Strongly minimal pseudofinite structures

Anand Pillay

October 16, 2014

In this note we point out that any strongly minimal pseudofinite structure (or set) is unimodular in the sense of [1], [5], [2], and hence measurable in the sense of Macpherson and Steinhorn [3], [2] as well as 1-based. The argument, involving nonstandard finite cardinalities, is straightforward. A few people asked about this issue in private conversations and communications, in particular Martin Bays - Pierre Simon, Dugald Macpherson - Charles Steinhorn (in MSRI, spring 2014), and more recently Alex Kruckman. So we thought it worthwhile to clarify the situation with a quick proof. Thanks to all the above people for discussions.

Recall the basic notions. A structure M is in language L is pseudofinite if every sentence true in M is true in some finite L-structure. Equivalently M is elementarily equivalent to an ultraproduct of finite L-structures. If M is pseudofinite and saturated say, then every definable set X in M has a "nonstandard finite cardinality" $|X|$ which is an element of a saturated elementary extension of $(\mathbb{N}, +, \times, <, ...)$, and the map taking X to $|X|$ satisfies the usual properties inherited from the finite setting.

Suppose $D = M$ is strongly minimal and saturated. D is said to be unimodular if whenever $a = (a_1, .., a_n)$ and $b = (b_1, .., b_n)$ are each independent n-tuiples from D and $a \in acl(b)$ (so also $b \in acl(a)$) then $mlt(a/b) = mlt(b/a)$.

Definable means possibly with parameters. We refer to [5] for basics of stability, Morley rank ($RM(-)$) etc.

Lemma 0.1. *Suppose D is strongly minimal, saturated and pseudofinite. Let X be a definable set in D. Let $b = |D|$. Then there is a polynomial $P_X(x)$ in one variable x with (standard) integer coefficients and positive leading coefficient, such that $|X| = P(b)$. Moreover $RM(X)$ equals the degree of $P_X(x)$.*

Proof. This is the main point and has maybe been observed before, although I have not seen anything. We prove the Lemma by induction on $RM(X)$ also using the fact that D^n has Morley rank n and Morley degree 1. If X is finite, then $|X| = |X|$. Suppose $RM(X) = n$ and $X \subseteq D^m$ (for some $m \geq n$). After writing X as a finite disjoint union of suitable definable sets, we may assume (using the induction hypothesis) that for some projection $\pi : D^m \to D^n$, and some positive integer t, $\pi(X)$ has Morley rank n and $\pi|X$ is t-to-one. So $|X| = t|\pi(X)|$. And $|\pi(X)| = |D^n| - |D^n \setminus \pi(X)|$. Now $|D^n| = b^n$, and $RM(D^n) \setminus \pi(X)$ has Morley rank $< n$. So we can apply the induction hypothesis to get the desired $P_X(x)$ and note that the leading coefficient of P_X is $t > 0$. $\square$

Now there are a few ways to proceed. We could use the pair $(RM(X), t_X)$ where t_X is the leading coefficient of P_X to show directly MS-measurability of D. Or directly obtain unimodularity. We will do the latter.

Corollary 0.2. *Supppse D is strongly minimal and pseudofinite. Then D is unimodular.*

Proof. We may assume D is saturated. Let $a, b \in D^n$ each be generic over $\emptyset$ with $acl(a) = acl(b)$. Let $k = mlt(b/a)$ and $\ell = mlt(a/b)$. We have to prove that $k = \ell$. Let $\psi(x, y)$ be an L-formula such that $\models \phi(a, b)$, $\psi(a, y)$ isolates $tp(b/a)$ and $\psi(x, b)$ isolates $tp(a/b)$. Let $\phi_1(x)$ be $\exists^{=k} y(\psi(x, y))$ and $\phi_2(y)$ be $\exists^{=\ell} x(\psi(x, y)$. Let $\chi(x, y)$ be the formula $\phi(x, y) \wedge \phi_1(x) \wedge \phi_2(y)$. So $\chi(x, y)$ is true of (a, b) in D. Let $Z \subseteq D^{2n}$ be the set defined by $\chi(x, y)$. We compute $|Z|$ in two ways. Let X be the projection of Z on the first n-coordinates, and Y the projection of Z on the last n coordinates. Then $|Z| = k|X| = \ell|Y|$. Note that other X and Y have Morley rank n hence by Lemma 0.1, there are polynomials $P(x)$, $Q(x)$ over $\mathbb{Z}$ of degree $< n$ such that $|X| = b^n - P(b)$ and $|Y| = b^n - Q(b)$. If by way of contradiction $k > \ell$ we have $(k - \ell)(b^n) = kP(b) - \ell Q(b)$. This is impossible, as the right hand side is an integral polynomial ofdegree $< n$ in b, for example by considering sufficiently large standard natural numbers b. So the Corollary is proved. $\square$

Remark 0.3. *(i) One can deduce by standard means that any pseudofinite theory of finite U-rank (i.e. every complete type has finite U-rank) is 1-based. See the proof of Proposition 3.5 in [2] for example. In particular all definable groups in such a theory are abelian-by-finite.*

(ii) There are examples of ω-stable non abelian-by-finite pseudofinite groups in [4].
(iii) We would tentatively conjecture that any regular type in a stable pseudofinite theory is locally modular???

References

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