Nash groups

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- This will also be an opportunity to discuss the model theory of the structure $(\mathbb{R}, +, \times)$, real algebraic groups, Lie groups, and the mutual interactions.
- This is in principle an old problem, that I began thinking about in the mid 1980's.
- But for various reasons, including psychological, it seems to be currently accessible.

Model theory I

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- A *definable set* in M is simply a subset of $M \times .. \times M$ "defined by" such an expression.

Model theory II

- For example a *real algebraic variety* $X \subseteq \mathbb{R}^n$ is (by definition) a set defined by a finite system of polynomial equations $P(x_1, .., x_n) = 0$ over $\mathbb{R}$ so is definable in the structure $(\mathbb{R}.+, \times)$.

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- The real exponential function is not semialgebraic.

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- So the original motivation was: describe the definable groups in $(\mathbb{R}, +, \times)$, equivalently semialgebraic groups, up to definable isomorphism.
- Note that any real algebraic group (both the underlying set and group operation are given by polynomials) is semialgebraic.
- On the face of it a semialgebraic group need not be a topological group: the interval $[0, 1)$ with group operation $+$ modulo 1, is a definable group, but is not, at least naively, a topological group.

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- Firstly if U is an open semialgebraic subset of $\mathbb{R}^n$, then a function $f : U \to \mathbb{R}$ is said to be Nash if f is both analytic and semialgebraic, which essentially means that there is a polynomial $P(\bar{x}, y)$ such that $P(\bar{x}, f(\bar{x})) = 0$ for all $\bar{x} \in U$. (e.g. $f(x) = +\sqrt{x}$ on $(0, \infty)$).

Nash manifolds II

- An (n-dimensional) Nash manifold is a Hausdorff topological space X with a covering by *finitely many* open sets V_i, each homeomorphic via some f_i to an open semialgebraic subset U_i of $\mathbb{R}^n$ such that the transition maps $f_i \circ f_j^{-1}$ between the open semialgebraic sets $f_j(V_i \cap V_j)$ and $f_i(V_i \cap V_j)$ are Nash (i.e each coordinate is Nash).

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- By a *locally Nash manifold* we mean as above, except that we allow the possibility of infinitely many V_i. And again this is intuitively a "locally definable" or "ind-definable" real analytic manifold.
- There is a natural notion of a Nash map between Nash manifolds, and a Nash group is a Nash manifold X with group operation given by a Nash map $X \times X \to X$ (i.e. a group object in the category of Nash manifolds).

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- Somewhat paradoxically real projective space is affine as a Nash manifold, via stereographic projection, whereby any real nonsingular quasiprojective variety is affine as a Nash manifold.
- Conversely it was proved that any affine Nash manifold is Nash isomorphic to a nonsingular real algebraic variety.
- $\mathbb{R}/\mathbb{Z}$ has naturally the structure of a Nash manifold: $[0, 1]$ with $0, 1$ identified, but it is non affine. In general little is known about arbitrary (not necessarily affine) Nash manifolds.

Nash groups I

We mention some early results:

Theorem 0.1

(P 1986) Any semialgebraic group G can be semialgebraically equipped with the structure of a Nash group (in particular a Lie group) unique up to Nash isomorphism. The category of semialgebraic groups coincides with that of Nash groups.

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Theorem 0.2

(Hrushovski-P, 1994, with model theory proof) Any Nash group G is locally, Nash isomorphic to a real algebraic group H in the sense that there is Nash isomorphism between open semialgebraic neighbourhoods of the identity of G and of H.

Nash groups II

- Earlier, in the late 80's Madden-Stanton had noted the one-dimensional case of Theorem 0.2 and used it to classify one-dimensional Nash groups (included in our main result, to be stated later). In particular there are many distinct incarnations of the 1-dimensional compact Lie group S^1 as Nash groups. Examples....

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- We used Theorem 0.2 to prove the following (Hrushovski-P, 1994, with corrected proof 2012):

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Note this already takes us out of the category of algebraic groups: the proper finite covers of $SL(2, \mathbb{R})$ are neither real algebraic nor linear.

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Corollary 0.5

The locally Nash groups are, up to local Nash isomorphism precisely the quotients of universal covers of real algebraic groups by discrete subgroups.

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- (ii) Classify the resulting Nash groups up to Nash isomorphism.
- We will answer question (i) in the commutative case, and give some examples showing some subtleties around (ii) compared to the one-dimensional case.

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- This passes to universal covers: in particular we have $0 \to L \to \tilde{G} \to_{\pi} \tilde{C} \to 0$.

Result

We have, with above notation the solution to (i) in the commutative case:

Theorem 0.6

(P-Starchenko, 2013) Let G be a commutative real algebraic group, and let Γ be a discrete subgroup of $\tilde{G}$ (maybe trivial). Then $\tilde{G}/\Gamma$ has a (compatible) structure of Nash group just if $\tilde{C}/\pi(\Gamma)$ is compact.

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- The universal cover of either $SO(2,\mathbb{R})$ or a real abelian variety does not have the compatible structure of a Nash group (a point that seems to have been overlooked by Madden-Stanton).
- $\tilde{G}/\Gamma$ has structure of Nash (definable) group if and only if $\tilde{G}$ has a "definable fundamental region" for Γ.

Fundamental region I

- A fundamental region Ω of $\tilde{G}$ for Γ is simply a neighbourhood Ω of 0 in $\tilde{G}$ such that every coset of Γ in $\tilde{G}$ meets Ω and both Ω and $\Omega + \Omega$ contain only finitely many elements of Γ.

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- The choice of a definable fundamental region in $\tilde{G}$ for Γ determines a Nash or definable group structure on $\tilde{G}/\Gamma$ (in the obvious way; explain).

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- The choice of a definable fundamental region in $\tilde{G}$ for Γ determines a Nash or definable group structure on $\tilde{G}/\Gamma$ (in the obvious way; explain).
- The interesting new phenomena, compared with the one-dimensional case, is that different choices of definable fundamental region may give non isomorphic Nash group structures on the same locally Nash group $\tilde{G}/\Gamma$.

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- $\Omega_1 = \{(x, y) : x \in \mathbb{R}, 1 \leq y \leq 2\}$.
- $\Omega_2 = \{(x, y) : 0 \leq x \leq 1, y \in \mathbb{R}_{>0}\}$.
- For the first choice L/Γ contains a definable copy of $(\mathbb{R}, +)$ but not of $(R_{>0}, \times)$.
- For the second choice it is the other way round, so they cannot be Nash isomorphic.