Model theory

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1 Introduction

Contemporary or modern (mathematical) logic was born at the end of the 19th century. Its origin is connected with mathematics rather than philosophy, and my article will likewise be informed by a mathematical culture although I will try make connections with philosophy and the philosophy of mathematics. Although mathematical logic emanates from a so-called Western intellectual tradition, it is now, like mathematics as a whole, a world subject with no essential national or cultural distinguishing marks.

Unfortunately I am not knowledgeable about philosophical and (early) mathematical traditions in the Indian subcontinent, so will not be able to make any serious comparative analyses. Also I am not trying here to give a proper history of model theory with appropriate references, bibliography, credits etc., but rather a description of how I see the subject now, with some minor commentary on historical developments. Also I will only be able to give a hint of the main technical notions and definitions in the subject. So I will point the reader towards a few basic texts, reviews, and historical accounts of the subject, where more details and as well as a detailed bibliography can be found, such as Hodges' textbook and history [3], [4] and Marker's textbook [5]. Another survey [7] by myself contains more technical details than the current article, and my book [6] from 1996 contains an exhaustive technical treatment of some of the themes I will discuss, but assuming a prior

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acquaintance with model theory. The volume [2] is a good reflection of the state of model theory around the beginning of the modern era (1971). It also contains an informative historical article by Vaught on model theory up to 1945. Finally the book [1] gives a readable account of some of the machinery behind one of the major modern successes of the applications of model theory (mentioned at the end of Section 6).

Among the strands in the early history of logic were identifications of correct standard forms of argument (the syllogisms) but also, with Gottfried Leibniz, the rather bold idea that one might in principle be able to settle all disputes by mechanical logical means. These were complemented by considerations of the nature of mathematical truths compared to empirical truths (e.g. Kant), as well as the beginnings of the mathematicization of logic (e.g. Boole).

So "logic" here is supposed to refer to intrinsic reasoning or truths, independent of experience. For example the statement that a thing is equal to itself is a truth of logic rather than experience, although philosophers such as Hegel (and also I guess many Indian philosophers) have commented on the vacuity and even conditionality of such truths. Likewise the fact that from "P implies Q", and P, we can deduce Q, is supposed to be valid on purely logical grounds, independent of which statements P and Q denote.

Rather than try to base all knowledge on logic, Frege and Russell, among others, attempted to show that all or at least major parts of *mathematical knowledge* can be founded on logic. Once one starts to investigate seriously such claims, one is forced to define one's terms, and find a formal framework within which to carry out the project. And this, in a sense, was behind the birth of modern logic. But another crucial factor was that dominant mathematicians of the time, such as Hilbert and Poincaré, were very caught up in "foundational" problems, not only around whether mathematics could be reduced to logic, but also about the justifications of the use of "infinitistic" methods and objects, outside the scope of normal intuition. As it turned out Gödel's work in the 1930's showed that not only the Frege-Russell-Whitehead project, but also a "second level" program of Hilbert to "reduce" infinitistic to finitistic methods, were doomed.

In spite of this failure of the logicist and Hilbert programs, the efforts of these late 19th and early 20th century logicians left a lasting impact on mathematics (and also philosophy). Firstly set theory as a universal language for mathematics was largely accepted (even though not all mathematical truths could be settled on the basis of accepted axioms about sets), and this contributed towards the possibility of mathematicians from different subdisciplines being able, at least in principle, to communicate in a precise and effective with each other. And of course the search for additional axioms for sets led to the rich subject of contemporary set theory. Moreover the "defining of one's terms" issue mentioned above led to precise mathematical treatments of notions such as truth, proof, and algorithm. It is interesting that model theory (truth), proof theory (proof) and recursion theory (algorithm), together with set theory, remain the four principal and distinct areas of contemporary mathematical logic. In any case modern mathematics, its language, and unity, are closely bound up with logic, although paradoxically logic has been somewhat marginalized within contemporary mathematics. Nevertheless, mathematical logic is now undoubtedly regarded as a bona fide part of mathematics and the various areas and subareas have their own internal programs and aims, which are continually being modified. But one can ask to what extent these investigations can have impacts on mathematics as a whole, as was the case at the beginning of the 20th century. I will try to convey something both of the "inner movement" of model theory, as well as its actual and potential wider impacts. To read this article profitably will require some mathematical background, but as mentioned above I will try to comment on the "philosophical" content and impact too.

2 Truth

The notion *truth in a structure* is at the centre of model theory. This is often credited to Tarski under the name "Tarski's theory of truth". But this "relative", rather than absolute, notion of truth was, as I understand it, already something known, used, and discussed. In any case, faced with the expression "truth in a structure" there are two elements to be grasped. Truth of what? And what precisely is a structure? An illuminating historical example concerns the independence of Euclid's "axiom of parallels" from his other axioms. A statement equivalent to this axiom of parallels is

(AP): given any line ℓ and point p not on ℓ there is exactly one line through p which is parallel to (does not intersect) ℓ.

The independence statement is that (AP) is *not* a logical consequence of a certain collection $\mathcal{A}$ of other axioms involving points and lines (such as that any two distinct points lie on a unique line). This was shown by finding a "model" of the set $\mathcal{A}$ of axioms in which moreover the statement (AP) is

false. The kinds of things here that are (or are not) true are statements such as (AP) or the statements (axioms) from $\mathcal{A}$. And the relevant structure or "model" consists of one collection P of objects which we call "points", another collection L of objects, called "lines", and a relation I of "incidence" between points and lines, thought of as saying that p lies on ℓ. Note that (AP) can be expressed, in a somewhat convoluted manner, as follows:
(*) for any p and for any ℓ such that *not* $pI\ell$, [there is ℓ' such that ($pI\ell'$ and it is not the case that there exists p' such that $p'I\ell$ and $p'I\ell'$) and for any ℓ'' such that $pI\ell''$ and it is not the case that there exists p' such that $p'I\ell''$ and $p'I\ell''$, $\ell'' = \ell'$].
So the structure constructed (a model of non Euclidean geometry) was one where the statements in $\mathcal{A}$ are true and the above statement (*) is *false*.
Already there is a considerable degree of abstraction in my presentation. The intuitive geometric notions of point and line are replaced by purely formal sets and relations. This is a typical example of a *structure* in the sense of model theory, logic, or universal algebra (or even Bourbaki), namely a universe of objects, together with certain relations between them. In the example the objects come in two sorts, "points" and "lines" and the only relation is I. Moreover statements such as (*) above, have a rather definite logical form. They involve the basic "variables" p, ℓ, as well as expressions (logical connectives) such as "and", "not", "for all", "there exists", as well as "equality". To check the truth or falsity of such an axiom in a structure, the "for all" and "there exists" connectives should range over objects in the structure at hand, and it is this kind of proviso which typifies "truth in a structure" as opposed to "absolute truth".
So at the basic level, model theory is concerned with two kinds of things, structures and formal sentences (or statements), as well as the relation (truth or falsity of a sentence in a structure) between them. Traditionally the expressions *syntax* (for formal statements) and *semantics* (for the interpretation of sentences in structures) were a popular way of describing model theory. The formal sentences in the example above belong to what is called *first order logic*, because the *for all*, and *there exists* expressions (or quantifiers) range over objects or elements of the underlying set of the structure (rather than subsets of the underlying sets for example). Higher order and/or infinitary logic involve quantifying over subsets or subsets of the set of subsets etc, and/or infinitely long sentences or expressions. There are also other variants, involving cardinality or probability quantifiers for example. These higher order or infinitary logics were extremely popular in the 1960's and 1970's, and

are still the subject of substantial research. However we will, in this article, concentrate on the first order case.

So, summarizing, a *structure* M is a set X equipped with some distinguished family $\mathcal{R}$ of *relations* on X, namely subsets of $X, X \times X$, $X \times X \times X$ etc. We also allow a family $\mathcal{F}$ of distinguished functions from $X \times X \times ... \times X$ to X. There are two typical kinds of examples. First of a combinatorial nature such as *graphs*. A graph is a set X (of "vertices") equipped with a binary relation $R \subset X \times X$, representing adjacency. Secondly, the structures of algebra, such as groups, rings, fields etc. For example a group is a set X equipped with a function $m : X \times X \to X$ satisfying the group axioms (associativity, and existence of an identity and inverses). Corresponding to a structure M is a formal first order language $L(M)$ within which one can express properties which may or may not be true in the structure M. For example, in the case of graphs the property that every element is adjacent to another element can be expressed by:

for all x there is y different from x such that $R(x, y)$,

or more formally

$$\forall x \exists y (x \neq y \wedge R(x, y))$$

Likewise in the case of groups the basic group axioms can be expressed in a first order manner, and by definition a group is a structure (with a single distinguished binary function) in which these axioms are true.

Commonly the notion that a (formal) sentence σ is true in a structure M, is also expressed by saying that M is a *model* of σ, as discussed at the beginning of this section. The formal notation is $M \models \sigma$.

What is called a *theory* in logic is some collection of sentences belonging to some first order language. An example of such is $Th(M)$ for M a given structure, namely the collection of all sentences in $L(M)$ which are true in M.

If M and N are structures for a common first order language (for example M, N are both graphs) it makes sense to ask whether M and N are *isomorphic*, meaning that there is a bijection between the underlying sets X, Y say of these structures which interchanges the distinguished relations. Being isomorphic means being the same to all intents and purposes. A weaker notion is *elementarily equivalence* meaning that any first order sentence true in M is true in N (and vice versa). The question of when elementarily equivalent implies isomorphic is a pervasive problem in model theory which will be discussed subsequently.

I mentioned at the beginning of this paragraph the idea that "truth in a structure" is a kind of relative rather than absolute truth. However I should make it clear that this is neither a notion of "truth in a possible world", nor "truth relative to a point of view", nor "approximate truth", although model-theoretic tools have been used to explore these latter notions.

3 Decidability

I want to distinguish at the beginning between those first order *theories* which I will call *foundational* and those which I will call *tame*. The foundational theories (such as the accepted axioms of set theory in the language with a "membership relation") are those which purport to describe all or large chunks of mathematics, and are connected to the origin of modern logic as described in section 1. Gödel proved that in general such foundational theories are *undecidable*. Namely there is no algorithm to decide whether or not a given (formal) statement, is or is not a consequence of the axioms. Among the important foundational theories is $Th(\mathbb{N})$ where the structure $\mathbb{N}$ consists of the set of natural numbers equipped with addition and multiplication. Undecidability of $Th(\mathbb{N})$ amounts to there being no algorithm or effective method for deciding which (first order) statements about $\mathbb{N}$ are *true*. The proof of this rests on Gödel's insight that arithmetic, namely the structure $\mathbb{N}$, is rich enough to represent reasoning and computation in a "first order" manner. So for example any effective procedure for deciding which first order statements or sentences are true in M would yield an effective procedure for deciding whether or not for any given computing device and any given input, there is a well-define output (which is known to be impossible). At the opposite end of the spectrum are the "tame" theories and/or structures, which are as a rule decidable. A typical example is real plane geometry. The real plane $P = \mathbb{R}^2$ is just a flat surface, as usually understood, stretching to infinity in all directions. The relevant structure has two sorts of objects, the set P of points of the plane, and the set L of straight lines in P, equipped with a single relation $I(p, \ell)$ expressing that the point p is on the line ℓ. It is a fact that the structure $M = (P, L, I)$ is decidable. Already one sees a distinction between "geometry", represented by the structure M, and arithmetic, represented by the structure $\mathbb{N}$. In addition to the real numbers there are other number systems which belong to geometry, such as the complex numbers and the p-adic numbers. And again the number systems themselves (fields), or plane geometry over those number fields, are decidable structures.

The distinction between "foundational" and "tame" theories is heuristic rather than mathematically precise. But model theory does have a number of precise notions other than decidability, which separate these classes of theories, and more generally provide other meaningful dividing lines between classes of first order theories and structures. Contemporary model theory has tended to concentrate on the tame region of mathematics, although exploration of the borderline or middle ground between tame and foundational is a fascinating topic.

4 Foundations revisited

As mentioned in the introduction the two main programs to build mathematics on, or recover mathematics from, logic, namely axiomatic or set-theoretic (Frege, Russell, Whitehead), and proof-theoretic (Hilbert), failed. But as one might expect, these programs have been preserved or resurrected in more modest fashions. The proof theory/set theory/recursion theory nexus has been the main environment for such endeavours. One of the popular programs is what is called "reverse mathematics", developed by Harvey Friedman and Steve Simpson among others. To go into detail here would be be too technical for the mature of this article. But briefly the idea is to recover certain parts of mathematics from certain parts of logic (and vice versa) at the level of theorems and axioms. The logical environment here is what is called *second order arithmetic*, although it is actually a first order theory. The kind of axioms considered are *set existence* axioms of a logical nature. It was recognized rather early that theorems of mathematics, such as the existence of solutions of differential equations, depend on such logical axioms of various levels of strength. The point of reverse mathematics is that often one can in turn derive the logical axiom from the mathematical theorem. So here the strength or content of an axiom of logic is expressed by an accepted theorem of mathematics. This gives a new sense in which logic explains mathematics, mathematics is recovered from logic, or even logic is recovered from mathematics. This subject of reverse mathematics has not been uncontroversial, but nevertheless the subject has had a pervasive influence around the proofs/sets/computability side of mathematical logic. One of the things I want to discuss is a kind of reverse mathematics at the level rather of logical properties and mathematical objects. The logical properties will come from model theory, and the mathematical objects from some basic kinds of geometry. The whole relationship will exist within "tame" mathematics, far from the foundational theories discussed earlier. This "model-theoretic" reverse mathematics was the creation of Boris Zilber. But there are a couple of provisos. First the relationships between logical properties and geometry, were just conjectural. Secondly these conjectured relationships turned out to be false. In the next section I will describe this model-theoretic reverse mathematics.

5 Categoricity

A natural property of a structure M for a first order language L is *categoricity*, which means that whenever N is elementarily equivalent to M then in fact N is isomorphic to M. Namely M is completely determined by the first order sentences which are true in M. Unfortunately (or fortunately) because of the *compactness theorem* of first order logic, a structure M will be categorical if and only if it (or rather its underlying set) is *finite*. (The compactness theorem states that a set Σ of first order sentences has a model if and only if every finite subset of Σ has a model.) As model theory typically deals with infinite structures, the next best thing is the notion of categoricity with respect to a cardinal number. So here some acquaintance with basic set theory, cardinal numbers and ordinal numbers, is required. The smallest infinite cardinal is $\aleph_0$ the cardinality of the set of natural numbers. The next bigger after that is $\aleph_1$. The cardinal numbers are all of the form $\aleph_\alpha$ for some ordinal α. As soon as M is infinite, there will (by the compactness theorem) be structures elementarily equivalent to M of any infinite cardinality. For κ an (infinite) cardinal, we will say that the structure M is κ-categorical if whenever M_1, M_2 are structures elementarily equivalent to M, both of cardinality κ, then M_1 and M_2 are isomorphic. By definition the property that M is κ-categorical, is a property of the first order theory $Th(M)$ of M.

It turns out that the case when $\kappa = \aleph_0$ is very special and in some sense a "singularity". The study of $\aleph_0$-categorical structures is equivalent (by considering automorphism groups) to the study of a certain class of infinite permutation groups, often called "oligomorphic" permutation groups. The model-theoretically more interesting notion is κ-categoricity, for *uncountable* κ, namely $\kappa > \aleph_0$ (or $\kappa = \aleph_\alpha$ for $\alpha > 0$). In this context we have the celebrated theorem of Michael Morley that a structure M is κ-categorical for *some* uncountable κ just if M is κ-categorical for *any* uncountable κ. Bearing in mind Morley's Theorem we use the expression *M is uncountably categorical* for "M is κ-categorical for some uncountable κ".

One of the key "number systems" in mathematics is the field $\mathbb{C}$ of complex numbers. We view this as a structure $(\mathbb{C}, +, \times)$ namely $\mathbb{C}$ equipped with addition and multiplication as distinguished functions. For different reasons related to *definability* which will be discussed later, this structure is sometimes identified with the subject *algebraic geometry*, the study of sets of solutions of systems of polynomial equations. What is relevant to our current discussion is that $(\mathbb{C}, +, \times)$ is an uncountably categorical structure: any structure elementarily equivalent to it will be an *algebraically closed field of characteristic* 0, $(F, +, \times)$, the isomorphism type of which is determined by its transcendence degree, which coincides with its cardinality if F is uncountable.

Another basic example of an uncountably categorical structure is a *vector space* V over a countable field F. The structure is $(V, +, \{f_r : r \in F\})$ where $f_r : V \to V$ is scalar multiplication by r. The structures elementarily equivalent to this are precisely the vector spaces over F, each of which is determined by its F-dimensions, which again agrees with its cardinality in the uncountable case. Again for definability reasons, this structure (or class of structures) is sometimes identified with linear geometry over F (sets of solutions of linear equations).

A third basic example is the set $\mathbb{Z}$ of integers (positive AND negative) equipped with the successor function f which takes x to $x + 1$. $Th(\mathbb{Z}, f)$ contains the information that the underlying set is infinite and that f is a bijection such that for each n $f^n(x) \neq x$ for all x (where f^n denotes f iterated n times). We leave it to the reader to check that again this structure is uncountably categorical. One can not really see any natural geometry attached to this structure.

The thrust of what came to be called Zilber's conjecture was that, in a technical sense which I do not want to go in to now, the above three structures (or rather their theories) are the *only* examples of uncountably categorical structures. So Zilber's conjecture was saying that some very fundamental structures of mathematics can be characterized by logic, namely through the notion of uncountable categoricity of their first order theory, and so in a sense this class of structures is"implicitly defined" by logic. This conjecture and in fact the general point of view giving rise to it, presents another possible fundamental relationship between logic and mathematics.

Zilber's conjecture turned out to be false. Ehud Hrushovski, in the late 1980's, found a combinatorial method for constructing new uncountably categorical structures which do not fit into the three cases described above. For now let us say that the first example above (the complex field) has a model-theoretic property called *nonmodularity*, the second example has a property *modularity and nontriviality* and the third a property *triviality*. What Hrushovski's examples gave were nonmodular structures which were not "essentially" algebraically closed fields. Zilber has since attempted to preserve at least the spirit of his original conjecture by trying to show that these new examples of Hrushovski also have a geometric origin and correspond to some classical mathematical objects. But what for me is more interesting is the fact that the original Zilber conjecture is valid in a range of very interesting and rich contexts, and carries with it new insights as well as analogies between different parts of mathematics. Some such examples will be discussed below.

6 Definable sets

An interest in the *definable sets* in a structure M has always been present in model theory. But since the 1980's the study of definable sets has moved to centre stage in the subject. In section 2, I introduced and discussed the notion σ is true in M, notationally, $M \models \sigma$, where M is a structure for a language L and σ is a first order sentence of that language. In particular I mentioned the sentence

$$\forall x \exists y (x \neq y \wedge R(x, y))$$

in the language of graphs expressing that every element is "adjacent" to another element.
However consider the expression

$$\exists y (x \neq y \wedge R(x, y))$$

which I will denote $\phi(x)$. It does not really make sense to ask whether this expression is true or false in a structure $M = (X, R)$, because it depends on what x refers to. But it *does* make sense to ask, given a structure M together with an element $a \in X$, if $\phi(x)$ is *true of* a in M, which in this specific case means to ask whether a is adjacent to some other element in M. We write

$M \models \phi(a)$ to mean that $\phi(x)$ is *true of* a in M. The set of such a, is a typical example of a *definable set* in M. The expression $\phi(x)$ above is called a *first order formula*, and x is called a *free variable* in the expression, because it is not controlled or quantified by a "for all" or "there exists". Likewise we can speak of formulas $\psi(x_1, x_2, .., x_n)$ of a first order language L, in any number of free variables. If M is a structure for such as language, then the set defined by ψ in M is, by definition:

$$\{(a_1, .., a_n) \in M^n : M \models \psi(a_1, .., a_n)\}$$

Sets as above, which are collections of finite tuples of the underlying set of the structure M, are precisely what we call *definable sets* in the structure M. There is a natural way of saying that a map (or function) between two definable sets is definable. Hence from a structure M we obtain a *category* $Def(M)$, the category of definable sets in M.

It has become useful to think of definability in a "geometric" rather than "combinatorial" way. For example consider the circle with centre $(0, 0)$ and radius 1 in the real plane. It is defined in the structure $(\mathbb{R}, +, \times, 0, 1)$ by the formula

$$\phi(x_1, x_2) : x_1^2 + x_2^2 = 1$$

Note that the formula $\phi(x_1, x_2)$ does not contain any "for all" or "there exists". It is a *quantifier-free* formula. On the other hand the formula

$$\psi(x) : (\exists y)(x = y^2)$$

does have a quantifier, and moreover defines, in $(\mathbb{R}, +, \times, 0, 1)$ the set of nonnegative elements of $\mathbb{R}$.

Many structures M of a "tame" nature often have a "quantifier-elimination" property that definable sets in the structure can be defined by formulas with not so many quantifiers ("for all", "exists"). This enables one to get a handle on $Def(M)$. In the case of $(\mathbb{C}, +\times, 0, 1)$ there is a full quantifier elimination, in the sense that all definable sets are defined without quantifiers. The consequence is that $Def((\mathbb{C}, +, \times, 0, 1))$ is "essentially" the category of "complex algebraic varieties". In the case of $(\mathbb{R}, +, \times, 0, 1)$ there is a relative quantifier elimination yielding that the category of definable sets is precisely the category of "semialgebraic" sets. Each of these categories (algebraic varieties, semialgebraic varieties) corresponds to a whole subject area of mathematics. These quantifier elimination results are associated with Abraham Robinson and Alfred Tarski. Moreover in the case of $(\mathbb{R}, +, \times, 0, 1)$ the relative quantifier elimination result lies at the foundations of *semialgebraic geometry*.

The (so far undefined) properties of nonmodularity, triviality, etc. from section 5, can be expressed or seen in the behaviour of definable families of definable sets. For example nonmodularity of $(\mathbb{C}, +, \times)$ is seen via the 2-dimensional family of lines in $\mathbb{C} \times \mathbb{C}$ (a 2-dimensional definable family of definable 1-dimensional subsets of $\mathbb{C} \times C$). Among other rich mathematical structures M where $Def(M)$ is tractable, are *differentially closed fields*, and *compact complex manifolds* (proved by Robinson, and Zilber, respectively) The mathematical sophistication increases here. But in both these cases the Zilber conjecture from section 5 is true, in suitable senses. Moreover, without going into definitions and extreme technicalities, the property of "nonmodularity" has definite mathematical meaning and consequences in these examples. Differentially closed fields are "tame" structures appropriate for or relevant to the study of ordinary differential equations in regions in the complex plane. Definable sets are essentially solution sets of differential equations. And the property of nonmodularity (of a definable set) is related to the complete integrability of the corresponding differential equation. For compact complex manifolds, a definable set is essentially a compact complex analytic variety, and nonmodularity is related to it being "algebraic" (biholomorphic to a complex algebraic variety).

Among the celebrated applications of model theory to other parts of mathematics is Hrushovski's proof of a certain "number theoretic-algebraic geometric" conjecture, the Mordell-Lang conjecture for function fields of positive characteristic, which makes essential use of the validity of the Zilber conjecture in "separably closed fields" (as well as using other model-theoretic techniques).

7 Miscellanea

What I have given so far is a discussion of a few themes in contemporary model theory, influenced by my own preoccupations. Here I will attempt to rectify the balance, mentioning other trends and themes (some of which are also close to my own work and interests).

Well into the 1970's it was a pretty common belief within the mathematical logic community that model theory consisted essentially of a collection of tools and techniques related to the fundamental notions of semantics and syntax, possibly enhanced by a few basic theorems. (This may also be suggested by the previous sections of the present paper.) In spite of the strength of logic and model theory in the Soviet Union, Poland, and other countries in Eastern and Western Europe, it must be said that in the 1950's and 1960's the (emerging) subject was dominated by two schools, one around Alfred Tarski in Berkeley, and the other one around Abraham Robinson in Yale. Both stressed the potential applications of model theory within other parts of mathematics (although we should note that already in 1940 the Soviet logician Malt'sev was applying the compactness theorem to obtain results in group theory). In the case of Robinson the intention was very clearly reflected in his pioneering work around nonstandard analysis, the theory of model companions, and applications to complex analysis, among other things. It was a little less clear what Tarski had in mind, in spite of his early and fundamental work on definable sets in the field of reals. But undoubtedly the group around Tarski, including Vaught, Morley and Keisler, set the stage for later developments in "pure" model theory. The 1960's and 1970's also saw a close relationship developing between model theory and set theory, with for example an intense investigation of infinitary and/or non first order logics, where Tarski and his group had a major influence. In fact around this time the conventional wisdom was that the future of model theory lay in its connection with set theory, in spite of Morley's work on categoricity (from the mid 1960's). It was Saharon Shelah who, building on the work of Morley, showed that (first order) model theory could be a subject with its own coherent and internal program. With the benefit of hindsight I would say that he raised the question of whether there could be a meaningful classification of first order theories (not explicitly involving decidability properties). Shelah tended to look for dividing lines among first order theories, as well as "test questions" which would be answered one way on one side of the dividing line and another way on the other side. The test questions which Shelah asked typically had a strong set-theoretic content, possibly resulting from the surrounding mathematical culture and influences (such as Tarski). One such test question, coming naturally out of Morley's work, was, for a given theory T, what could be the function $I(T, -)$ which for a cardinal κ gives the number of models of T of cardinality κ, up to isomorphism. Shelah's investigation (and solution) of this problem involved a series of dividing lines among first order theories, the first of which was "stable versus unstable". The property of "stability" for a first order theory T (or structure M) vastly generalizes the property of uncountable categoricity from section 5. A rough definition of stability of T is that no linear ordering is definable on any infinite set in a model of the theory T (so the real field $(\mathbb{R}, +, \times)$ is *unstable*). Shelah and other model theorists developed a considerable machinery for constructing structures, classifying structures, and also studying and classifying definable sets in structures, under a general assumption of stability. This is called stability theory. Although Zilber's conjectures were not originally formulated within the generality of stable theories, it is stable theories that provide the right environment for these notions. The integration of these different points of view is often called geometric stability theory (or even geometric model theory).

There are two conclusions to this part of the story. Firstly, that in spite of the heavily set theoretic appearance of Shelah's work in model theory (up to and including the present) it actually has a strong geometric content with amazing mathematical insights. Secondly, it is now uncontroversial that model theory exists as a subject in and for itself, and part and parcel of the subject is its strong connections to other parts of mathematics.

An important and respected tradition in model theory, to which both Robinson and Tarski contributed seriously, and which is already referred to above, is the model-theoretic and logical analysis of specific concrete structures and theories. But the issue is which notions or "bits of theory" are guiding the analysis. Decidability and quantifier elimination were historically major such notions. Valued fields have been studied logically for a long time. Again the mathematical sophistication increases here, but I just want to comment that fields equipped with a valuation are another context in which"infinitesimals" appear in mathematics. The work of Ax-Kochen-Ershov on the first order theory of Henselian valued fields (late 60's), followed by Macintyre's quantifier elimination theorem for the field $\mathbb{Q}_p$ of p-adic numbers (mid 70's) represented and led to another major interaction of model theory with algebraic geometry and number theory. More recently, this logical analysis of valued fields has been increasingly informed by notions from stability theory, even though the structures under discussion are unstable.

In the early to mid 1980's, several model theorists (including myself) tried to develop a theory, analogous to stability theory, based on abstracting definability properties in the *unstable* structure $(\mathbb{R}, +, \times)$. This came to be called o-minimality. This has been another successful area with close contacts to real analytic geometry. But even here, the connection with stability theory has recently turned out to be much more than an "analogy".

I have restricted myself in the bulk of this article to first order logic and model theory, where the syntax is of a restricted form. But more general logics, involving infinitely long expressions, and/or quantifiers other than "there exists" and "for all", continue to be investigated. At the same time, "finite model theory", the study of the connection between semantics and syntax when we restrict ourselves to finite structures, has seen a fast development and is now integrated into computer science.

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