# LMS Prospects in Mathematics: Logic and Model Theory.

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December 19, 2012

# Logic I

- Modern mathematical logic developed at the end of the 19th and beginning of the 20th centuries with the so-called foundational crisis or crises.
- There was a greater interest in mathematical rigour, and a concern whether reasoning involving certain infinite quantities was sound.
- In addition to logicians such as Cantor, Frege, Russell, major mathematicians of the time such as Hilbert and Poincaré participated in these developments.
- Out of all of this came the beginnings of mathematical accounts of higher level or "metamathematical" notions such as set, truth, proof, and algorithm (or effective procedure).

# Logic II

- It is interesting that these four items are still at the base of the main areas of mathematical logic: set theory, model theory, proof theory, and recursion theory, respectively.
- One of the greatest recursion theorists was Alan Turing, founder, in a sense, of the modern computer, whose anniversary was celebrated this year.
- These areas of logic are still concerned with foundational questions, such as the search for new axioms for sets.
- But there are also close connections that have developed between logic and other areas. For example proof theory has close connections to category theory, computer science, and more recently homotopy theory (from topology).
- And set theory has close connections with combinatorics and analysis among other things. I will discuss model theory later.

# UK activity

- Set theory has a strong but relatively small representation in the UK, with research going on at East Anglia and Bristol.
- Recursion theory is currently only pursued in Leeds.
- There is research in proof theory (in the broad sense) in Cambridge and Leeds, but also in many Computer Science departments throughout the UK.
- Model theory is well-represented in the UK with strong research groups in East Anglia, Leeds, Manchester, Oxford, and London (Queen Mary).
- Please contact me at pillay@maths.leeds.ac.uk if you would like some more information information, in particular names of people, references for the material below...

# Model theory and definability I

- I want to say something about some trends in model theory, which are related in various ways to my own current research.
- What is model theory? I would personally define it it to be the "study and classification of first order theories", but there is a lot to handle here and it tends to emphasize the purer aspect of the subject.
- Others might define it as the study of "definability" in structures and classes of structures, which is maybe more appropriate for the applied aspects of the subject, and connects up with "truth".
- By an (abstract) structure we mean simply a set $X$ equipped with a collection of relations $R$ on $X$ (i.e. subsets of $X, X \times X$,..) and functions $f$ from $X \times .. \times X \to X$. Written $(X, R_1, R_2, .., f_1, f_2, ..)$

# Model theory and definability II

- This is not particularly controversial and has nothing to do with logic per se. A group $(G, \cdot)$ is a structure, as is any ring, field, ordered set $(X, <)$, or graph $(X, R)$.
- Our number systems $(\mathbb{N}, +, \cdot, 0)$, $(\mathbb{Z}, +, \cdot, ..)$, $(\mathbb{R}, +, \cdot, 0, 1)$, $(\mathbb{C}, +, \cdot, ..)$ are familiar structures.
- An area called "universal algebra" studies abstract structures $(X, f_1, f_2, ...)$, and among other things, and solutions of systems of "equations" $f_1(\bar{x}) = f_2(\bar{x})$ in $X^n$. Trying in the process to place results from usual algebra (groups, rings,..) in a more abstract or general setting.
- Model theory studies, more generally, solution sets in an abstract structure $(X, .....)$, of "first order formulas", and such solution sets are what we call *definable sets* in the relevant structure.

# Model theory and definability III

- If $(G, \cdot)$ is a group, and $a \in G$ then the collection of elements of $G$ which commute with $a$ is the solution set of an "equation", $x \cdot a = a \cdot x$.
- However $Z(G)$, the centre of $G$, which is the collection of elements of $G$ which commute with every element of $G$, is "defined by" the first order formula $\forall y (x \cdot y = y \cdot x)$.
- In the structure $(\mathbb{R}, +, \cdot, -)$ the ordering $x \leq y$ is defined by the first order formula $\exists z (y - x = z^2)$.
- The expression "first order" means that in addition to equations we allow quantifiers $\forall$ and $\exists$ which range over individuals of the structure at hand.
- Our familiar number systems already provide quite different behaviour or features of definable sets.

# Model theory and definability IV

- In the structure $(\mathbb{N}, +, \times, 0)$, subsets of $\mathbb{N}$ definable by formulas $\phi(x)$ which begin with a sequence of quantifiers $\exists y_1 \forall y_2 \exists y_3 ... \forall y_n$ get more complicated as $n$ increases.
- The collection of definable subsets of $\mathbb{N}$ is called the arithmetic hierarchy, and already with one existential quantifier we can define "noncomputable" sets. (Negative solution to Hilbert 10.)
- Whereas in the structure $(\mathbb{R}, +, \cdot)$, the hierarchy collapses, one only needs one block of existential quantifiers to define definable sets. Moreover the definable sets have a geometric feature: they are the so-called semialgebraic sets.
- I.E. Finite unions of subsets of $\mathbb{R}^n$ of form $\{\bar{x} : f(\bar{x}) = 0 \wedge \bigwedge_{i=1,..k} g_i(\bar{x}) > 0\}$ where $f$ and the $g_i$ are polynomials with coefficients from $\mathbb{R}$.

# $O$-minimality I

- It follows from the description above that the definable sets in $(\mathbb{R}, +, \cdot)$ (i.e. semialgebraic sets) have good topological properties (if you know what that means): such as, any definable subset of $\mathbb{R}^n$ has finitely many connected components.
- Equivalently (for nontrivial reasons),
  (*) every definable subset of $\mathbb{R}$ is a finite union of intervals and points.
- In the early to mid 80's, we took (*) as a *definition* of an $o$-minimal structure on $\mathbb{R}$.
- Namely, let $(\mathbb{R}, +, \cdot, f_1, f_2, ..)$ be a structure. We call the structure $o$-minimal if (*) holds (for sets definable in the sense of the structure $(\mathbb{R}, +, \cdot, f_1, f_2, ..)$).

# $O$-minimality II

- This has been called "tame topology" by van den Dries. For example weird examples such as space filling curves can not be definable in an $o$-minimal structure.
- A big industry in $o$-minimality has developed over the years, including $o$-minimal economics. Theorems concerning analysis and topology are proved under a general assumption of $o$-minimality.
- Moreover structures such as $(\mathbb{R}, +, \cdot, exp)$, where $exp$ is the real exponential function, were proved to be $o$-minimal. (One of the big all time theorems in model theory, proved by Wilkie).
- Recently there have been surprising applications of $o$-minimality to diophantine geometry, which will finish my talk.

# Diophantine Geometry I

- We will talk about the field $\mathbb{C}$ of complex numbers, the "spaces" $\mathbb{C}^n$ and subsets $V$ of $\mathbb{C}^n$ which are solution sets of finite systems of polynomial equations $P(x_1, .., x_n) = 0$ (where $P$ has complex coefficients).
- This is already maybe a bit abstract. But nevertheless such point sets $V$ are called *algebraic varieties*, and the business of algebraic geometry is to try to describe and classify them.
- Note that such an algebraic variety is a definable set in the structure $(\mathbb{C}, +, \cdot)$.
- If the polynomials have coefficients from $\mathbb{Z}$ or $\mathbb{Q}$, or even if not, it is natural to consider points of $V$ whose coordinates are in $\mathbb{Q}$ (or in some number field).

# Diophantine Geometry II

- We call such points *rational points* of $V$, and diophantine geometry is about the structure and number of rational points on algebraic varieties (which is a fancy way of saying number and structure of solutions in $\mathbb{Q}^n$ of systems of polynomial equations over $\mathbb{Q}$).
- A key theme is trying to describe those varieties which have *many* rational points. (If $V$ is an algebraic curve defined by a single polynomial equation $P(x_1, x_2) = 0$ then "many" means just "infinitely many".) Mordell conjecture. Faltings.
- We will consider a variant of this problem, which is thematically: if $V$ has many special points then $V$ is special (where of course the definitions of special are crucial).

# Diophantine Geometry III

- "Special case": Let $\mathbb{C}^*$ be $\mathbb{C} \setminus 0$ (which is a group under complex multiplication). A torsion point on $\mathbb{C}^*$ is a point $x$ such that $x^m = 1$ for some $m$ (which in particular lies on the unit circle).
- $(\mathbb{C}^*)^n$ (Cartesian product) is also a group, and a torsion point of $(\mathbb{C}^*)^n$ is just an $n$-tuple $(x_1, .., x_n)$ of torsion points of $C^*$.
- In this context a "special point" in $(\mathbb{C}^*)^n$ is a torsion point, and a special subvariety $V$ of $(\mathbb{C}^*)^n$ is one defined by a finite system of equations of the form $x_1^{k_1} x_2^{k_2} ... x_n^{k_n} = 1$ (i.e. $V$ is a subgroup of $(\mathbb{C}^*)^n$).
- So in this set-up: Theorem. If $V$ has many special points then $V$ is special.

# Diophantine Geometry IV

- The Theorem is well-known, but it was noticed (Zannier) that results (Pila-Wilkie) in $o$-minimality could be used to give a new proof, and since then new results in the thematic context have been obtained by $o$-minimal methods.
- Let me mention finally how $o$-minimality and definability are related to the Theorem above.
- First a theorem of Pila-Wilkie that if $(\mathbb{R}, +, \cdot, f_1, f_2, ..)$ is an $o$-minimal structure, and $X \subseteq \mathbb{R}^n$ is definable, then almost all, in an asymptotic sense, rational points in $X$ are contained in positive-dimensional semialgebraic subsets of $X$.
- Secondly the fact that $(\mathbb{R}, +, \cdot, f, g)$ is an $o$-minimal structure, where $f, g$ are the restriction of $sin, cos$ respectively to any bounded interval.

# Diophantine Geometry V

- And moreover that (identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ in the usual way) the map $f(x_1, .., x_n) = (cos(2\pi x_1), sin(2\pi x_1), ..., cos(2\pi x_n), sin(2\pi x_n))$ has the property that the torsion points of $(\mathbb{C}^*)^n$ are in the image of $f$ on $[0, 1]^n \cap \mathbb{Q}^n$.
- A third ingredient is a result on transcendence or algebraic independence properties of the exponentials of rational functions, due to Ax. But it goes a bit beyond the scope of this talk, so I will leave it at that.
- Thanks.