Logical Completeness, Form, and Content: an archaeology

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A thesis that is inconceivable now seemed obvious in the ninth century, and it somehow endured until the fourteenth century. Nominalism, which was formerly the novelty of a few, encompasses everyone today; its victory is so vast and fundamental that its name is unnecessary. No one says that he is a nominalist, because nobody is anything else. But we must try to understand that for the people of the Middle Ages reality was not men but humanity, not the individuals but mankind, not the species but the genus, not the genera but God.[1]

1. *Les Mots et les choses*

The "signifying function" of words in the sixteenth century, according to Foucault, depends not upon our acquaintance with them or with their use, "but with the very language of things" (*Foucault 1970*, p. 59). For its inhabitants,

> [t]here is no difference between marks and words in the sense that there is between observation and accepted authority, or between verifiable fact and tradition. The process is everywhere the same: that of the sign and its likeness, and this is why nature and the world can intertwine with one another to infinity, forming, for those who can read it, one vast single text. (p. 34)

Duret's (*1613*) etymology is essentialist: "Thus the stork, so greatly lauded for its charity towards its father and mother, is called in Hebrew *chasida*, which is to say, meek, charitable, endowed with pity ..." (p. 36). And conversely, the nature of a thing is prescribed by its formal features: its name, its shape. That ingesting a walnut will relieve a headache is written into the fruit, in its resemblance with the human brain (p. 26). "The names of things were lodged in the things they designated, just as strength is written in the body of the lion, regality in the eye of the eagle, just as the influence of the planets is marked upon the brows of men ..." (p. 36). The link does not require proof, is in fact closed off from critique. Aldrovandi, Grégoire, all the thinkers of the Renaissance, assumed such associations prior to their investigations. They were "meticulously contemplating a nature which was, from top to bottom, written," for they inhabited "an unbroken tissue of words and signs, of accounts and characters, of discourse and forms" (p. 40). By the end of the Renaissance, the framework is laid out before the gaze that it directs. Crollius (*1624*) is able to write, "Is it not true that all herbs, plants, trees, and other things issuing from the bowels of the earth are so many magic books and signs?"

[1] Jorge Luis Borges, "From allegories to novels."

(p. 26). But one cannot fix one's spectacles while wearing them, and the "prose of the world" cannot literally be questioned.

In the seventeenth century, we are told, things are different. Language is believed to be arbitrary, its relationship to the world contingent on the details of its fallible design and conventional use. "As a result," Foucault urges, "the entire *episteme* of Western culture found its fundamental arrangements modified. And, in particular, the empirical domain which sixteenth century man saw as a complex of kinships, resemblances, and affinities, and in which language and things were endlessly interwoven—this whole vast field was to take on a new configuration" (p. 54). When each sign had been assumed to be true to the world it resembled—when the world consisted entirely of signs—we faced the possibly endless task of reading all its naturally endowed features in order to know all that it expressed. But now each sign can be known in full and is, in fact, completely known, for it is an artifact precisely as the world is not. We can no longer wonder whether we have fully grasped the sign, for there is nothing in it that we did not put there. We are instead in doubt about what before could not meaningfully be questioned: whether our systems of signs adequately fit the world.

> The simultaneously endless and closed, full and tautological world of resemblance now finds itself dissociated and, as it were, split down the middle: on the one side, we shall find the signs that have become tools of analysis, marks of identity and difference, principles whereby things can be reduced to order, keys for taxonomy; and, on the other, the empirical and murmuring resemblance of things, that unreacting similitude that lies beneath thought and furnishes the infinite raw material for divisions and distributions. On the one hand, the general theory of signs, divisions, and classifications; on the other, the problem of immediate resemblances, of the spontaneous movement of the imagination, or nature's repetitions. And between the two, the new forms of knowledge that occupy the area opened up by this new split. (p. 58)

Foucault illustrates one form of knowledge made possible by this rupture. Don Quixote, he says, "wanders off on his own" in this newly dug chasm, between the written word and the world of things (p. 48). Whereas before, "[i]n the sixteenth century, the fundamental supposition was that of a total system of correspondence"—this system underlying and making possible the investigation of the world, shielding itself thereby from both the need and the possibility of proof—"[f]rom now on, every resemblance must be subjected to proof by comparison ..." (p. 55). For Cervantes, "[e]ach exploit must be a proof" (p. 47).

What Foucault is able to recover are two fundamentally opposed stances towards truth itself, each an independent *réseau de nécessités* simultaneously constraining thought and providing for its possibility. But the sets of assumptions are fundamentally incommensurable, he feels, so that the hypothesis of a line of reasoning leading, through ingenious inventions, from the one to the other is hopelessly naive. Borges acknowledges that "the passage ... from realism to nominalism required several centuries," but, humorously, he identifies a single moment at the end of that progression at which the passage occurred: "That day in 1382 when Geoffrey Chaucer, who perhaps did not believe he was a nominalist, wished to translate a line from Boccaccio into English, *E con gli occulti ferri i Tradimenti*[2], and he said it like this: 'The smyler with the knyf under the cloke.' " To illustrate the later passage from the Renaissance to the Classical distinction between language and world, Foucault deploys no such comical device. He belabors Borges's punchline:

> Establishing discontinuities is not an easy task even for history in general. And it is certainly even less so for the history of thought. We may wish to draw a dividing-line; but any limit we set will perhaps be no more than an arbitrary division made in a constantly mobile whole. We may wish to mark off a period; but have we the right to establish symmetrical breaks at two points in time in order to give an appearance of continuity and unity to the system we place between them? Where, in that case, would the cause of the existence lie? Or that of its subsequent disappearance and fall? What rule could it be obeying by both its existence and its disappearance? If it contains a principle of coherence within itself, whence could come the foreign element capable of rebutting it? Generally speaking, what does it mean, no longer to be able to think a certain thought? Or to introduce a new thought? (p. 50)

Foucault identifies Cervantes as the first narrator of a new set of assumptions about language and the world. For him "the written word and things no longer resemble one another," and for this reason *Don Quixote* is the first work of modern literature (p. 48). But he also points to Descartes, to Bacon, to the *Logique de Port-Royal*, each in a different way assisting language in its "withdrawal from the midst of beings themselves" as it "entered a period of transparency and neutrality" (p. 56). There is no single author of the break with sixteenth century thought. Observations accumulate until, as Dewey put it, through their combined stress on the basic assumptions that made them possible, "the cabinets of science break of their own dead weight" (*1896*, p. 96) The breakage makes possible other observations, which alternatively exacerbate the rupture or help shape a new basic way of thinking.

Foucault's "archaeology" of the categories of thought is sweeping and provocative. But when the question about the adequacy of language to the world recurs in modern logic's concrete terms, the patterns of reconfiguration Foucault described are recognizable. Logic's metamorphosis from philosophical topic to mathematical subject was enabled by the identification of proofs with texts and the scientific study of tangible inscriptions. The written word can be surveyed, its properties observed. Questions of right reasoning, even of truth, can now be submitted to combinatorial tests. Logic is in its "period of transparency and neutrality" whereas before it had centered around the dependencies of thoughts and propositions on one another (*Ibid.*). But to legitimize this scientific turn one must show that modern logical systems really do encode the rules

[2] "And treachery with hidden weapons."

of inference and criteria of truth that they leave behind in their "withdrawal from the midst of beings themselves" (*Ibid.*).

Kurt Gödel answered this question for a fragment of logical language when he proved, in his *1929* thesis, that "the restricted functional calculus" suffices to prove all universally valid first-order formulas. In *1930* Gödel said that the question he answered arises immediately for anyone who sets out to study this calculus (p. 103). Whether some such system could be shown to be "complete" in this sense had been explicitly posed over a decade before. Its analogue was proved for a system of propositional logic around the same time: by Paul Bernays in his *1918 Habilitationsschrift* and again in *1921* by Emil Post. In 1967 Gödel reflected on the fact that the technical achievements underlying his completeness theorem were present already in the work of Herbrand and Skolem, although neither writer managed to see that the result was essentially in his hands, and wrote to Hao Wang, "The blindness . . . of logicians is indeed suprising" (*Wang 1996*, p. 240).

Gödel attributed the "blindness" of his predecessors to their lacking the appropriate attitude "toward metamathematics and toward non-finitary reasoning" (Ibid.). Others have suggested that the principal obstacle in the way to the completeness theorem was a failure to appreciate the value of restricting one's attention to first-order quantification. "When Frege passes from first-order logic to a higher-order logic," van Heijenort (*1967*) writes, "there is hardly a ripple" (p. 24). Moore emphasizes instead the "failure" on the part of Gödel's contemporaries "to distinguish clearly between syntax and semantics" (*1988*, p. 126). While each of these accounts points to a substantial aspect of Gödel's characteristic approach to the study of logic, it is here suggested that "logical blindness" is reciprocatingl. Gödel never considered that others' logical vision might be, rather than defective, simply different—that their inability to see their way to the completeness theorem derived from their focus being held elsewhere. But thinking about logic in the terms that defined Gödel's contribution was not universal, perhaps not even common, in the early twentieth century. His early writing plays a major part in an implicit argument that the correspondence of proof and truth, of logical form and content, is a proper way of thinking about logical completeness. While the theorem contained in Gödel's thesis is a cornerstone of modern logic, its far more sweeping and significant impact is the fact that, through its position in a network of technical results and applications, the way of thinking underlying the result has come to seem definitive and necessary, to the extent that we have managed to forget that it has not always been with us.

The details of three separate notions of logical completeness, their histories, and the technical problems that they point to, should illustrate the contingency of the familiar conception of logical completeness as the correspondence of form and content. That conception, the one Gödel helped to drive into our basic understanding of logic, will be seen to have originated alongside a second one based on a coordination of analytic and synthetic reasoning. Each of these modern conceptions can, moreover, be traced back to a common ancestor, a question about the relationship between the metaphysical dependence of truths on one another and the subjective lines of reasoning that lead from judgement to judgement. Far from diminishing the significance of Gödel's achievements, this new vision should showcase the profound conceptual clarification and reconfiguration of fundamental notions that operate between the lines of Gödel's thesis. Gödel's work is revolutionary not because of what questions he answered so much as because he managed, in answering them, to demonstrate the value of a particular way of asking them.

2. Bolzano's question

Bernard Bolzano engaged in the profound study of two distinct notions of logical consequence over several decades in the early nineteenth century. The work most remembered and highly regarded by modern logicians, because of its striking resemblance to twentieth century set-theoretical definitions of consequence, concerns the *Ableitbarkeit* ("derivability") relation. In his 1837 masterpiece, *Wissenschaftslehre*, Bolzano in fact defines a network of concepts—validity, compatibility, equivalence, and derivability—in terms of one another in a way very similar to contemporary presentations. Here is his definition of the last of these:

> Let us then first consider the case that there is a relation among the compatible propositions A, B, C, D, $\ldots$ M, N, O, $\ldots$ such that all the ideas that make a certain section of these propositions true, namely A, B, C, D, $\ldots$, when substituted for i, j, $\ldots$ also have the property of making some other section of them, namely M, N, O, $\ldots$ true. The special relationship between propositions A, B, C, D, $\ldots$ on the one side and propositions M, N, O, $\ldots$ on the other which we conceive of in this way will already be very much worthy of attention because it puts us in the position, in so far as we once know it to be present, to be able to obtain immediately from the known truth of A, B, C, D, $\ldots$ the truth of M, N, O, $\ldots$ as well. Consequently I give the relationship which subsists between propositions A, B, C, D, $\ldots$ on the one hand and propositions M, N, O, $\ldots$ on the other the title, a relationship of *derivability* [*Ableitbarkeit*]. And I say that propositions M, N, O, $\ldots$ would be derivable from propositions A, B, C, D, $\ldots$ with respect to the variables i, j, $\ldots$, if every set of ideas which makes A, B, C, D, $\ldots$ all true when substituted for i, j, $\ldots$ also makes M, N, O, $\ldots$ all true. (§155)

Although this notion of derivability prefigures modern definitions of logical consequence in many ways, there are several evident disparities between Bolzano's concept and our own. For one thing, Bolzano requires all the propositions involved in the *Ableitbarkeit* relationship to be "compatible" with one another. The result is analogous to a stipulation, absent from modern logical theory, that formulas be jointly-satisfiable in order to stand in a relationship of logical consequence with one another. One result of this unfamiliar requirement is that, for Bolzano, nothing at all is derivable from a self-contradictory proposition, whereas in modern logical theory all formulas are consequences of an unsatisfiable one. It is also noteworthy that Bolzano attends to propositions, not formulas, and to their reinterpretations over a fixed domain of ideas.[3] This is a more conservative approach to modality than the modern one, wherein not only may the extensions of predicate symbols and constant symbols vary, but so too may the underlying set of objects. Furthermore, the individuation of "ideas" with respect to which one may vary one's interpretation is made imprecise by the focus on "propositions" and their constituents in place of the modern focus on formulas and the symbols they contain. Bolzano's stance on these matters, though it appears peculiar from a modern point of view, was not whimsical. He maintained his position consistently over many years. However, the modern notion is not conceptually distant from Bolzano's on these points and can meaningfully be seen as a refinement or adjustment of his definition.

Nevertheless, a strong contrast must be drawn between Bolzano's *Ableitbarkeit* relation and the modern notion of logical consequence, if not in terms of their technical details, in terms of the sort of relationship their authors took themselves to be defining. The logical consequences of a formula, on the modern view of things, are solely determined by the existence and details of certain set-theoretical structures, quite independently of our access to them or ability to draw inferences based on them. Bolzano's *Ableitbarkeit* relation, by contrast, is procedural: There is nothing "out there" over and above particular deductions that we might perform that determines any special relationship between the propositions that bear this relation to one another. In Bolzano's preferred terminology, the relationship corresponds to no "objective dependence" of propositions on one another (*Bolzano 1810*, II.§12). It is merely the case that *we* are able "to obtain immediately from the known truth of A, B, C, D, $\ldots$ the truth of M, N, O, $\ldots$ as well" and furthermore are able to do this in an *a priori* manner. When he wrote (*1837*, §155), "[fo]r the sake of variety, I shall also sometimes say that propositions M, N, O, $\ldots$ *follow from* or can be *inferred* or *concluded* from the set of propositions A, B, C, D, $\ldots$," he did not explain the phenomenon of inferring correctly in terms of a metaphysical relationship between propositions that our inferences might track. He treated right reasoning as primitive, the variation of ideas in propositions as part of the inferential process.

Of course, Bolzano was not claiming that propositions only stand in the *Ableitbarkeit* relation with one another after someone has in fact carried out a logical deduction. It is an objective and eternal fact, for Bolzano, whether or not such a relationship attains. So what is his point in saying that the relationship is only subjective, that a truth obtained in this way is a "mere conclusion [*bloßer Schlußsatz*]" and not a "genuine consequence [*eigentliche Folge*]" (§200)?

According to Bolzano, the fact of propositions A, B, C, D, $\ldots$ standing in the *Ableitbarkeit* relation to propositions M, N, O, $\ldots$ comes to no more than our ability rightly to infer the latter from the former: The correctness and incorrectness of our inferences is

[3] Bolzano's doctrine of ideas and propositions "in themselves" is notoriously mystifying. The relations among them are "objective," but in some sense they do not "exist." Nothing in the present analysis of Bolzano's thought depends on a choice among the several disambiguations of these notions that have been proposed. On the contrary, the fact that Bolzano offers no clear account of these central notions will prove important.

objective. It is typically erroneous, however, to call the latter the consequence of the former, because one proposition being the consequence of another is itself a feature of reality independent of what knowledge we may have of this fact and the use, in reasoning, we might make of it. Indeed, throughout his logical investigations, Bolzano's considerably more sustained focus was devoted, not to the *Ableitbarkeit* relation, but to the theory of this objectively significant consequence relation, a theory he called "*Grundlehre.*"

Bolzano's 1810 *Beyträge* is the definitive exposition of this theory of ground and consequence. In §2 of part II of that booklet, Bolzano wrote:

> [I]n the realm of truth, i.e. in the sum total of all true judgments, a certain *objective connection* prevails which is independent of our actual and *subjective recognition* of it. As a consequence of this some of these judgements are the grounds of others and the latter are the consequences of the former. To represent this objective connection of judgements, i.e. to choose a set of judgements and arrange them one after another so that a consequence is represented as such and conversely, seems to me to be the proper *purpose* to pursue in a scientific exposition. Instead of this, the purpose of a scientific exposition is *usually* imagined to be the greatest possible *certainty* and *strength of conviction.*

This consequence relation, which Bolzano called *Abfolge*, is at the center of a robust philosophical account of mathematical truth. Its influences are legion. A proof, according to Bolzano, must track the objective *Abfolgen* between propositions. Individual mathematical truths therefore have at most one proof (§5). Moreover, the division of mathematical truths into those that have proofs and those basic truths for which no proof can be given is not a matter of convention but is objectively determined and there for us to discover (§13). The distinguishing features of an axiom are, accordingly, not its self-evidence, but its ontological role as ground for other truths and the absence of any proposition serving in the capacity of its ground (§14). Conversely, and most importantly for Bolzano, the self-evidence of a mathematical fact is no reason not to seek a proof for it, for a proof will uncover its grounds, which are typically unrelated to the (good) reasons we might have for accepting the fact as true (§7).

One might reasonably wonder why this hypothesized network of objective relationships should more properly be the focal point of "scientific exposition" than the simple discovery of mathematical facts. Bolzano provided several justifications for the shift in perspective that will serve to illustrate further his concept of consequence.

Primarily, and most often, Bolzano points to an inherent value in coming to understand the structure of the hierarchy of facts. This hierarchy is a feature of the world forever off limits to researchers who "stop short" at certainty. Behind this incentive is the idea that proofs, of the special sort that Bolzano seeks, are explanatory: A fact's grounds are the reason why that fact is true. In some sense they constitute their consequences, and therefore being more than "*Gewissmachungen*" that assure us of a truth, proper proofs are "*Begründungen*, i.e., presentations of the objective reason for the truth concerned" (*Bolzano 1817*, Preface, §I). Science should not simply record but also explain facts.

There is also an aesthetic value to Bolzano's proofs. Through them, one is able to see one's way to a mathematical truth without recourse to ideas and terms that are "off topic." "[I]f there appear in a proof *intermediate concepts* which are, for example, *narrower* than the subject, then the proof is obviously defective; it is what is usually otherwise called a μετάβασις εὶς ̓άλλο γένος[4]" (*1810*, II.§29). Thus, although Bolzano did not actually define the *Abfolge* relation or specify, in any but a few select cases, what the unprovable basic truths are, he disclosed a highly non-trivial fact about the *Grundlehre*: Every non-basic fact is grounded in other facts about one and the same concepts that the consequent, non-basic fact is about.

Bolzano further hinted that the conceptual purity of his proofs affords a scientific advantage, in that it will facilitate the discovery of new truths. In his 1804 *Betrachtungen*, after claiming that "one must regard the endeavor of unfolding all truths of mathematics down to their ultimate grounds ... as an endeavor which will not only promote the *thoroughness* of education but also make it *easier*," Bolzano wrote:

> Furthermore, if it is true that if the first ideas are clearly and correctly grasped then much more can be deduced from them than if they remain confused, then this endeavor can be credited with a *third* possible use—the *extending* of the science. (par. 3)

In the continuation of this passage and again in the *Beyträge* he recounted episodes in the history of mathematics when attempts to prove facts about which there was already no doubt led to the discovery of new truths.

Bolzano's first justification of the centrality of the *Abfolge* relation strikes modern readers as arcane. As Rolf George has remarked, "[i]t seems absurd to try to decide whether two lines intersect because they share a common point, or have a point in common because they intersect," and yet Bolzano insisted that "nobody who has a clear concept of ground and consequence will deny that the first proposition is not objectively grounded in the second, but the second in the first" (*George 1971*, xxxvii). Meanwhile, the second justification seems ideological and the third speculative. Are impure proofs really defective in some way? Pure proofs more explanatory? Might the seemingly disorganized methods of expert mathematicians not actually have some scientific advantage over Bolzano's contrived efforts to track the objective dependencies of truths? Bolzano did not address these individual points in a way that would persuade many skeptics. However, he emphasized a fourth advantage to his standard of proof with greater scientific promise: If we are not careful in mathematical reasoning to uncover the objective grounds for our claims, then the impurity of our demonstrations will lead to disorder, threatening circularity in our reasoning. "For example," Bolzano wrote,

> in [Lagrange's] *Théorie des fonctions analytiques, No. 14*, the important claim that the function $f(x+i)$, with i continuously variable, can in general be expressed

[4] "crossing to another kind"

$$f(x+i) = (x) + ip + i^2q + i^3r + \dots$$

> is derived from a *geometrical* consideration: namely from the fact that a continuously curved line which cuts the x-axis has no smallest ordinate. Here one is in a real *circulus vitiosus*, because only on the assumption of the purely arithmetical assertion about to be proved can it be shown that every equation of the form $y = fx$ gives a continuously curved line. (*1810*, §29)

The same point appears to be the impetus of his celebrated *1817* proof of the intermediate value theorem. He began that paper by examining the geometrical demonstrations of this theorem by Euler, Laplace, and others, and observed that "such a geometrical proof is, in this as in most cases, really circular. For while the geometrical truth to which we refer is ..., extremely *evident*, and therefore needs no *proof* in the sense of *confirmation*, it none the less needs *justification*." This alone does not substantiate his charge of circularity. But Bolzano proceeded to ask us to consider "the objective reason" why that geometircal truth attains and remaked:

> Everyone will, no doubt, see very soon that this reason lies in nothing other than that general truth, as a result of which every continuous function of x which is positive for one value of x, and negative for another, must be zero for some intermediate value of x. And this is just the truth which is to be proved. (Preface, §I)

Bolzano was thus not simply after a glimpse at the alleged causal links among truths. There is a clear scientific reason why the theorems of mathematics should be justified through an analysis that discloses their objective grounds: Because such an analysis will track the objective dependencies of truths on one another, there will be no risk that the evidence such analysis uncovers for a truth will itself depend, in a circular fashion, on that truth. When he examined Euclid's *Elements* he observed the "dissimilar objects" dealt with therein:

> Firstly triangles, that are already accompanied by circles which intersect in certain points, then *angles*, adjacent and vertically opposite angles, then the *equality* of triangles, and only much later their *similarity*, which however, is derived by an atrocious detour [*ungeheuern Umweg*], from the consideration of parallel lines, and even of the *area* of triangles, etc.! (*1810*, Preface)

The Euclidean method is thus unsatisfactory by Bolzano's lights, but because it draws its sting from the threat of circularity, this dissatisfaction is no mere matter of preference. For Bolzano it is a call for bold revision in mathematics: "But if one considers," he continued, how well-composed the *Elements* is, "and if one reflects how every successive proposition, with the proof with which Euclid understands it, necessarily requires that which precedes it, then one must surely come to the conclusion that the reason for the disorder must be fundamental: the entire method of proof which Euclid uses must be incorrect."

In developing the *Grundlehre*, Bolzano advanced logical theory in ways comparable in scope to his work on *Ableitbarkeit* but oriented in a different direction. The initial insight seems to have occurred to him rather early. In his youthful *1804* pamphlet he wrote, "I must point out that I believed I could not be satisfied with a completely strict proof *if it were not even derived from concepts* which the thesis proved contained, but rather made use of some fortuitous alien, *intermediate concept* [*Mittelbegriff*], which is always an erroneous μετάβασις εὶς 'άλλο γένος[5]" (Preface, par. 4). That proofs should be free from such intermediate concepts and the concomitant "atrocious detours" in reasoning (attributed to Euclidean methods) was inspired by the desire to capture the objective ground and consequence relations in the world, to produce proofs that were topically pure and therefore free from circularity, but the significance of the notion of analyticity that Bolzano developed is not tied down to those ambitions.

In §17 of part II of the *Beyträge*, Bolzano had distinguished analytic and synthetic truths according to the Kantian criterion of conceptual containment (the predicate of an analytic, and not a synthetic, truth contains its subject.) In §31 he extended this to a distinction between analytic and synthetic proofs. He did not explicitly define the notions, but they can be reconstructed from context: A proof is analytic if its derived formula contains, in its compound concepts, all the simple concepts that appear elsewhere in the proof. Remarkably, Bolzano suggested that "the whole difference between these two kinds of proof [analytic and synthetic] is based simply on the *order and sequence* of the propositions in the exposition." Thus Bolzano rediscovered the formidable ontological burden that he placed on proofs reflected in a rather mundane feature of those proofs' written appearance.

This observation is supported by the rudiments of a theory of proof transformation, outlined in §20. Because every compound proposition is built out of a subject and predicate which depend on the individual concepts of which it is composed, the proposition itself, if true, "is actually also a derivable, i.e. provable proposition." Moreover, its single proof begins with only simple propositions about the simple concepts contained in the compound, proved proposition. One of Bolzano's great discoveries is that the rules of inference in a proper analytic proof that lead from these simple propositions to the proved proposition are other than the patterns of syllogistic reasoning to which logicians in his day devoted so much attention. To make this point, he embarked on a fine classification of inference rules:

> [I]t should now be discussed at length *how many simple and essentially different kinds of inference there are*, i.e. how many ways there are that a truth can depend on other truths. It is not without hesitation that I proceed to put forward my opinion, which is so very different from the usual one. Firstly, concerning the *syllogism*, I

[5] "crossing to another kind"

> believe there is only a single, simple form of this, namely *Barbara* or Γραμματα in the *first* figure. (§12)

Bolzano suggested a modification of *Barbara*'s classical presentation, based on the natural way of reasoning and continued, "However, these are small matters.—Every other figure and form of the syllogism seems to me to be either not essentially different from *Barbara* or not completely simple." There is no explicit presentation in the *Beyträge* of criteria for simplicity and identity of inference rules, but clearly suggested in these remarks is the idea that *Barbara* in the first figure is all one would have to use in order to derive any proposition derivable from the full gamut of classical syllogistic forms. These ideas alone are substantial in the history of logic, but what Bolzano proceeded to describe is even more so. "But on the other hand," he wrote, "I believe that there are some *simple kinds of inference* apart from the syllogism." Among his examples is the inference from "*A is* (or *contains*) *B*" and "*A is* (or *contains*) *C*" to "*A is* (or *contains*) [*B et C*]." "[I]t is also obvious," Bolzano claimed "that according to the necessary laws of our thinking the first two propositions can be considered as *ground* for the third, and not conversely" (§12).

After illustrating a couple of other such rules, which similarly establish compound clauses within the sub-sentential structure of propositions, Bolzano noted a crucial difference between his new "analytical" rules and the syllogism, clearly based on his rich notion of *Abfolge*: The syllogism rule is not reversable—its premises in no way follow from its conclusion—but the analytical rules each are. For this reason, the reverse of each analytical inference "could seem like an example of another kind of inference ....

> But I do not believe that this is a [proper] *inference* .... I can perhaps *recognize* subjectively from the truth of the *first* of these three propositions the truth of the two others, but I cannot view the first *objectively* as the *ground* of the others. (§12)

Thus propositions with compound concepts can be proved in a way that charts the *Abfolge* hierarchy, i.e., purely analytically, from propositions containing only simple concepts. "On the other hand," Bolzano wrote in a long note to §20, "how propositions with simple concepts could be proved other than through a syllogism, I really do not know."

In §27, drawing from the observed features of the analytical inference rules, Bolzano argued for the following claim: "If several propositions appearing in a scientific system have the same subject, then the proposition with the more compound predicate must follow that with the simpler predicate and not conversely." His "proof" is just one line, after which he wrote, "This truth has forced itself particularly clearly upon those who have thought about the nature of a scientific exposition." He continued:

> Moreover, it is obvious here that we cannot extend our assertion *further*, and instead of the expression, "*the proposition with the more compound predicate*," put the more general one, "*the proposition with the narrower predicate*." For whenever we make an inference by a syllogism one of the premises (namely the so-called *major*), with

> just the same subject as the conclusion, has the predicate (namely the *terminus medius*) which is *narrower* than that of the conclusion: *S contains M*, *M contains P*, therefore *S contains P*, where the concept *M* must obviously be narrower than *P* because otherwise the proposition, *M contains P*, could not be true, and yet the judgement, *S contains M*, must be considered as one preceding the judgement, *S contains P*.

In other words, Bolzano recognized that his proofs, because their propositions are ordered so as to track the objective *Abfolgen* in the world, would have a form of what modern logicians call a subformula property were it not for the ubiquity of the syllogism rule. Even with this rule, though, every proof has a related property. Given the normalizing techniques discussed in §20, typical proofs may generally be written so that they begin with several syllogisms devoted to establishing the needed simple truths from which to infer, purely from analytic rules, their more compound consequence. In §30 Bolzano touched on this again. There he seems to be saying that even within the syllogistic part of proofs, there is a determined single way to proceed.

In the preface of *1817a* Bolzano describes a "purely analytic procedure" differently, as one in which a derivation is performed "just through certain changes and combinations which are expressed *by a rule completely independent* of the nature of the designated quantities." This description points to the features of analyticity emphasized by the eighteenth century algebraists, who sought to extend algebraic techniques to mechanics, geometry, and other disciplines. Laplace, for example, in chapter 5 of book V of his *Exposition du système du monde* had written:

> The algebraic analysis soon makes us forget the main object [of our researches] by focusing our attention on abstract combinations and it is only at the end that we return to the original objective. But in abandoning oneself to the operations of analysis, one is led by the generality of this method and the inestimable advantage of transforming the reasoning by mechanical procedures to results inacessible to geometry. (*Kline 1972*, p. 615)

Similarly Lagrange (*1788*, preface) declared, "The methods which I expound in [*Mécanique analytique*] demand neither constructions nor geometrical or mechanical reasonings, but solely algebraic operations subjected to a uniform and regular procedure" (*Ibid*). Nowadays one reflexively associates these "mechanical procedures" with derivability and conceives of logical consequence as residing in the "object of our researches"—on the semantic half of this divide. One wonders whether one's formulas are adequate to their intended interpretations, whether these mechanical procedures in fact trace the interrelationships among the objects of our researches, these latter being "the original objective." Bolzano could only have it the other way around: His *Grundlehre* revealed that the analytic calculus traces facts back their their ultimate, constituitive grounds, and he shared Laplace's suspicion that these same dependencies might be inacessible by geometrical or other traditional mathematical inferences. Should one side of this divide prove inadequate, it could only be the latter—this being rightly designated as mere derivability—because "by abandoning oneself to the operations of analysis" one accesses the objective dependencies among truths.

Bolzano's concepts of *Ableitbarkeit* and *Abfolge* each deserve detailed reconstruction and appreciation. Their influence on our current understanding of logic is significant, yet unlikely fully known. Doubtless, too, in serving aims that no longer drive modern investigations—precisely, that is, where they have failed to be influential—they are windows into possible lines of thought that history, for one reason or another, simply did not preserve. Perhaps their greatest legacy lies in neither of them individually, though, but in Bolzano's desire to establish a correspondence between them.

Whereas the *Abfolge* relation holds only between truths, false propositions may stand in the relationship of *Ableitbarkeit* with one another so long as (1) under some substitution of ideas they all are true and (2) under all substitutions that make some one part of them true, so too is the second part. For this simple reason, one cannot conclude from the derivability of some proposition that one has uncovered the grounds, in the premises of this derivation, of a proposition. More crucially, the same conclusion cannot be drawn even when all the propositions in the derivation are true. This is evident from the fact that derivability is obviously reflexive and often symmetric, whereas according to Bolzano no truth is its own ground (*1837*, §204), and no two truths could mutually ground one another (§211). The *Ableitbarkeit* relation is not, in Bolzano's idiom, "subordinate" to the *Abfolge* relation.

Observing this gap between his two notions of consequence, Bolzano wrote, in §162 of his *Wissenschaftslehre*:

> Not every relationship of *derivability*, then, is so constituted that it also expresses a relationship of ground to consequent holding between its propositions when they are all true. But the relationship of derivability that does posses this characteristic will doubtless be sufficiently worthy of attention to deserve a designation of its own. I will therefore call it a *formal ground-consequence* relation [*formalen Abfolge*] ..."

Given the nature of these relations, the converse question seems more relevant: Is every *Abfolge* relation representable with a derivation? In the terms Bolzano coined, is every *Abfolge* relation in fact a *formalen Abfolge* relation? If not, then the very idea of placing this conception of logical consequence at the center of all scientific exposition is puzzling. Whatever the merits of knowing the objective grounds of a mathematical fact, no science can be devoted to this task without some sort of method for the discovery of such grounds. Yet Bolzano managed only to indicate a handful of unrelated instances of this fundamental relation, appealing haphazardly to theological, moral, and intuitive ideas to do so. On the other hand, if from its grounds a truth can always, in principle, be derived, then the process-centered *Ableitbarkeit* relation is seen to be adequate to trace the objective dependencies of truths on one another. This alone would not establish that the theory of derivability would allow us to discover the grounds of every proposition.

The problem, again, is that far too many things are similarly derivable. But it would guarantee that the objective consequence relation does not lie beyond the reach of human inferential practices. All such consequences would be captured in a derivation, and from further considerations about the features of a derivation—whether, for example, it unfolds into an analytic proof—one would be able to determine whether it has turned up the ultimate reasons for the proposition's truth.

Bolzano devoted §200 to this question. The section is entitled, "Is the relation of ground and consequence subordinate to that of derivability?" Here is how he explained the point:

> If truths are supposed to be related to each other as ground and consequence, they must always, one might believe, be derivable from one another as well. The relation of ground and consequence would then be such as to be considered a particular species of the relation of derivability; the first concept would be subordinate to the second.

Tempting as this belief might be initially, after a little reflection it is untenable. Why should the ultimate reasons for the myriad truths of mathematics always be related to them so that from consideration of the variation of ideas in each, one can reliable infer from them their objective dependencies? Why should the formal features of propositions give us any access to the shimmering reality beyond? Few contemporary writers share Bolzano's confidence in our intuitions about the realm of objective dependencies, but even Bolzano recognized the unconvincingness of speculation on this issue. "Probable as this seems to me," he concluded, "I know no proof that would justify me in looking upon it as settled." One can diagnose the difficulty more definitely, as not simply the absence of a proof but indeed the lack of any clear idea of what a proof might look like. Bolzano's two theories of logical consequence are themselves not precise enough for their correspondence with one another to be subject to proof. All the same, the question is at the center of Bolzano's thought. The procedural *Ableitbarkeit* relation provides a calculus of inference. The ontological *Abfolge* relation is a feature of the world absolutely independent of our ability to reason about it. By establishing that these notions correspond we would ensure that the logical structure of the world is accessible, that some line of thought could trace the dependencies of truths, that the reasons behind the complex facts of reality are discoverable and comprehensible.

3. Gentzen's answer

The single point at which all of Bolzano's logical investigations are focused is, from the modern point of view, deeply suspect. Logic is blind to considerations of truth, to say nothing of ultimate explanations for why some statements are in fact true. Indeed, Bolzano's own development of *Ableitbarkeit* was a turn away from factual truth, towards distinguishing those statements that could be true from those that could not, towards identifying statements that rise and fall together no matter what the world is like. "Mathematics," he wrote, "concerns itself with the question, *how must things be made in order that they should be possible?*" unlike metaphysics which "raises the question, *which things are real?*" (*1810*, I.§9). But in the end, it was the objective grounding of truths that drove him, and if the theory of *Ableitbarkeit* cannot be shown to trace the world's *Abfolgen*, it loses much of its scientific interest. Modern logicians, by contrast, have no expectation that their craft will uncover ultimate grounds. Many do not even believe in such things. What remains of Bolzano's intricate scheme for writers who do not share his metaphysical aspirations?

In the century after Bolzano's work, logicians managed to uproot nearly the whole apparatus that he developed from its ontological setting. Astonishingly, not only did they preserve many of the details of his theory in the process, but the metaphysical character of Bolzano's thought proved in many ways to be the principal obstacle to its development.

The previous section exposed, at the center of Bolzano's logic, an unanswerable question about the correspondence of the *Ableitbarkeit* and *Abfolge* relations. There is no pre-theoretical guarantee that ultimate explanations of facts should be discoverable by humans, but if each *Abfolge* is in fact a *formalen Abfolge*, then, at least in principle, every grounding relation could be traced by a logical inference. The problem is that so little is known about the *Abfolge* relation that there is no way to tell whether propositions so related to one another also stand in the relation of *Ableitbarkeit*.

One peculiarity of this dilemma is that Bolzano's own development of the *Grundlehre* revealed a strikingly manageable system of analysis, whereas the reportedly inference-based theory of *Ableitbarkeit* is quite difficult to implement. The proofs in the *Grundlehre* are chains of applications of a single syllogism rule together with a collection of analytical rules for "combining concepts." Given a true proposition, the task of building its proof-tree is generally not daunting. One merely identifies the complex concepts that it contains and looks up the rules needed to establish such a concept in the subject or predicate position. The immediate grounds of this proposition can then simply be constructed from the template that the relevant rule provides. Those grounds will themselves be truths, so that their immediate grounds can be discovered in the same fashion. The only reason this procedure is not a fully adequate means for discovering a proposition's grounds is that often no straightforwardly analytic rule is applicable. Occasionally, as Bolzano noted, all the concepts in a truth are simple, and the way from it up to its ultimate grounds can only be charted by the syllogism rule. Unlike the rules that Bolzano introduced, the syllogism rule does not provide any hints as to the needed premises for a given consequence. As Bolzano explained, the predicate of the major premise and the subject of the minor premise, the so-called *terminus medius*, is not contained in the consequence and can only be guessed at. This is a serious obstacle, of course, to a fully mechanizable system of proof. But even partial mechanizability, of the sort provided by the analytical rules of inference, puts the *Abfolge* relation more within our reach than the relation of *Ableitbarkeit*. To determine "derivability," one must be ingenious at every turn. Which ideas ought one vary? Which new ideas ought one replace them with? And how, more dauntingly, can one tell that in a particular case all such replacements will yield the same result?

It is thus odd to consider the *Ableitbarkeit* relation as anthropomorphically grounded and the *Abfolge* relation as somehow beyond us. This categorization only makes sense given Bolzano's metaphysical distinction between *Gewissmachungen* and *Begründungen*, for the systematic study of each relation reveals that the former cannot feasibly be discovered, whereas discovering the latter is often largely a matter of routine calculation. Unhindered by Bolzano's emphasis that the *Abfolge* relation obtain only among truths, one might sensibly refer to this as a derivability relation and Bolzano's *Ableitbarkeit* as a definition of logical consequence. Such, at least, was the way things seemed to the twentieth century logician Gerhard Gentzen.

Gentzen is remembered primarily for two deep contributions to logic: the design of logical proof calculi that make perspicuous the flow of reasoning in a proof and the use of transfinite induction to calibrate the consistency strength of formal mathematical theories. It is fairly well known how these achievements relate to one another. Gentzen's logical calculi are susceptible, because of their division of inference rules into structural and analytic sorts, to tightly controlled proof-transformations. One can thus reason about the provability of certain formulas by considering how to transform alleged proofs of such formulas into certain canonical forms. Transfinite induction is used to track the transformation, and the perspicuous features of the canonical forms allow one quickly to rule out the possibility of such a proof being written down. Thus, for example, if one thinks of consistency as the unprovability of a contradiction, one can determine that a mathematical theory is consistent by reflecting on the fact that any proof in that theory of a contradiction could be rewritten in a special, but obviously impossible, form.

Underlying Gentzen's logical achievements, however, is a more fundamental contribution to modern thought. He introduced a notion of logical completeness that simultaneously made possible these scientific results and solidified a conception of logic completely dissociated from Bolzano's ontological scheme. Gentzen's idea is observable in his dramatic recasting of Bolzano's question.

Gentzen's conception of logical completeness originates in his first paper from *1932*. The "formal definition of provability" in that paper consists of "sentences" of the form $M \longrightarrow v$, where v is an "element" and M is a "complex" (a non-empty set of finitely many elements). Gentzen's typographic convention is that concatenation of letters represents their set-theoretical union. Sentences can also be written with the elements of a complex displayed, thus: $u_1 u_2 \ldots u_n \longrightarrow v$. Because complexes are sets, the same element cannot appear multiple times in the same complex, and the order in which the elements of a complex are listed is immaterial. Gentzen referred to the complex at the left of a sentence's arrow symbol as its antecedent and to the lone element on the right of the arrow symbol as the succedent. He defined tautologies to be those sentences whose antecedent is the singleton set containing the same element that appears in the sentence's succedent, and he called sentences whose antecedent contains the element in the succedent "trivial."

Prior to Gentzen, much attention was devoted to the distinction between categorematic and syncategorematic terms—the parts of language that signify on their own and those "logical particles" that serve merely to bind significant bits of language together so that they signify, not singly, but as a composite. A convincing definition of this distinction proved elusive, but, it was thought, a fully articulated notion of logical consequence depended on such a definition. Even writers who shifted the locus of logical relationships from sentences to propositions sought some such distinction: The logical consequences of a proposition are those propositions that follow from it by dint of their structure, this structure determined by their syncategorematic parts. Consider Bolzano's definition of *Ableitbarkeit*: "the relation among the compatible propositions A, B, C, D, ... M, N, O, ... such that all the ideas that make a certain section of these propositions true, namely A, B, C, D, ..., when substituted for i, j, ... also have the property of making some other section of them, namely M, N, O, ... true." To ever determine whether propositions stand in this relation with one another, one has to know how finely one can carve them up into "ideas" and also how the parts of the propositions that are not "ideas" do their binding. As one burrows ever more deeply into a proposition, one finds sentential connectives, modal qualifiers, sub-sentential particles (quantifiers, etc.), and with each discovery the specification of a proposition's truth conditions, in terms of the contribution that these particles make to its meaning, is increasingly complex (and subject to debate). "To be sure," Bolzano wrote, "this distinction has its ambiguity, because the domain of concepts belonging to logic is not so sharply demarcated that no dispute could ever arise over it" (*1837*, §148).

Gentzen's response to this conundrum was to abandon the search for any final word as to the "logical" parts of a sentence and to develop a fully general account of logical consequence independent of such considerations:

> We say that a complex of elements *satisfies* a given sentence if it either does not contain all antecedent elements of the sentence, or alternatively, contains all of them and also the succedent of that sentence. ... We now look at the complex K of all (finitely many) elements of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ and $\mathfrak{q}$ and call $\mathfrak{q}$ a *consequence* of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ if (and only if) every subcomplex of K which satisfies the sentences $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ also satisfies $\mathfrak{q}$. (p. 33)

Gentzen did not propose that the very simple structure of his sentences captured everything of logical importance about how propositions are related to one another. On the contrary, he wanted his sentences to display minimal logical structure—just enough to meaningfully unpack the basic notion of logical consequence. That notion, he claimed, does not depend on what the elements of a sentence are, and therefore ought to be studied in a setting that pried no more deeply into the structure of sentences than necessary. The result is at once free of controversy and applicable to a wide variety of topics. Gentzen suggested a few: interpret the elements as events and read $u_1 u_2 \ldots u_n \rightarrow v$ as "the happening of events $u_1, u_2, \ldots u_n$ causes the happening of v"; interpret the arrow as a containment relation and read the same sentence as "any collection that contains $u_1, u_2, \ldots u_n$ also contains v"; read the same sentence as "an object with the properties $u_1, u_2, \ldots u_n$ also has the property v." Among these he also mentions the more expected reading, "if the propositions $u_1, u_2, \ldots u_n$ are true, then the proposition v is also true." Gentzen remarked, "[o]ur considerations do not depend on any particular kind of informal interpretation of the 'sentences,' since we are concerned only with their formal structure," and evidently the modicum of formal structure displayed in his "sentences" suffices for an adequate definition of our shared concept of logical consequence (*Ibid.*).

It is fair to ask why Gentzen did not treat sentences themselves as elemental and say simply that one sentence is the consequence of some others if any situation in which the latter are all true, so too is the former. The problem with this approach is that the informal notion that one arrives at cannot be "formalized." Gentzen viewed the above definition, not as a semantic analysis of the notion of logical consequence, but as a clarification of that informal notion preliminary to its formalization. There is, for example, no distinction in kind between the complexes that might satisfy a sentence and the parts of that sentence. In place of a theory about these complexes and their elements, Gentzen designed a logical proof system that formally captures the intuitive notion. He specified two inference rules for his system, which he called "THINNING" and "CUT":

$$\frac{L \longrightarrow v}{ML \longrightarrow v}\ \textit{thinning} \qquad\qquad \frac{L \longrightarrow u \qquad Mu \longrightarrow v}{LM \longrightarrow v}\ \textit{cut}$$

Then he defined a "proof" of a sentence $\mathfrak{q}$ from the sentences $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ to be "an ordered succession of inferences (i.e., THINNINGs and CUTs) arranged in such a way that the conclusion of the last inference is $\mathfrak{q}$ and that its premises are either premises of the $\mathfrak{p}$'s or tautologies" (p. 31). Gentzen wrote these proofs in tree-form. Here is an example proof of $Kf \longrightarrow d$ from $c \longrightarrow e$, $ef \longrightarrow a$, and $ac \longrightarrow d$, where $K = bc$:

$$\cfrac{\cfrac{\cfrac{c \longrightarrow e}{bc \longrightarrow e}\ \textit{thinning} \qquad ef \longrightarrow a}{Kf \longrightarrow a}\ \textit{cut} \qquad ac \longrightarrow d}{Kf \longrightarrow d}\ \textit{cut}$$

In section 4, Gentzen wrote:

> Our formal definition of provability, and, more generally, our choice of the forms of inference will seem appropriate only if it is certain that a sentence $\mathfrak{q}$ is "provable" from the sentences $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ if and only if it represents informally a consequence of the $\mathfrak{p}$'s. (p. 33).

Theorem I of *Gentzen 1932* states that the proof system is "correct."

Theorem I. *If a sentence $\mathfrak{q}$ is provable from the sentences $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ then it is a consequence of them.*

Proof. Observe first that the conclusion of a thinning of a sentence is a consequence of that sentence: Suppose the complex K satisfies $L \longrightarrow v$. Either it does not contain every element in L or else it contains v. Either way, K also satisfies $ML \longrightarrow v$. Similarly, the conclusion of a CUT of two sentences is a consequence of those sentences: Any K that does not satisfy $LM \longrightarrow v$ must contain all the elements in L and all the elements in M but not v. If K furthermore contains u, then it fails to satisfy $Mu \longrightarrow v$; otherwise it fails to satisfy $L \longrightarrow u$. Observe, now, that every tautology is a consequence of every sentence, and every sentence is a consequence of itself. From this last observation, it follows that every initial sentence in a proof from $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$ is a consequence of $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$. By the first observation and the evident transitivity of consequence, if the premises of a THINNING or a CUT are consequences of $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$, then so too is their conclusion. Therefore, every sentence in a proof of $\mathfrak{q}$ from $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$ is a consequence of $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$, and in particular $\mathfrak{q}$ is. □

Gentzen established the converse in Theorem II, where he in fact showed that proofs of a specific "normal form" alone suffice to exhibit all the consequences among sentences. Normal proofs are proofs of the form:

$$
\begin{array}{c}
\dfrac{\mathfrak{r}_2 \qquad \dfrac{\mathfrak{r}_1 \quad \mathfrak{s}_1}{\mathfrak{s}_2}\,\textit{cut}}{\cdot}\,\textit{cut} \\
\cdot \\
\cdot \\
\dfrac{\dfrac{\mathfrak{r}_{n-1} \qquad \dfrac{\mathfrak{r}_{n-2} \qquad \mathfrak{s}_{n-2}}{\mathfrak{s}_{n-1}}\,\textit{cut}}{\mathfrak{s}_n}\,\textit{cut}}{\mathfrak{q}}\,\textit{thinning}
\end{array}
$$

That is, such proofs are chains of applications of CUT followed by a single, terminal application of THINNING.

Theorem II. *If a sentence* $\mathfrak{q}$ *is a consequence of the sentences* $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$*, then there exists a normal proof of* $\mathfrak{q}$ *from* $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$*.*

Proof. Observe first that every trivial sentence obviously has such a normal proof (a proof with zero CUTs is normal). Suppose, therefore, that $\mathfrak{q}$ is non-trivial and of the form $L \longrightarrow v$. Let $\mathfrak{S}$ be the set of all non-trivial sentences with succedent v for which there is a normal proof from $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$ without THINNING. Notice that, because the conclusion of a CUT never contains an element that did not appear in its premises, $\mathfrak{S}$ is finite. Clearly if the antecedent of some $\mathfrak{s}$ in $\mathfrak{S}$ is entirely contained in L, there would be a normal proof of $\mathfrak{q}$ from $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$ (to construct it, just add one THINNING at the bottom of the normal proof of $\mathfrak{s}$ from $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$). We will show that this must be the case.

Suppose that no $\mathfrak{s}$ in $\mathfrak{S}$ has an antecedent that is contained in L. Define a sequence of complexes $L = M_1, M_2, \dots, M_k = N$ recursively: If every $\mathfrak{p}$ in $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$ is satisfied in M_i, let M_i be N. Otherwise, choose one such $\mathfrak{p}$ $(O \longrightarrow u)$. Its succedent u does not belong to M_i, and we define $M_{i+1} = uM_i$. Notice that this sequence is necessarily finite, because the complex consisting of all the finitely many elements among $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$ necessarily satisfies them all. We will show that N is a counter-example to the claim that $\mathfrak{q}$ is a consequence of $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$. N satisfies $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$, and it contains L, so we need only show that N does not contain v.

An inductive argument shows that in fact each M_i satisfies all the sentences in $\mathfrak{S}$ and does not contain v.

Base: $M_1 = L$ does not contain v, because of the assumption that $\mathfrak{q}$ is not trivial. Furthermore, if $M_1 = L$ did not satisfy some $\mathfrak{s}$ in $\mathfrak{S}$, then the antecedent of $\mathfrak{s}$ would be contained in L, contrary to our current assumption.

Induction: Assume that M_i satisfies all the sentences in $\mathfrak{S}$ and does not contain v, and consider $M_{i+1} = uM_i$ where $O \longrightarrow u$ is a sentence from among $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$ that M_i does not satisfy. O belongs to M_i, because M_i does not satisfy $O \longrightarrow u$. By hypothesis v does not occur in M_i, and therefore v does not occur in O. Now, were M_{i+1} to contain v, then v would be u. But in that case, $O \longrightarrow u$ would be in $\mathfrak{S}$ despite not being satisfied by M_i, contrary to the hypothesis. Therefore M_{i+1} does not contain v.

Suppose now that there is a sentence in $\mathfrak{S}$ that M_{i+1} does not satisfy. Any such sentence must have the form $Pu \longrightarrow v$, with P contained in M_i (again, u is the succedent of the sentence $O \longrightarrow u$ that M_i does not satisfy, the element appended to M_i to arrive at M_{i+1}.) Consider the CUT:

$$\frac{O \rightarrow u \qquad Pu \rightarrow v}{OP \rightarrow v}\ cut$$

The sentence $OP \longrightarrow v$ belongs to $\mathfrak{S}$, because (1) it has the succedent v, (2) both $Pu \longrightarrow v$ (as a member of $\mathfrak{S}$) and $O \longrightarrow u$ (as a sentence from among $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$) have normal proofs from $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$ without THINNING, (3) v belongs neither to O nor to P. But O and P both belong to M_i, although v does not, so M_i does not satisfy $OP \longrightarrow v$, contrary to the hypothesis. □

This proof system thus formally realizes the concept of logical consequence in a way that is abstracted from considerations about the specific features of propositions by virtue of which they follow from one another. Because of the normal form component of Gentzen's proof, one can in fact say more: The logical consequences of a set of sentences are each derivable from them with the single rule CUT, perhaps followed by a weakening of the derived assertion with the rule THINNING. In other words, Gentzen showed that CUT is the formal inference rule underlying the intuitive consequence relation.

In his *1932* paper, Gentzen applied this proof system to questions about the independent axiomatizability of theories. The results, though not well-known, are significant, and the proofs are elegant. The deeper significance of Gentzen's "formal definition of derivability," however, is that it provides a setting for a precise formulation of a notion of logical completeness. This is the use that Gentzen made of his system in his *1934-35 Untersuchungen*. The idea is simple. First replace the elements of the proof system with formulas from a specific branch of logic. In the *Untersuchungen* Gentzen studied the first-order predicate calculus. Then add to the "structural rules" CUT and THINNING new rules associated with the logical particles of these formulas. These "logical rules" formalize the inferences that lead to and from propositions containing the associated logical particles. Technical considerations motivate a few new structural features (complexes are replaced by finite sequences, new structural rules are added to eliminate redundancies or rearrange formulas in a sequence). The resulting system is called a "sequent calculus." Gentzen illustrated how he arrived at his logical rules from an empirical study of mathematical proofs.[6] But the question immediately arises, whether or not rules arrived at in this way fully capture the meanings of the logical particles they are associated with. Can one extract with these rules every logical consequence of some premises? If so, then it is reasonable to say that the rules associated with the logical connectives are complete in the sense that they fully capture those connectives' meanings—but how can one show that this is the case? The sequent calculus allows this question to be posed precisely: Can everything derivable in fact be derived without CUT, or is the CUT rule an essential ingredient in some logical derivations?

In §3 of the *Untersuchungen* Gentzen answered this completeness question for both classical (LK) and intuitionistic (LI) sequent calculi formulations of predicate logic with his famous "*Hauptsatz*": "Every LI- or LK- derivation can be transformed into another LI- or LK- derivation with the same endsequent, in which no CUTs occur." It is noteworthy that the LI and LK results are essentially the same—these calculi differ only in the structure of their sequents—whereas other conceptions of completeness, wherein syntactic calculi are coordinated with semantic theories, require essentially different proofs for classical and intuitionistic logic. It is furthermore noteworthy that several features of Gentzen's logical writing indicate that he viewed the *Hauptsatz* as a completeness result. That evidence is presented in *Franks 2010*. One detail relevant to the current discussion is the fact that although Gentzen worked after, and in full awareness of, Gödel's results from 1929–1931, he seemed not to appreciate or acknowledge Gödel's own completeness theorem.

Gentzen's formulation of logical completeness has an additional theoretical feature that results from the nature of the logical rules in his system. Those rules are "analytic" in the same sense that the rules of Bolzano's *Grundlehre* are: Their premises contain no ideas that their conclusion does not contain. In Gentzen's modern terminology, the formulas in a premise of such a rule are all subformulas of the formulas in its conclusion. Every formula in a derivation without the CUT rule therefore occurs as a subformula in the endsequent. Gentzen put the point like this:

> [T]hese properties of derivations without cuts may be expressed as follows: [Their formulae][7] become longer as we descend lower in the derivation, never shorter. The

[6]Gentzen first codified empirically observable inference rules in his natural deduction calculus. The sequent calculi rules are the result of rewriting the natural deduction rules so that the analytic rules associated with logical particles become disentangled from the synthetic operation DETACHMENT or, in the final analysis, CUT.

[7]Szabo has "The S-formulae," preserving Gentzen's technical terminology.

> final result is, as it were, gradually built up from its constituent elements. The proof represented by the derivation is not roundabout in that it contains only concepts which recur in the final result. (p. 88)

Cut-free proofs thus are analytic in Bolzano's sense. Gentzen echoed Bolzano's aversion to the "atrocious detours" found in synthetic proofs: A cut-free proof, he wrote, "makes no detour" [*er macht keine Umwege*] (p. 69). One can even read Gentzen's metaphor of "constituent elements" as a reflection of Bolzano's idea that the *Abfolge* relation traces the reasons why facts obtain.

It is instructive, though, to observe the departures from and inversions of Bolzano's ideas that led Gentzen to his solution of the completeness problem. In Gentzen's systematization of logic, Bolzano's *Abfolge* relation, which had been an objective feature of the transcendent world, is replaced by logical rules immanent in mathematical practice. Because these rules are on display in actual mathematical reasoning, and not hidden away in the causal structure of the realm of propositions, they are observable. To discover them, Gentzen had only to devise the right conceptual grid to place on mathematical discourse. Bolzano's *Ableitbarkeit* relation, meanwhile, is simplified into a concrete definition of logical consequence. But whereas Bolzano conceived of the *Ableitbarkeit* relation as anthropomorphic rather than ontological, and therefore faced the challenge of establishing a correspondence between one relation in the realm of objective dependencies and a second relation about derivability, Gentzen analyzed logical consequence directly into his proof system so that the analytic rules live alongside the synthetic CUT operation. There is no difference in kind between logical consequence and derivability, and completeness is no longer a matter of establishing a correspondence between the immanent and the transcendent. The logical rules of the sequent calculus are complete, both individually (by capturing the meanings of the logical particles they govern) and collectively, because proofs involving the CUT rule can be transformed into purely analytic proofs.

A surprising outcome of Gentzen's answer to the completeness question of Bolzano's *Wissenschaftlerhe* is that it serves also as an elucidation of a central problem in his *Beyträge*. In that earlier work, we saw, Bolzano was occupied with the question of whether or not "the propositions appearing in a scientific system" always are ordered so that the propositions with the more compound predicate follow those with the simpler predicate. In §27 he claimed that this is the case, but that one cannot also say that propositions with wider predicates follow those with the narrower predicates. The reason, he claimed, is that the syllogism rule disrupts the analyticity of proofs yet is for all we know unavoidable. This is because the analytical rules that he observed are not applicable to multiple "simple sentences," and it is possible that such sentences are found both among the groundless, basic truths and also among the grounded, consequent truths. In §20 Bolzano observed that there is no way to reason one's way to simple propositions other than with the syllogism. So if there are such truths that are not basic, in Bolzano's sense, then there are truths that do not have purely analytical proofs. Obviously Gentzen's logical investigations do not touch on such ontological matters as which facts are basic.

However, his cut-elimination theorem does corroborate Bolzano's claim, by showing that CUT is essential only when reasoning from axioms.[8] It is eliminable otherwise.[9]

The ontological dimension of Bolzano's thought cannot be easily evaluated. It elevated the *Abfolge* relation to a transcendent realm whose study proved too esoteric for scientific tools. This rendered the question of the adequacy of the *Ableitbarkeit* relation unanswerable and left vague the limits of normalization—preventing Bolzano from recognizing the subformula property of his analytic proofs. For these reasons, Gentzen's deflation of the transcendental aspect of logic appears to be just what was needed to put forward a scientifically respectable, and answerable, question of logical completeness. Indeed, Gentzen not only managed to pose the question of logical completeness in terms that admit precise solution, but he managed to show that Bolzano's questions about completeness and analyticity of proofs are in the final analysis the same question. It is thus tempting to fault Bolzano's ontological preoccupations as the major obstacle in the way of modern logic. On the other hand, it must be stressed that nothing like the question of logical completeness was posed in any form prior to Bolzano. The chasm that Gentzen filled, with his *1932* analysis of logical consequence and his *1934–35 Hauptsatz*, is the one that Bolzano dug.

4. Gödel's answer

Unlike Gentzen who closed the gap between *Abfolge* and *Ableitbarkeit* by projecting the latter relation into his proof system, today's prevailing conception of logical completeness preserves Bolzano's metaphysical divide between consequence and derivability. But whereas Bolzano, following Laplace and Lagrange, saw in the mechanical manipulation of signs a way of leaving intuition behind and tracking the objective dependencies among truths, this activity is today associated with derivability. *Ableitbarkeit*, which Bolzano considered anthropomorphically grounded, is recast in terms of set-theoretic semantics as the consequence relation that transcends effective processes with formal signs. One asks whether all the logical consequences of a set of sentences are in fact derivable from those same sentences. Bolzano's question is reversed.

This reversal cannot be attributed without further ado to Gödel, who never described logical consequence as a semantic notion. Gödel defined "*logischen Folgens*" as "being formally provable in finitely many steps," and he nowhere departed from this usage. Indeed, Gödel did not formally define any semantic notions in his papers on logical completeness, appealing instead to intuitive ideas of satisfiability and validity that suffice for his purposes. Even Tarski, whose later work on logical consequence eventually coalesced into the definition familiar today, throughout his papers from the 1920s meant by the "consequences" of a set of sentences the larger set of sentences formally derivable from

[8] It is interesting to note that CUT is Gentzen's slight modification of the rule Hertz called SYLLOGISM in the context of closely related research. See (*Hertz 1929*), which Gentzen cites.

[9] In fact, an extension of Gentzen's work, due to Gasai Takeuti, shows that in sequent calculus proofs from axioms, only "anchored" CUTs are essential; "free" CUTs are eliminable. Anchored CUTs are instance of the CUT rule, at least one of whose premises is a descendant, in the proof-tree, of an axiom.

them.[10] Most remarkably, although today a simple rearrangement of ideas leads from what Gödel proved—that every universally valid, first-order formula is provable—to the statement of "strong completeness"—the derivability from a set of sentences of all their logical consequences—the first characterization of logical completeness in these terms did not appear until *Robinson 1951*—two decades after Gödel's thesis.[11] If Gödel neither inhabited nor articulated the modern syntax/semantics distinction, how did he manage to prove the theorem that made this nineteenth century relic respectable once again?

Although Gödel's work is well-known, it is not usually read against the backdrop of the shifting configuration of ideas that surrounded it. Gödel, however, reflected on such matters in his late writing and considered the currents of thought relevant factors, not always positive, in the attainment of new results. In *Gödel 1961* he characterized twentieth-century work in the foundations of mathematics in terms of *apriorism* and empiricism—grouping the first of these notions together with theology, idealism, and metaphysics as being "on the right" of a dialectic with empiricism, skepticism, and positivism on its left. "Now it is a familiar fact," he wrote, "even a platitude, that the development of philosophy since the Renaissance has by and large gone from right to left—not in a straight line, but with reverses, yet still, on the whole" (p. 375). However, he cautioned, "by its nature mathematics is very recalcitrant in the face of the *Zeitgeist*" (p. 379). Modern thought has managed to annex more and more questions into the domain of empirical investigation, often against deep-seated presuppositions that certain matters could not be so addressed. But even in the twentieth century, the opinion persisted that questions of mathematics are perhaps uniquely immune to this trend, are quintessentially *a priori* in nature, so that their reformulation in empirical terms will always be a mutilation of their original, properly mathematical sense. Referring specifically to the question of logical completeness, David Hilbert remarked that "[u]p till now we have come to the view that these rules suffice only through experiment [probieren]" and called for a mathematical proof that would provide more than evidence (*Hilbert 1929*, p. 140).[12]

Although he made his own affinity for rightward thinking clear and decried the "rabid" progression of thought "on the left" as overly pessimistic and a disservice to the nobility of human reason, Gödel acknowledged a certain propriety in the *Zeitgeist* and declared a partial allegiance to it: "Now one can of course by no means close one's eyes to the great advances which our time exhibits in many respects, and one can with a certain justice assert that these advances are due just to this leftward spirit in philosophy and

[10] See for example the first five papers in *Tarski 1956*.

[11] See *Dawson 1993* for more information about the late acknowledgement of the primacy of the semantic consequence relation and the role of Robinson's book. It is worth noting that even Robinson's contribution was widely dismissed. In his *1953* review, Goodstein wrote "Only the first fifth of the book, which is devoted to an extension of Gödel's completeness theorem..., has any bearing on the foundations of mathematics, and the remaining four-fifths may be read without reference to this first part which could with advantage have been omitted."

[12] A passage in *Hilbert and Ackermann 1928*, evidently based on notes prepared by Bernays in 1917, makes the same point. Why is the question still unsolved? Because at present "[i]t is only known purely empirically that this axiom system suffices for all applications."

world-view" (p. 377). If we try to salvage mathematics by ignoring this fact, the result will not fit with the world we inhabit, i.e., the world as we view it. Gödel therefore advised a "workable combination" of empiricism and *apriorism* that "avoids both the death-defying leaps of idealism into a new metaphysics as well as the positivistic rejection of all metaphysics" (p. 387).

It has been suggested that this stage of Gödel's thought succeeded a more stridently "rightward" temperament that informed his youthful achievements. But Gödel's thesis can profitably be read as an exercise in the dialectic between the modern *Zeitgeist* and mathematical recalcitrance. What distinguished Gödel from other logicians was not solely his adherence to the classical sense of objective mathematical truth—other writers of his day resisted the encroachment of empiricism in mathematics. Gödel seems, rather, to have been uniquely attentive to both voices in this dialectic: He addressed the completeness question in a way only possible by heeding each.

This is evident especially in the reference Gödel made, in the introductory remarks to his thesis, to the conviction, shared by Hilbert, Poincáre, and other leading scientific voices, that ontological and semantic theories about existence and truth could be eliminated from scientific discourse, these concepts redefined nominalistically. Hilbert gave this idea a wide audience when, in his 1900 address to the International Congress of Mathematicians, he remarked:

> if it can be proved that the attributes assigned to the concept can never lead to a contradiction by the application of a finite number of logical processes, I say that the mathematical existence of the concept (for example, of a number or a function which satisfies certain conditions) is thereby proved. ... Indeed, when the proof for the compatibility of the axioms [of real numbers in analysis] shall be fully accomplished, the doubts which have been expressed occasionally as to the existence of the complete system of real numbers will become totally groundless.

Hilbert's most detailed clarification of this "consistency implies existence" doctrine came in the course of a 1897–1902 correspondence with Gottlob Frege about the foundations of geometry and axiomatic method. Frege had reported his view about the relationship between truth and consistency:

> I call axioms propositions that are true but are not proved because our knowledge of them flows from a source different from the logical source, a source which might be called spatial intuition. From the truth of the axioms it follows that they do not contradict each other. (*Frege 1980*, p. 39)

Recorded in these words is a vestige of Bolzano's *Grundlehre*: Axioms are not a matter of convention, but inherently unprovable facts. They are objectively true, the ground of the truth of other mathematical claims, and just as Bolzano knew that a proof that tracked the *Abfolgen* among propositions could not be circular, Frege cites the truth of each individual axiom as the reason that they cannot contradict one another. In his reply, Hilbert objected to each point:

> I found it very interesting to read this sentence in your letter, for as long as I have been thinking, writing, and lecturing on these things, I have been saying the exact opposite: if the arbitrarily given axioms do not contradict each other with all their consequences, then they are true and the things defined by the axioms exist. For me this is the criterion of truth and existence. (p. 40)

Clearly Hilbert's intention to treat consistency as "the criterion of truth and existence" depends on the completeness of his logical rules. If those rules are not strong enough and sufficiently many, then a system could be free of contradiction for the irrelevant reason that the derivational apparatus is too meagre. In that case one would not want to conclude that the things defined by its axioms exist. Gödel observed, however, that the doctrine that mathematical existence and truth be reduced to the syntactic condition of consistency is an obstacle to a completeness proof. Logical completeness, he wrote, "can easily be seen to be equivalent to the following: Every consistent axiom system consisting of only [first-order formulas] has a realization." He continued: "one might perhaps think that the existence of the notions introduced through an axiom system is to be defined outright by the consistency of the axioms and that, therefore, a proof [of the existence of a model, based on those axioms' formal consistency] has to be rejected out of hand." That proof requires one to demonstrate the link between consistency and existence—a task hard to appreciate by anyone who believes that mathematical existence can simply be defined in this manner. Thus the completeness proof that Hilbert sought would elude anyone who identified existence and consistency at a conceptual level. But Gödel objected further to Hilbert's idea that any consistent set of axioms implicitly defines a system of things as not merely requiring proof, but simply false. If axioms are supposed to fix their reference uniquely in the sense of being categorical, then they can only do this if they comprise a syntactically complete theory. For if a system, say first-order PA, is not syntactically complete, then there is some sentence S such that both PA$+S$ and PA$+\neg S$ are consistent and, by the completeness theorem, satisfiable. In that case, PA has two distinct models and therefore cannot meaningfully be said to define any one structure. "This definition," Gödel concluded, "manifestly presupposes ... that every mathematical problem is solvable." (Gödel's most famous achievement is his (*1931*) demonstration that this presupposition is false, i.e., that many theories including PA are syntactically incomplete.) Nominalism is self-refuting because its attempted redefinition of truth and existence depends on a theorem that falsifies those definitions.

Gödel did not present this refutation of the "left-leaning" attempt to supplant metaphysical truth with syntactic conditions as reason to return to Frege's traditional approach to questions of truth and consistency. Nor did he draw from his discovery a fully developed reversal of Bolzano's consequence/derivability scheme. He worked within the modern framework, preserving the syntactic notion of consequence, and merely investigated a tension internal to Hilbert's thought: between the doctrine that there is nothing more to truth than formal consistency and the idea that logical completeness must be proved. Thus the configuration of ideas familiar to logicians today, far from being necessary, is of rather recent vintage. Gödel set in motion a gradual reconfiguration of basic assumptions about logical form and content that began in the only possible place—where everyone around him was already standing. The semantic definition of consequence followed years later, the reformulation of completeness as a full reversal of Bolzano's question later still.

The scientific merits of reconcieving logical completeness are well-known. If Gödel did not envision anything quite like modern set-theoretical semantics, he nevertheless showed that the realization of completeness as a correspondence of truth and proof had a further consequence: the compactness of first-order languages, what has perhaps been the single most applicable result in model-theory. But not only has Gödel's theorem shaped proof-theory and model-theory in the decades since Gödel's work. The legitimacy of each as a branch of logic and the observed coincidences of their central concepts (e.g., theorems of Beth and Craig) rest on the fundamental correspondence that Gödel established.

Gödel's refusal to be swept away by nominalism played an essential role in his work on logical completeness. But even as he resisted the ideological currents surrounding formalism, Gödel was attentive to the formalisms themselves, the problems they raised, their merits and limitations. He wrote to Wang in 1972 that "Wittgenstein's negative attitude towards symbolic language is a step backward" (*Wang 1996*, p. 174). Gödel managed to be drawn into neither the nominalistic nor the metaphysical conception of logic so deeply as to be unable to appreciate its rival. In his hands, in fact, the two conceptions were not rival: Their hidden, partial affinity was the subject of his work.

5. A yellow rose

Gödel's vindication of the old metaphysical concept of mathematical truth against the encroachment of nominalism was doubly unlikely. Most obviously, as Gödel himself stressed, the momentum of positivism and empiricism that had accumulated since the Renaissance was, by the twentieth century, formidable. Scientific progress in the large, and modern logical research in particular, thrived on overturning *a priori* theories. The ontological character of Bolzano's thought, for example, was an obstacle to its scientific development. Even as it framed a clear and gripping completeness question, it stood in the way of several fundamental features of modern logic—the focus on formulas instead of propositions, the association of derivability with mechanizable processes, the disregard of factual truth—and ultimately, also, rendered its own central question unanswerable. Gödel noted that "the preceding rightward philosophy" was "excessive" and oriented in "the wrong direction," and, despite his ideological misgivings, he credited the stubborn sobriety of empiricism with correcting these faults (*1961*, p. 381).

Gödel's tenacity in the face of the *Zeitgeist* and his ability to inaugurate an entire research program based on a resistance against the prevailing scientific temperament is therefore striking. It is even more incredible in light of the conventionality of the formulation of completeness in terms of form/content correspondence. Students of logic today take Gödel's formulation to be *the* definition of logical completeness and see Gödel's theorem as inevitable. But a satisfactory reformulation of Bolzano's question did not require opposing the spirit of the age. The progression of ideas led very naturally to Gentzen's results, which show definitively that completeness can be dissociated from metaphysical truth and expressed as a property internal to a logical system. It is a marvel that a great scientific achievement of the twentieth century would involve reinstating the centrality of *a priori*, transcendent truth when the same question can be answered in the century's preferred terms.

Some readers might be tempted to fasten on the plurality of conceptions of logical completeness and the evitability of Gödel's accomplishment as evidence against the importance of his theorem and the correctness of the point of view that made it possible: If Gödel's conception of logic has viable rivals, then it is not the right way of looking at things in any objective sense. But closer consideration supports the opposite verdict. Yes, logic could well have developed around an alternative conception of completeness. The question Gödel answered was not "out there"—intelligible and pressing to everyone who worked in the field—but was rather an artifact of a particular way of looking at things that the current of modern thought made all but impossible. Gödel not only overcame these challenges, but he communicated his discovery with such clarity and force as to introduce a whole way of thinking so fundamental and useful that it has come to seem, in very short time, unavoidable. Gödel's theorem is monumental, not because it solved an eternal problem, but because it pins down a way of looking at logic that we might not otherwise have been acquainted with, yet could not today live without.

One cannot sail against the wind by ignoring it, only by reading it especially carefully. So too Gödel did not re-varnish the reputation of objective mathematical truth by turning his back on the tendencies of modern thought—he showed that empirical, formal logical investigations depend on this scorned fantasy.

> Then the revelation occurred: Marino saw the rose as Adam might have seen it in Paradise, and he thought that the rose was to be found in its own eternity and not in his words; and that we may mention or allude to a thing, but not express it; and that the tall, proud volumes casting a golden shadow in a corner were not—as his vanity had dreamed—a mirror of the world, but rather one thing more added to the world.[13]

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[13] Jorge Luis Borges, "A yellow rose."

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