Syntactic Interchange

Intermediate Logic

The Interchange Theorem says that if \( \psi \) results from replacing the subformula \( \phi_s \) of \( \phi \) with \( \psi_s \), then if \( \phi_s \) and \( \psi_s \) are equivalent so are \( \phi \) and \( \psi \). Unlike substitution, interchange requires neither that the subformula \( \phi_s \) be a single sentence letter nor that all occurrences of \( \phi_s \) in \( \phi \) be replaced. In this respect its conditions are more liberal. But also unlike substitution, which allows replacement with any formula whatsoever, interchange has the stringent requirement that the sub-formula one is replacing be equivalent to the formula one is replacing it with. So in terms of viable choices of \( \psi_s \), interchange is far more conservative.

This second point of contrast between substitution and interchange is worthy of close consideration. A formal axiomatic system cannot have an “interchange rule” without compromising its purely syntactic character. For one cannot know whether or not the rule can be applied in the course of constructing a proof without first considering certain semantic relationships among formulas, viz. which formulas are equivalent to which. The substitution rule poses no such difficulty. For although one must take care in its application to substitute only for sentence letters and to do so in a thorough-going manner, these conditions are explicit in the form of the formula one is looking at and thus can be checked without attending to what the formulas or any of their constitutive parts mean. (Compare this distinction with the difficulty one faces when considering how to restrict the substitution rule in the formulation of the calculus of proofs from arbitrary assumptions.)

Thus interchange is a semantic phenomenon. Our efforts to coordinate the syntax and semantics of truth-functional logic will be greatly aided if something of this phenomenon can be recovered in syntactic terms, though. For that reason, in this note we endeavor to formulate a syntactic correlate of the interchange theorem and to prove that it is true. At the end of the note,
we point out three special cases of “syntactic interchange” that are useful in the proof of the completeness of system \textit{Peirce}.

To say that two formulas “\(\phi\)” and “\(\psi\)” are equivalent is to say that “\(\phi \equiv \psi\)” is valid. This bi-conditional will be valid when (since \textit{Peirce} is sound) and only when (assuming that \textit{Peirce} is complete) its translation into the language of \(\{\supset, \bot\}\) is provable in \textit{Peirce}. One way to express the Interchange Theorem in syntactic terms, then, is to say that if “\(\psi\)” results from replacing the sub-formula “\(\phi_s\)” of “\(\phi\)” with “\(\psi_s\),” then if the appropriate translation of “\(\phi_s \equiv \psi_s\)” is provable in \textit{Peirce} then so is the appropriate translation of “\(\phi \equiv \psi\).” This claim is equivalent to the Interchange Theorem under the assumption of the completeness of \textit{Peirce}. Thus, by proper use of this claim, one might be able to simulate certain properties of interchange in the syntactic setting. Whether or not one will be able to do this depends on whether or not system \textit{Peirce} is in fact complete. Our tactic will be to assume that the simulation works in order to foster the appropriate intuition about what we will need in order to prove completeness. If our assumption is correct, it will then demonstrably be so once the completeness of \textit{Peirce} is proved.

The first move along these lines is to reformulate the syntactic interchange claim as a statement even more of which falls within the scope of a provability clause. This is done as follows: The conditional “if the appropriate translation of ‘\(\phi_s \equiv \psi_s\)” is provable in \textit{Peirce} then so is the appropriate translation of ‘\(\phi \equiv \psi\),” will be reformulated in the object language as a pair of formulas, both of which we claim (in the metalanguage) are provable in \textit{Peirce}. By availing ourselves of the equivalence of “\((p \land q) \supset r\)” and “\(p \supset (q \supset r)\),” this also allows us to avoid use of awkward direct translations of bi-conditionals. Finally, to enhance the applicability of the theorem, we allow multiple substitutions. Syntactic Interchange thus reads:

\textbf{Syntactic Interchange:} If “\(\psi\)” results from replacing zero or more occurrences of the sub-formula “\(\phi_s\)” of “\(\phi\)” with “\(\psi_s\),” then

1. \(\vdash P (\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\phi \supset \psi))\), and
2. \(\vdash P (\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\psi \supset \phi))\).

(1) is the syntactic correlate of the claim that if “\(\phi_s\)” and “\(\psi_s\)” are equivalent, then “\(\phi\)” implies “\(\psi\).” (2) is the syntactic correlate of the claim that if “\(\phi_s\)” and “\(\psi_s\)” are equivalent, then “\(\psi\)” implies “\(\phi\).”
To prove this result, we proceed by induction on the number of occurrences of the symbol ‘⊃’ in ‘φ’.

**Base:** Suppose ⊃ does not occur in φ. Then either φ is a single sentence letter or φ is ⊥. Therefore the only sub-formula of φ is φ itself. There are two cases to consider.

- If ψ results from making zero replacements, then φ is ψ. We therefore have proofs:

\[
\begin{align*}
q \supset (p \supset (s \supset s)) & \quad \text{theorem 10} \\
(\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\phi \supset \psi)) & \quad \text{sub}
\end{align*}
\]

\[
\begin{align*}
q \supset (p \supset (s \supset s)) & \quad \text{theorem 10} \\
(\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\psi \supset \phi)) & \quad \text{sub}
\end{align*}
\]

- Otherwise, ψ results from making one replacement, so φ is φₕ and ψ is ψₕ. Thus we have proofs:

\[
\begin{align*}
p \supset (q \supset p) & \quad \text{axiom 2} \\
(\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\phi \supset \psi)) & \quad \text{sub}
\end{align*}
\]

\[
\begin{align*}
p \supset (s \supset s) & \quad \text{theorem 9} \\
(\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\psi \supset \phi)) & \quad \text{sub}
\end{align*}
\]

**Induction:** Assume that syntactic interchange holds for formulas containing ⊃ n times or fewer, and suppose that φ has n+1 occurrences of ⊃. If the number of replacements made to generate ψ is zero or if the entire formula is replaced, then this is identical to the base case. Otherwise, notice that φ has the form φ₁ ⊃ φ₂ and that ψ has the form ψ₁ ⊃ ψ₂ (the displayed ⊃ is the main connective), where ψ₁ is the result of replacing zero or more occurrences of φₕ in φ₁ with ψₕ and ψ₂ is the result of replacing zero or more occurrences of φₕ in φ₂ with ψₕ. Then since φ₁ and φ₂ each have n or fewer ⊃ₕ’s, the induction hypothesis applies to them, generating
Thus we have proofs:

1. \((\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\phi_1 \supset \psi_1))\)
2. \((\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\psi_1 \supset \phi_1))\)
3. \((\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\phi_2 \supset \psi_2))\)
4. \((\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\psi_2 \supset \phi_2))\)

The proof of the Syntactic Interchange Theorem is complete.

When one verifies that a formula is valid, one checks that it is true under all interpretations of its sentence letters. Slightly more formally, one substitutes truth values for the sentence letters in the formula and then resolves this interpreted formula by appealing to the definitions of its connectives and replacing interpreted sub-formulas with truth values repeatedly until what remains is a single truth value. This is graphically represented by filling out one row of the formula’s truth table. To prove completeness, one must show that every formula, each of whose interpretations resolves thus to T, is provable. Peirce is equipped to express this, for the zero-place connective “⊥” is part of its language and is also a truth-value. Peirce also can express truth with the formula “⊥ ⊃ ⊥.” Thus the completeness question is the question

1. \((\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\phi_1 \supset \psi_1))\)
2. \((\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\psi_1 \supset \phi_1))\)
3. \((\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\phi_2 \supset \psi_2))\)
4. \((\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\psi_2 \supset \phi_2))\)
whether every formula that resolves to “⊥ ⊃ ⊥” whenever “⊥” is substituted for some of its sentence letters and “⊥ ⊃ ⊥” is substituted for the rest is provable. Syntactic Interchange is used to formalize the notion of resolution. (If these heuristics are too vague for your taste, trust that they will be made more precise later in the note on the completeness theorem.)

Since resolution steps are always replacements of sub-formulas with “⊥” or “⊥ ⊃ ⊥,” three special cases of syntactic interchange are particularly important.

**Special Case 1:** If “ψ” results from replacing zero or more occurrences of the sub-formula “φs” of “φ” with “⊥ ⊃ ⊥,” then

1. ⊢ P φs ⊃ (φ ⊃ ψ), and
2. ⊢ P φs ⊃ (ψ ⊃ φ).

```
(φs ⊃ ψs) ⊃ (((ψs ⊃ φs) ⊃ (φ ⊃ ψ))
(φs ⊃ (⊥ ⊃ ⊥)) ⊃ (((⊥ ⊃ ⊥) ⊃ φs) ⊃ (φ ⊃ ψ))
((p ⊃ (s ⊃ s)) ⊃ ((q ⊃ p) ⊃ r)) ⊃ (p ⊃ r)
(((φs ⊃ (⊥ ⊃ ⊥)) ⊃ (((⊥ ⊃ ⊥) ⊃ φs) ⊃ (φ ⊃ ψ)))) ⊃ (φs ⊃ (φ ⊃ ψ))
φs ⊃ (φ ⊃ ψ)
```

This is intuitive enough: If “φs” is replaced with “T,” then if “φs” is true and ψ are equivalent.

**Special Case 2:** If “ψ” results from replacing zero or more occurrences of the sub-formula “φs” of “φ” with “⊥,” then

1. ⊢ P (φs ⊃ ⊥) ⊃ (φ ⊃ ψ), and

```
(φs ⊃ ψs) ⊃ (((ψs ⊃ φs) ⊃ (ψ ⊃ φ))
(φs ⊃ (⊥ ⊃ ⊥)) ⊃ (((⊥ ⊃ ⊥) ⊃ φs) ⊃ (ψ ⊃ φ))
((p ⊃ (s ⊃ s)) ⊃ ((q ⊃ p) ⊃ r)) ⊃ (p ⊃ r)
(((φs ⊃ (⊥ ⊃ ⊥)) ⊃ (((⊥ ⊃ ⊥) ⊃ φs) ⊃ (ψ ⊃ φ)))) ⊃ (φs ⊃ (ψ ⊃ φ))
φs ⊃ (ψ ⊃ φ)
```

This is intuitive enough: If “φs” is replaced with “T,” then if “φs” is true φ and ψ are equivalent.
\[
(\phi_s \supset \psi_s) \supset ((\psi_s \supset \phi_s) \supset (\phi \supset \psi)) \\
(\phi_s \supset \bot) \supset ((\bot \supset \phi_s) \supset (\phi \supset \psi)) \\
(q \supset ((\bot \supset p) \supset r)) \supset (q \supset r) \\
((\phi_s \supset \bot) \supset ((\bot \supset \phi_s) \supset (\phi \supset \psi))) \supset ((\phi_s \supset \bot) \supset (\phi \supset \psi)) \\
(\phi_s \supset \bot) \supset (\phi \supset \psi)
\]

This is similarly intuitive: If \(\phi_s\) is replaced with “F,” then if \(\phi_s\) is false \(\phi\) and \(\psi\) are equivalent. Note that Peirce expresses “\(\phi_s\) is false” with the formula \(\phi_s \supset \bot\).

**Special Case 3:** If \(\psi_1\) results from replacing zero or more occurrences of the sub-formula \(\phi_s\) of \(\phi\) with \(\bot\) and \(\psi_2\) results from replacing zero or more occurrences of the sub-formula \(\phi_s\) of \(\phi\) with \(\bot \supset \bot\) then \(\vdash_p \psi_2 \supset (\psi_1 \supset \phi)\).

\[
(\phi_s \supset \bot) \supset (\psi_2 \supset \phi) \\
\phi_s \supset (\psi_1 \supset \phi) \\
(((p \supset q) \supset (s \supset r)) \supset ((p \supset (t \supset r)) \supset (s \supset (t \supset r)))) \\
(((\phi_s \supset \bot) \supset (\psi_2 \supset \phi)) \supset ((\phi_s \supset (\psi_1 \supset \phi))) \supset (\psi_2 \supset (\psi_1 \supset \phi)) \\
(\phi_s \supset (\psi_1 \supset \phi)) \supset (\psi_2 \supset (\psi_1 \supset \phi)) \\
\psi_2 \supset (\psi_1 \supset \phi)
\]

\(\text{SI}\) \hspace{1cm} \(\text{sub}\) \hspace{1cm} \(\text{theorem 19}\) \hspace{1cm} \(\text{mp}\)