restriction to  $C$ ) of  $\mathcal{G}$ . This means that (i) the set of objects of  $\mathcal{G}(C)$  is  $O(C)$ , those points of  $O$  whose coordinates lie in  $C$ , and (ii) for  $a, b \in O(C)$ , the set of morphisms between  $a, b$  in  $\mathcal{G}(C)$  is precisely  $Mor(a, b)(C)$ . Then

**Lemma 5.1.**  *$\mathcal{G}(C)$  is a (quantifier-free,  $\emptyset$ -definable) groupoid in the structure  $C$ , but is not necessarily connected.*

*Proof.* Obvious: note that if  $a \in O(C)$ ,  $H_a(C)$  is nonempty as it contains at least the identity, but there is no reason a priori that  $Mor(a_1, a_2)(C)$  is nonempty for  $a_1 \neq a_2 \in O(C)$ .  $\square$

The next remark allows us to consider any (small) substructure of a model of  $T$  as a subfield of  $\mathcal{C}$  (over  $C_K$ ) in an unambiguous fashion vis-a-vis  $K$ .

**Remark 5.2.** *If  $C_1$  and  $C_2$  are subfields of  $\mathcal{C}$  containing  $C_K$  and  $i$  is an isomorphism between  $C_1$  and  $C_2$  over  $C_K$ , then  $i$  is an elementary map in the sense of  $DCF_K$  (i.e. in the sense of the structure  $\mathcal{U}$  as a differential field with constant symbols for elements of  $K$ ).*

*Proof.* By 1.7. □

**Definition 5.3.** *By a good substructure  $(C, Y)$  of a model of  $\tilde{T}$  we mean a definably closed substructure such that  $Y$  is nonempty.*

As  $\tilde{T}$  has quantifier elimination, the elementary type of a good substructure  $(C, Y)$  in an given model of  $\tilde{T}$  depends only on the isomorphism type of  $(C, Y)$ .

**Proposition 5.4.** *Fix a (small) substructure  $C$  of a model of  $T$  which by Remark 5.2, we will assume to be a substructure of  $\mathcal{C}$ . There is a natural one-one correspondence between:*

- (a) *the good substructures  $(C, Y)$  of models of  $\tilde{T}$ , in fact of  $(\mathcal{C}, \mathcal{Y})_{L_{\partial, K}}$ , up to isomorphism over  $C$ ,*
- (b) *The strongly normal extensions  $L$  of  $K$  over  $C$ , up to  $K(C)$ -isomorphism (as differential fields),*
- (c) *The set of connected components of the groupoid  $G(C)$  from above.*

*Proof.* We first discuss (a) and (b). If  $(C, Y)$  is a definably closed substructure of  $(\mathcal{C}, \mathcal{Y})$ , then  $Y = bG(C)$  for any  $b \in Y$ , and  $K(C)(b)$  is a strongly normal extension of  $K$  over  $C$  (equal to  $K(C)(b_1)$  for any other  $b_1 \in Y$ ). Conversely if  $L$  is a strongly normal extension of  $K$  over  $C$ , then  $(C, \mathcal{Y}(L))$  is a good substructure whose isomorphism type over  $C$  clearly depends only on the isomorphism type of  $L$  over  $K(C)$ . Suppose  $(C, Y_1)$  and  $(C, Y_2)$  are isomorphic over  $C$ . Let  $b_1 \in Y_1$  and let  $b_2$  be the image of  $b_1$  under such an isomorphism. Then (by QE)  $b_1$  and  $b_2$  have the same type over  $K(C)$  in  $\mathcal{U}$  so the differential fields they generate over  $K(C)$  are isomorphic.

We now discuss (b) and (c). If  $L$  is a strongly normal extension of  $K$  over  $C$  (i.e. a strongly normal extension of  $K(C)$ ) and  $b \in \mathcal{Y}(L)$ , then  $f(b) = a$  is a tuple from  $C = C_L$ , so  $a \in O(C)$ . Conversely if  $a \in O(C)$  and  $b$  is any point of  $\mathcal{Y}$  with  $f(b) = a$  then we can appeal to Lemma 3.3 to see that  $K(C)(b)$  is a strongly normal extension of  $K$  over  $C$ .

Now suppose that  $b_1, b_2 \in \mathcal{Y}$ , with  $a_i = f(b_i) \in G(C)$  for  $i = 1, 2$  and suppose that  $L_1 = K(C)(b_1)$  is isomorphic to  $L_2 = K(C)(b_2)$  over  $K(C)$ . We may assume  $L_1 = L_2$ . But then  $b_2^{-1}b_1 \in G(C)$  and by the definitions in section 4 of the groupoid, is in  $Mor(a_1, a_2)(C)$ . The converse is the same: If  $Mor(a_1, a_2)(C) \neq \emptyset$ , then for some  $b_1, b_2 \in \mathcal{Y}$  such that  $f(b_i) = a_i$ , then  $b_1 = b_2 d$  for some  $d \in G(C)$ . So  $K(C)(b_1) = K(C)(b_2)$ . □

We now come back to the interpretation  $\omega$  of  $\tilde{T}$  in  $ACF_K$  from section 2. The interpretation is quantifier-free, namely takes quantifier-free formulas of the language of  $\tilde{T}$  to quantifier-free formulas in the language of  $ACF_K$ . either by definition or because  $ACF_K$  has quantifier elimination. So  $\omega^*$  acts on substructures: if  $L$  is a subfield of  $\mathcal{U}$  containing  $K$ , then  $\omega^*(L)$  is  $(L, G(L))$  considered as an  $L_{K,\partial}$ -substructure of  $(\mathcal{U}, G)_{L_{\partial,K}}$ . Clearly the isomorphism type of  $\omega^*(L)$  depends only on the isomorphism type of  $L$ . (One can thus consider  $\omega$  as an interpretation of the universal part of  $\tilde{T}$  in the universal part of  $ACF_K$ .)

**Lemma 5.5.** *For any field  $L$  extending  $K$ ,  $\omega^*(L) = (L, G(L))$  is a good substructure (of a model of  $\tilde{T}$ ).*

*Proof.*  $\omega^*(L)$  is definably closed as  $L$  is and also  $G(L)$  is nonempty, as it contains at least the identity. □

We know that  $\omega$  operates on formulas and sentences. Moreover  $\omega$  preserves the quantifier-complexity. Fix an arbitrary theory  $T_1$  in the language of  $ACF_K$ . Then  $\omega^{-1}(T_1)$  is by definition the set of sentences  $\sigma$  in the language of  $\tilde{T}$  such that  $\omega(\sigma) \in T_1$ . So

**Remark 5.6.**  $\omega^{-1}(T_1)$  is (up to logical closure) precisely the set of  $L_{\partial,K}$ -sentences which are true in  $\omega^*(L)$  whenever  $L \models T_1$ .

We will restrict our attention to universal  $T_1$ , in which case  $\omega^{-1}(T_1)$  is also a set of universal sentences, in fact those of the form  $\forall x(\phi(x))$  for those  $L_{\partial,K}$  formulas  $\phi(x)$  such that the sentence  $\forall..(\omega(\phi))$  is in  $T_1$ .

**Proposition 5.7.** *Let  $T_1$  be a universal theory in the language of  $ACF_K$  containing  $(ACF_K)_{\forall}$ . Let  $C$  be a field in between  $C_K$  and  $\mathcal{C}$ . Let  $L$  be a strongly normal extension of  $K$  over  $C$ , and let  $(C, Y)$  be the good substructure corresponding to it by Proposition 5.4. Then  $L$  (as a structure in the language of  $ACF_K$ ) is a model of  $T_1$  if and only if the  $L_{\partial,K}$ -structure  $(C, Y)$  is a model of  $\omega^{-1}(T_1)$ .*

*Proof.* Suppose first that  $L \models T_1$  (and  $C = C_L$ ). We want to show that  $(C, Y) = (C, \mathcal{Y}(L))$  is a model of  $\omega^{-1}(T_1)$ . Now  $(C, Y)$  is an  $L_{\partial,K}$  substructure of  $(\mathcal{U}, G)$ , so an  $L_{\partial,K}$ -substructure of  $(L, G(L))$ . But the latter is a model of the universal theory  $\omega^{-1}(T_1)$ , by 5.6. So therefore is  $(C, Y)$ , as required.

Conversely, suppose  $(C, Y) \models \omega^{-1}(T_1)$ . We may assume that  $(C, Y)$  is an  $L_{\partial, K}$ -substructure of  $(\mathcal{C}, \mathcal{Y})$ , and  $L$  the differential field generated by  $K(C)(Y)$ . Let  $\forall \bar{x}(\chi(\bar{x}))$  be a sentence in  $T_1$ , where  $\chi$  is quantifier-free. Let  $\bar{d}$  be a tuple of the right length from  $L$ . We want to see that  $\models \chi(\bar{d})$ . Now  $\bar{d} = f(\bar{c}, \bar{e})$  where  $\bar{c}$  is a tuple from  $C$ ,  $\bar{e}$  a tuple from  $Y$ , and  $f$  a  $K$ -rational function. Now clearly  $\forall \bar{z}\bar{w}(\chi(f(\bar{z}, \bar{w})))$  is in  $T_1$  (where  $\bar{z}, \bar{w}$  range over suitable varieties). Therefore this sentence, suitably rewritten is in  $\omega^{-1}(T_1)$ . So  $\models \chi(f(\bar{c}, \bar{e}))$ , whereby  $\models \chi(\bar{d})$  as required.  $\square$

We can apply the same ideas to the interpretation  $\omega_a$ . Remember that for  $a \in O$ ,  $\omega_a$  is an interpretation of  $\tilde{T}$  in  $T_a$ . The interpretation directly gives an isomorphic copy of  $(\mathcal{C}, \mathcal{Y})$  as an  $L_{\partial, K}$ -structure (quantifier-free) definable over  $a$  in  $\mathcal{C}$ , which we call  $\omega_a^*(\mathcal{C})$ , namely  $(\mathcal{C}, G(\mathcal{C}))$  with  $L_{\partial, K}$  structure induced by  $(id, h_b)$  for some/any  $b \in \mathcal{Y}$ , and definable in  $\mathcal{C}$  over  $a$ . If  $C$  is a subfield of  $\mathcal{C}$  containing  $C_K$  and  $a$  then  $(C, G(C))$  will be a substructure of the  $L_{\partial, K}$ -structure  $(\mathcal{C}, G(\mathcal{C}))$ , which we call  $\omega_a^*(C)$ . Again the isomorphism type of the  $L_{\partial, K}$ -structure  $(C, G(C))$  depends only on the isomorphism type of  $C$ . So for any field  $C$  containing  $C_K$ , we have the well-defined  $\omega_a^*(C)$ , substructure of a model of  $\tilde{T}$ .

**Remark 5.8.** *Note that  $\omega_a^*(C)$  is a good substructure (of a model of  $\tilde{T}$ ). Moreover it is isomorphic over  $C$  to any good substructure  $(C, Y)$  of  $(\mathcal{C}, \mathcal{Y})$  such that  $Y$  contains some  $b$  such that  $f(b) = a$  (assuming  $C \leq \mathcal{C}$ ).*

**Corollary 5.9.** *Let  $T_1$  be as in Proposition 5.7. Then there is a family  $\Psi_{T_1}(x)$  of universal formulas in the language of  $T$  such that for any field  $C$  containing  $C_K$  and  $a \in O(C)$ ,  $C \models \Psi_{T_1}(a)$  iff (after embedding  $C$  in  $\mathcal{C}$ ) the strongly normal extension  $L$  of  $K$  over  $C$  corresponding to the good substructure  $\omega_a^*(C)$  is a model of  $T_1$ .*

*Proof.* By Proposition 5.7,  $L$  is a model of  $T_1$  iff the  $L_{\partial, K}$ -structure  $\omega_a^*(C)$  is a model of  $\omega^{-1}(T_1)$ . But the latter holds if and only if  $(C, a)$  is a model of  $\{\omega_a(\sigma) : \sigma \in \omega^{-1}(T_1)\}$  and this precisely means that  $a$  realizes  $\Psi_{T_1}(x)$  in  $C$  where  $\Psi_{T_1}(x)$  is the partial type in the language of  $T$  obtained by replacing the parameter  $a$  by  $x$  in the set of sentences  $\{\omega_a(\sigma) : \sigma \in \omega^{-1}(T_1)\}$ . (This last point is easy: we simply express the set  $\omega^{-1}(T_1)$  of sentences asserting something about the  $L_{\partial, K}$ -structure  $(C, G(C))$  which is definable over  $a$  in  $C$  by a corresponding set of formulas of the language of  $T$  asserting something about  $a$  in  $C$ .)  $\square$

## 6 Proofs of Theorems 1.4, 1.5, 1.6

Here we put things together.

We assume throughout that  $C_K$  is existentially closed in  $K$  as a field. Let  $S$  be  $Th(C_K)$  in the language of  $T$  (i.e. in the language of fields with constants for  $C_K$ ). To say  $C_K$  is existentially closed in  $K$  means precisely that  $K$  is a model of  $S_\forall$ , so  $K$  embeds, over  $C_K$ , in some elementary extension  $M$  of  $C_K$ . Fix such  $M$  and  $C_K \leq K \leq M$ , and let  $T_1$  be the universal theory of  $M$  in the language with constant symbols for elements of  $K$ . Let  $S$  be the complete theory of  $M$  in the same language. Note that unless  $S$  has quantifier elimination  $T_1$  depends on the embedding of  $K$ . In any case  $T_1$  contains  $S_\forall$ , so producing a strongly normal extension which is a model of  $T_1$  yields a model of  $S_\forall$ .

We will fix  $\mathcal{R}$ , a saturated model of  $S$  which we can and will assume to be contained in  $\mathcal{C}$ . Let  $\Psi(x) = \Psi_{T_1}(x)$  the the set of formulas in the language of  $S$  from Corollary 5.9.

**Proposition 6.1.**  *$\Psi(x)$  is consistent with  $S$  (in particular realized in  $\mathcal{R}$ ).*

*Proof.* We have fixed an embedding of  $K$  in a model  $M$  of  $S$  (in particular  $M$  is a field containing  $K$ ). So  $M$  is a model of  $T_1$ . Then by 3.6  $\omega^*(M) = (M, G(M))$  with its  $L_{\partial, K}$  structure is a model of  $\omega^{-1}(T_1)$ . Let  $b \in G(M)$ . Then  $f(b) = a \in M$ . At this point we embed  $(M, G(M))$  as a good substructure  $(C, Y)$  of  $(\mathcal{C}, \mathcal{Y})$ . By Proposition 5.7 (and 5.4) the corresponding strongly normal extension of  $K$  over  $C$  is a model of  $T_1$ . By Corollary 5.9,  $a$  realizes  $\Psi(x)$  in  $C$ .  $\square$

Note that with our current notation and assumptions ( $C_K$  existentially closed in  $K$  and  $K \leq M \models Th(C_K)$ , and  $T_1 =$  universal theory of  $M$  in language of  $ACF_K$  we have the following) which contains Theorem 1.4.. Simply choose  $C$  to be either  $\mathcal{R}$  itself, or a small elementary substructure of  $\mathcal{R}$  containing  $C_K$  in which  $\Psi(x)$  is realized, and apply 5.9.

**Proposition 6.2.** *There is an elementary extension  $C$  of  $C_K$  with  $C_K \leq C \leq \mathcal{C}$ , and a strongly normal extension  $L$  of  $K$  over  $C$  which is a model of  $T_1$ , in particular a model of  $S_\forall$ .*

Without additional assumptions we may not be able to choose  $C = C_K$  in Proposition 6.2 But in case we can, we have the following uniqueness statement.  $T_1$  is an the first paragraph of this section.

**Proposition 6.3.** *Suppose That  $L_1$  and  $L_2$  are strongly normal extensions of  $K$ , which are both models of  $T_1$  and moreover have a common embedding into a model of  $T_1$ . Then  $L_1$  and  $L_2$  are isomorphic over  $K$  as differential fields.*

*Proof.* We make use of the “forgetful” interpretation  $\omega$ . In the  $L_{\partial,K}$ -structure  $(\mathcal{C}, \mathcal{Y})$  we have, for  $a_1, a_2 \in \mathcal{O}$  that  $Mor(a_1, a_2)$  is the set of  $h_{b_2} \circ h_{b_1}^{-1}$ , for  $b_1, b_2 \in \mathcal{Y}$ . So the “same” is true in the elementary extension  $(\mathcal{U}, G(\mathcal{U}))$ . We assume from the start that  $L_1, L_2$  are differential subfields of  $\mathcal{U}$ . Let  $b_1, b_2$  be in  $\mathcal{Y}(L_1), \mathcal{Y}(L_2)$  respectively. Let  $a_i = f(b_i)$  for  $i = 1, 2$ . Then  $a_i \in \mathcal{O}(C_K)$  for  $i = 1, 2$ . Now  $b_i \in G(L_i)$  for  $i = 1, 2$ . Now by assumption  $L_1$  and  $L_2$  have a common embedding into a model of  $T_1$  and hence also into a model  $M$  of  $T_2$ . There is no harm assuming that  $K \leq M \leq \mathcal{U}$  (where  $K$  is the given embedding of itself in  $\mathcal{U}$ ) and  $L'_1, L'_2$ , both subfields of  $M$  are the images of  $L_1, L_2$  under the embedding. Then for some  $b'_1 \in G(L'_1)$  and  $b'_2 \in G(L'_2)$  we have  $f(b'_1) = a_1$  and  $f(b'_2) = a_2$ . Hence  $(\mathcal{U}, G(\mathcal{U})) \models h'_{b_2} \circ h'_{b_1} \in Mor(a_1, a_2)$ . But  $h'_{b_2} \circ h'_{b_1} \in G(M)$ . So  $Mor(a_1, a_2)(M) \neq \emptyset$ . As  $C_k$  is an elementary substructure of  $M$ ,  $Mor(a_1, a_2)(C_K) \neq \emptyset$ , so by Proposition 5.4 (correspondence between (b) and (c)),  $L_1$  and  $L_2$  are isomorphic over  $K$  (as differential fields).  $\square$

Theorem 1.5 follows from Proposition 6.3. In the context of the statement of Theorem 1.5, choose  $M$  to be an elementary extension of  $C_K$  containing  $K$  and into which both  $L_1$  and  $L_2$  embed, over  $K$ . Let  $S$  be the theory of  $M$  which names for elements of  $K$  and  $T_1$  its universal part. Now apply 6.3.

To prove Theorem 1.6 we have to know more about  $\mathcal{G}(\mathcal{R})$ , the restriction of  $\mathcal{G}$  to the theory  $S$ , under the additional assumptions on  $C_K$ . We will use the following which should be considered folklore.

**Fact 6.4.** *Suppose  $k$  is a field (of characteristic 0) such that  $k$  has finitely many extensions of degree  $n$  for any  $n$  (i.e.  $k$  has Serre’s property (F)). Let  $G$  be an algebraic group over  $k$ . Working in the theory ACF let  $\mathcal{H}$  be a family of principal homogeneous spaces for  $G$ , each definable over  $k$ , and uniformly so. Then up to isomorphism over  $k$  (as  $G$ -spaces)  $\mathcal{H}$  is finite.*

*Proof.* Note that for  $X, Y \in \mathcal{H}$  the set of  $G$ -isomorphisms between  $X$  and  $Y$  is a  $k$ -definable set, and uniformly so. So  $X$  and  $Y$  are isomorphic over  $k$  just if this set has a  $k$ -rational point. So if the conclusion fails, then we

can find elementary extensions  $K$  of  $k$  such that  $H^1(K, G(K^{alg}))$  is arbitrary large in cardinality. But by the proof of Theorem 1.1 in section 2.3 of [1] there is a bijection between  $H^1(k, G(k^{alg}))$  and  $H^1(K, G(K^{alg}))$ , so we get a contradiction.  $\square$

**Corollary 6.5.** *Suppose  $k$  has property (F). Let  $\mathcal{G}$  be a connected groupoid definable over  $k$  in ACF. Then  $\mathcal{G}(k)$  has finitely many connected components.*

*Proof.* Fix  $a \in O(k)$  and let  $H_a = \text{Mor}(a, a)$ . The family of  $\text{Mor}(a, b)$  for  $b \in O(k)$  is a uniformly definable family of definable over  $k$  torsors for  $H_a$ . By 6.4, there are only finitely many, up to isomorphism over  $k$ . Now if  $\text{Mor}(a, b)$  and  $\text{Mor}(a, c)$  are isomorphic over  $k$  via  $\chi$ , then for any  $h \in \text{Mor}(a, b)$ ,  $\chi(h) \circ h^{-1} \in \text{Mor}(b, c)$  is defined over  $k$ , so  $\text{Mor}(b, c)(k)$  is nonempty. Hence we have finitely many connected components in  $\mathcal{G}(k)$ .  $\square$

We now return to the main context and notation. We can apply compactness as in 3.7 (see also Lemma 2.4 of [4]).

**Lemma 6.6.**  *$\Psi(x)$  is equivalent modulo the theory  $S$  to a single formula  $\psi(x)$ .*

*Proof.* Let  $a_1, \dots, a_k$  be realizations of  $\Psi$  which are representatives of the connected components of  $\mathcal{G}(\mathcal{R})$  which meet  $\Psi$ . So the following holds in  $\mathcal{R}$ : for all  $a$  realizing  $\Psi$  for some  $i = 1, \dots, k$  there is  $x \in \text{Mor}(a, a_i)$ . We can apply compactness to replace  $\Psi(x)$  by some formula  $\psi(x)$ . But then for all  $a$  satisfying  $\psi$  in  $\mathcal{R}$ , there is in  $\mathcal{R}$  an isomorphism between  $a$  and some  $a_i$  hence the strongly normal extension of  $K(\mathcal{R})$  corresponding to  $a$  is isomorphic to that corresponding to  $a_i$ , so is also a model of  $T_1$ . By 5.9  $\psi$  is equivalent to  $\Psi$  in  $S$ .  $\square$

*Proof of Theorem 1.6.* So  $\psi(x)$  is a formula over  $C_K$  consistent with  $S$ , so is realized in  $C_K$ , giving us (by 5.9 and 6.6) a strongly normal extension of  $K$  which is a model of  $T_1$  hence also a model of  $S_{\forall}$ .

## 7 Additional remarks and examples

In the body of this paper we have focused on strongly normal extensions coming from logarithmic differential equations on algebraic groups over the constants (one reason being that this is a direct generalization of equations  $Y' = AY$  on  $GL_n$  which subsume linear differential equations). When the base field  $K$  is algebraically closed these equations completely account for the strongly normal theory (via Kolchin's theory of  $G$ -primitives). But for more general  $K$  one needs to consider logarithmic differential equations on a  $G$  torsor  $V$ , where  $G$  is over  $C_K$  and  $V$  as well as the equation is over  $K$ . This is explained in the last section of Kolchin's book [6] on  $V$ -primitives, which the reader is referred to. But we just want to say that all the results of the paper hold for this more general class of equations, as long as  $V$  has rational points in suitable fields. For example, Theorem 1.3 becomes: Assume  $C_K$  is existentially closed in  $K$  and  $dlog_V(-) = A$  is a logarithmic differential equation on  $V$  over  $K$  where  $G$  is an algebraic group over  $C_K$  and  $V$  a  $G$ -torsor over  $K$ . Assume that  $V(L) \neq \emptyset$  for some elementary extension of  $C_K$  containing  $K$ . Then there is a strongly normal extension of  $K$  for the equation. With similar statements for Theorems 1.4, 1.5, 1.6.

We should also say that the methods of this paper extend to other situations, several derivations, difference equations etc.

We mentioned already that papers [3] and especially [1] make strong and essential use of Tannakian categories as presented in [2]. The first author of the current approach has given a model theoretic account and interpretation of the Tannakian formalism in [5]. The set-up in the current paper is different: we work in a possibly saturated differentially closed field containing the ground field. Nevertheless there are analogies, worth pointing out. Our theory  $\tilde{T} = Th(\mathcal{C}, \mathcal{Y})$  is analogous to the Tannakian category of  $\partial$ -modules over  $K$  generated by the given linear differential equation. Our interpretation  $\omega$  of  $\tilde{T}$  in  $ACF_K$  is analogous to the forgetful functor to the category of vector spaces over  $K$ . Our interpretations  $\omega_a$  of  $\tilde{T}$  in  $T_a$  are analogous to the fibre functors to the category of vector spaces over  $k = C_K$ .

### 7.1 Examples

The following examples illustrates some of the results and arguments in the paper. Let  $C$  be some field of constants,  $K = C(t)$  (where  $t' = 1$ ). First consider the linear differential equation  $y' = y/2t$ . This is already a logarithmic



differential equation on  $GL_1$ , so we seek nonzero solutions. If  $b \in \mathcal{U}$  is such a solution then  $b^2/t = a$  say is a constant, and  $b$  is a solution of the polynomial equation  $z^2 = tc$  over  $C(a)$ . We can take the function  $f$  from 3.7 to be  $f(b) = b^2/t$ , so  $b$  generates a usual Galois extension of  $K(C)$ . The image of  $f$  equals  $O$  is precisely the set of nonzero elements. All the Galois groups  $H^\sharp$  and  $H_a$  coincide with  $\mathbb{Z}/2\mathbb{Z}$ . And for  $a, b \in O$ ,  $Mor(a, b)$  is the solution set of  $az^2 = b$ . Of course there is always a Picard-Vessiot extension over  $K$ , as  $O(C) \neq \emptyset$ . Two such Picard-Vessiot extensions  $K(b_1)$  and  $K(b_2)$  are isomorphic over  $K$  if and only if  $Mor(a_1, a_2)(C) \neq \emptyset$  where  $a_i = b_i^2/t$ , and by above this is equivalent to  $a_1/a_2$  being a square in  $C$ . For the existence problem, the linear differential equation  $y'' = -y$  is a better example. The function  $f$  can be taken to be  $f(b) = b^2 + (b')^2$ . So  $O = Im(f)$  is the set of nonzero elements. Assuming  $C$  to be real closed, and  $K(b)$  is a PV extension of  $K$  then  $K(b)$  is formally real if and only if  $a = f(b)$  is not a square in  $C$ .

## 7.2 Internal covers and interpretations

The model-theoretic ideas underlying our approach above to the issues of existence, uniqueness etc., of strongly normal extensions come from the theory of internality and definable automorphism groups (as mentioned several times above). Some aspects of the context above are very specific to the case at hand, for example the “forgetful” interpretation  $\omega$ . Also aspects of the general theory are simplified by our set-up where the solution set of the relevant equation is already a torsor for an algebraic group in the constants. On the other hand the definable groupoid, the interpretations  $\omega_a$ , and results such as 5.4 are valid in a very general context. So we give a brief account and overview of the general context, leaving more details and elaborations to a future paper. This section may require from the reader a bit more fluency with the language of model theory. But remember that now by definable we mean without parameters..

The general construction of definable groupoids from internality data appears in [4]. We will be elaborating slightly on this material. The notion of “internality” is straightforward: Work in some structure  $M$ , let  $P, Q$  be definable sets. Then  $Q$  is internal to  $P$  if for some tuple  $b$  of parameters from  $M$  there is a  $b$ -definable surjection from some  $P \times \dots \times P$  to  $Q$ . So we obtain a  $b$ -definable equivalence relation  $E$  on  $P^n$  and a  $b$ -definable bijection  $h_b$  between  $P^n/E$  and  $Q$ . When no parameters  $b$  are required, there is nothing more to be said. But otherwise (under a saturation assumption on  $M$ ),

$Aut_M(Q/P)$  the group of permutations of  $Q$  induced by automorphisms of  $M$  fixing  $P$  pointwise, will be nontrivial and type-definable. To put this in a context where we can talk about interpretations, we choose the following set up, which is already a simplification of the most general situation:

$T$  and  $\tilde{T}$  are (possibly many-sorted) complete theories in languages  $L, \tilde{L}$ . We will assume that  $T$  has elimination of imaginaries, equivalently we identify  $T$  with  $T^{eq}$ . We will also assume (after Morleyizing) that both  $T$  and  $\tilde{T}$  have quantifier elimination in their respective languages. We will often identify a first order formula or sort, with the set it defines in some (possibly saturated) model. Assume  $\tilde{L}$  is  $L$  together with a new sort  $Q$  and possibly additional relations. We assume that  $T \subseteq \tilde{T}$ . We take  $N$  to be a saturated model of  $\tilde{T}$ , and let  $M$  be its “reduct” to  $L$ , so a saturated model of  $T$ . In particular the  $L$ -sorts  $P$  in  $M$  are also sorts in  $N$  so we can speak of a subset of  $P$  being definable in  $N$ , with or without parameters.

**Assumption I.**  $T$  is “stably embedded” in  $\tilde{T}$ .

What this means is that any subset  $X$  of a sort  $P$  in  $M$  which is definable in  $N$  is definable in  $M$  ( $M$  has no new induced structure from  $N$ ) but also if  $X \subseteq P$  is definable with parameters in  $N$  then it is definable with parameters in  $M$ . We deduce that if  $X$  is definable in  $N$  over parameter  $b$  then it is definable in  $M$  over  $dcl(b) \cap M$ .

**Assumption II.** In the structure  $N$ ,  $Q$  is internal to some sort  $P$  of  $M$ .

So for some parameter  $b \in N$  we have a definable bijection  $h_b$  between  $Q$  and some set  $R$  definable with parameters in  $M$ . Note that the latter set will be definable in  $M$  over  $dcl(a) \cap M$ .

The following basic result appears in many places, e.g. [10]

**Fact 7.1.** *There is a group  $H^+$  type-definable (without parameters) in  $N$  and a definable (without parameters) in  $N$  action of  $H^+$  on  $Q$ , isomorphic to the action of  $Aut_N(Q/M)$  on  $Q$ .*

According to our assumptions so far,  $\tilde{T}$  is an *internal cover* of  $T$ , using notation from [4].

**Assumption III.**  $H^+$  is definable (and of course the action too).

We can now obtain a definable groupoid  $\mathcal{G}$  (also  $\mathcal{G}^+$ ) associated to the internal cover  $\tilde{T}$  of  $T$ . This will depend on a choice of a definable set  $X$  in  $N$  (or in  $\tilde{T}$ ) which parameterizes some uniformly definable family of bijections between

$Q$  and definable (with parameters) sets in  $M$ . Our assumptions easily imply that such  $X$  exists.  $X$  will be in general definable in  $N^{eq}$ . So we have a uniformly definable family of definable bijections  $h_b$  (for  $b \in X$ ) of bijections between  $Q$  and definable sets  $Q_a$ ,  $a$  varying with  $b$ .

The same ideas as in the proof of Corollary 3.7, together with Assumption III yield:

**Lemma 7.2.** *There is a definable function  $f$  from  $X$  to some sort in  $M$  such that for each  $b \in X$ ,  $tp(b/f(b))$  implies  $tp(b/M)$ .*

It follows that we can rechoose the family  $Q_a$  of images such that  $a = f(b)$ . We may assume that for each  $a \in Im(f)$  the formula  $f(y) = a$  isolates  $tp(b/M)$  for each  $b \in X$  such that  $f(b) = a$ .

Let  $O = Im(f)$  a definable set in  $M$ .  $O$  is the set of objects of  $\mathcal{G}$  and  $O \cup \{+\}$  the set of objects of  $\mathcal{G}^+$ .  $Mor_{\mathcal{G}^+}(+, a)$  is the set of  $h_b$  for  $b \in f^{-1}(a)$  and  $Mor_{\mathcal{G}^+}(a, -)$  is the set of inverses. For  $a_1, a_2 \in O$ ,  $Mor_{\mathcal{G}}(a_1, a_2) = Mor_{\mathcal{G}^+}(a_1, a_2) = \{h_{b_2} \circ h_{b_1}^{-1} : f(b_1) = a_1, f(b_2) = a_2\}$ .  $Mor_{\mathcal{G}^+}(+, +)$  is  $H^+$ . So  $\mathcal{G}$  is definable in  $T$ , and  $\mathcal{G}^+$  in  $\tilde{T}$ . Both are connected. We write  $\mathcal{G}$  as  $\mathcal{G}_X$  to express the dependence on the choice of the definable set  $X$  in  $\tilde{T}$ .

Note that for  $a \in O$ , any of the bijections  $h_b : Q \rightarrow Q_a$  induces an interpretation of  $\tilde{T}$  in  $T_a$ , which we again call  $\omega_a$ .

Before stating the next result we recall the “category” of interpretations. Suppose  $\omega_1, \omega_2$  are interpretations of  $T_1$  in  $T_2$ . A morphism from  $\omega_1$  to  $\omega_2$  is a definable function (or collection of definable functions) in  $T_2$  which yield, for any model  $M$  of  $T_2$  an elementary embedding of  $\omega_1^*(M)$  in  $\omega_2^*(M)$ . If the function has an inverse which is also a morphism we call it an isomorphism. (So an isomorphism between  $\omega_1$  and  $\omega_2$  is a definable function in  $T_2$  which yields an isomorphism between  $\omega_1^*(M)$  and  $\omega_2^*(M)$  for any model  $M$  of  $T_2$ .) Finally we call  $\omega$  a bi-interpretation of  $T_1$  and  $T_2$  if it is an interpretation of  $T_1$  in  $T_2$ , and there is an interpretation  $\rho$  of  $T_2$  in  $T_1$ , such that both interpretations  $\rho \circ \omega$  and  $\omega \circ \rho$  (of  $T_1$  in itself and  $T_2$  in itself) are isomorphic to the identity.

Back in our context, note that the identity  $i$  gives a (tautological) interpretation of  $T$  in  $\tilde{T}$ . For  $B$  a definably closed subset of a model of  $\tilde{T}^{eq}$  (which can be taken to be a definably closed subset of  $N^{eq}$ ), let  $A = B \cap M$ , then the identity induces an interpretation which we call  $i_B$  of  $T_A$  in  $T_B$ , which is “full” in the sense that all the induced structure on  $M$  coming from parameters  $B$  in  $N$  comes from parameters  $A$  in the structure  $M$ . In analogy with

Section 5 we may want to call  $B$  a “good” substructure” of a model of  $\tilde{T}^{eq}$  if  $i_B$  is a bi-interpretation. It is not hard to see that  $i_B$  is a bi-interpretation just if,  $N \subseteq dcl(B, M)$ .

Let us now fix a definably closed subset  $A$  of  $M$ . Let  $Hom_T(\tilde{T}, T_A)$  be the category whose objects are interpretations of  $\tilde{T}$  in  $T_A$  whose restriction to  $T$  is the identity, and whose morphisms are isomorphisms of interpretations whose restriction to  $T$  is the identity.

Let  $\omega \in Hom_T(\tilde{T}, T_A)$ . Then for  $M$  a model of  $T$  containing  $A$  we obtain  $\omega^*(M)$ , a model of  $\tilde{T}$ , living inside  $M$ . By  $\omega^*(A)$  we mean the (definably closed) substructure  $\omega^*(M) \cap A$  of  $\omega^*(M)$ . By QE of the theories  $T$  and  $\tilde{T}$   $\omega^*(A)$  depends only on  $A$ .

We have the following general version of Proposition 5.4.

**Proposition 7.3.** *Let  $\mathcal{G} = \mathcal{G}_X$  be the groupoid, definable in  $T$ , defined above. Let  $A$  be a definably closed subset of a model of  $T$ . Then there is equivalence of categories between  $\mathcal{G}(A)$  and the full subcategory of  $Hom_T(\tilde{T}, T_A)$  whose objects are the interpretations  $\omega$  such that  $X(\omega^*(A)) \neq \emptyset$*

*Proof.* Let  $r = r_A$  be the map from  $\mathcal{G}(A)$  to  $Hom_T(\tilde{T}, T_A)$ , defined by: for  $a \in O(A)$ ,  $r(a) = \omega_a$ , and acting naturally on morphisms. (Note that  $\omega_a$  is an interpretation of  $\tilde{T}$  in  $T_a$  so also in  $T_A$ .)

We restrict ourselves to showing that  $X(\omega_a^*(A)) \neq \emptyset$  whenever  $a \in O(A)$ . (The rest follows easily.) Now the interpretation  $\omega_a$  is induced (by definition) from some (any) bijection between  $Q$  and  $Q_a$  of the form  $h_b$ , for  $b \in X$ . Now the isomorphism (over  $A$ ) between  $N$  and  $\omega_a^*(M)$  induced by  $h_b$  takes  $h_b$  itself to a bijection between  $Q_a$  and itself which is the identity  $1 \in H_a$ . So the latter is a point of  $X(\omega_a^*(A))$ . □

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