Effective Prime Uniqueness

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Prime Models

Definition

A countable model $\mathcal{A}$ is *prime* if for all models $\mathcal{B}$ if $\mathcal{A} \equiv \mathcal{B}$ then $\mathcal{A} \preccurlyeq \mathcal{B}$.

- Let T be the first order theory of $\mathcal{A}$. So $\mathcal{A}$ elementary embeds into every model $\mathcal{B}$ of T.
- T will always be a complete first order theory in a countable language.
- We want to explore the notion of prime structures. First classically then effectively.

Types

Definition

Given a model $\mathcal{A}$ and a tuple $\vec{a} \in |\mathcal{A}|$ the *type* of a, $p(\vec{a})$ is the theory of $(\mathcal{A}, \vec{a})$.

Lemma

If $\mathcal{A} \preccurlyeq \mathcal{B}$ via f then, for all $\vec{a} \in |\mathcal{A}|$, the types of $\vec{a}$ and $f(\vec{a})$ are the same.

Definition

A type $p(\vec{a})$ is principal iff there is a formula $\phi(\vec{a})$ such that, for all $\sigma(\vec{a}) \in p(\vec{a})$, $T \vdash \phi(\vec{a}) \to \sigma(\vec{a})$. ($\phi$ is called an atom of T and the complete formula of $p(\vec{a})$.)

Atomic Models

Theorem (Omitting Types)

If a type $p(\vec{a})$ is nonprincipal there is a model $\mathcal{A}$ omitting it.

Definition

A model $\mathcal{A}$ is atomic if all its types are principal.

Corollary

All prime models are atomic.

First Isomorphism Result

Lemma

If $\mathcal{A}$ and $\mathcal{B}$ are atomic models of T then they are isomorphic.

Proof.

Build the isomorphism f stagewise using a back and forth argument making sure the types of the domain and range remain the same.

(Forth) Assume that f_{2s} is a partial finite function with domain $\vec{a}$ such that the types of $\vec{a}$ and $f_{2s}(\vec{a})$ are the same. At stage $2s+1$, let d be the first element of $\mathcal{A}$ not in $\vec{a}$, **let** $\phi(\vec{a}, d)$ **be the complete formula of** $p(\vec{a}, d)$, then let $f_{2s+1}(d)$ be the first element of $\mathcal{B}$ not in $f_{2s}(\vec{a})$ such that $\mathcal{B} \models \phi(f_{2s}(\vec{a}), f_{2s+1}(d))$.

(Back) Similar. □

Two Useful Corollaries from the Isomorphism Result

Corollary

Atomic models are prime.

Corollary (Prime Uniqueness)

Two prime models are isomorphic.

Again the key idea is finding the complete formula.

Complete formulas

Lemma

$\{\phi | \phi$ is a complete formula$\}$ is Π^T_1.

Proof.

Check all σ does $T \vdash \phi \to \sigma$ or $T \vdash \phi \to \neg\sigma$. (Recall T is complete.) □

Lemma (Folklore)

There are computable complete theories T (T is called decidable) where $\{\phi | \phi$ is a complete formula$\}$ is Π^0_1-complete.

Sketch.

Use unary predicates R_i and $R_{i,s}$ to model $\phi_i(i)$ and $\phi_{i,s}(i)$. □

There are also decidable T where the complete formulas are computable. This is also dependent on the language.

Atomic Theories

Definition

A theory T is *atomic* if for every formula σ there is an atom ϕ of T such that $T \vdash \phi \to \sigma$.

Lemma (AMT)

Every atomic theory has an atomic model.

AMT was explored by Hirschfeldt, Shore and Slaman. It turns out that this does not hold effectively, is incomparable to WKL and is properly below ACA.

Question

Does prime uniqueness hold effectively?

Yes. Hence this talk and corresponding paper.

Effectively Prime and Atomic Models

Definition

Let T be a decidable theory and $\mathcal{A}$ a decidable model of T.

- The model $\mathcal{A}$ is *effectively prime*, if for every decidable model $\mathcal{M} \models T$, there is a computable elementary embedding $f : \mathcal{A} \to \mathcal{M}$. Note that f need not be uniformly computable in $\mathcal{A}$ and/or $\mathcal{M}$.
- The model $\mathcal{A}$ is *effectively atomic* if there is a computable function g that accepts as an input a tuple $\vec{a}$ from $\mathcal{A}$ (of any length) and outputs a complete formula $\varphi(\vec{x})$ so that $\mathcal{A} \models \varphi(\vec{a})$. Again g need not be uniformly computable in $\mathcal{A}$.
- The model $\mathcal{A}$ is *uniformly effectively prime* if there is a partial computable function Φ so that, given $\mathcal{M} \models T$, $\Phi(\mathcal{M})$ halts and outputs the code of a computable elementary embedding $f : \mathcal{A} \to \mathcal{M}$. Again Φ need not be uniformly computable in $\mathcal{A}$.

Effective Corollaries from the Isomorphism Result

Corollary

If two decidable models $\mathcal{A}$ and $\mathcal{B}$ of the same decidable theory T are both effectively atomic, then the classical back and forth construction produces a computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Corollary

Effectively atomic implies uniformly effectively prime.

Effectively prime = effectively atomic = uniformly effectively prime

Theorem (RCA$_0$)

There is a Turing functional $\Phi(\mathcal{A}, e)$ *such that if* T *is decidable and* $\mathcal{A} \models T$ *is a decidable model then either, for some* e, $\Phi(\mathcal{A}, e)$ *witnesses that* $\mathcal{A}$ *is effectively atomic or there is a decidable* $\mathcal{M} \models T$, *such that there is no computable elementary embedding of* $\mathcal{A}$ *into* $\mathcal{M}$.

Cannot be improved!

Lemma (Folklore)

For all Φ*, there in an effectively atomic* $\mathcal{A}$ *such that* $\Phi(\mathcal{A})$ *does not witness that* $\mathcal{A}$ *is effectively atomic.*

Sketch.

Use the recursion theorem for a code of $\mathcal{A}$. Work in the language of infinitely many unary relations and depending on $\Phi(\mathcal{A})$ the resulting model has nothing in any of these relations or exactly one of the relations splits the model into 2 infinite parts. □

Hence the “obvious” notion of “uniformly effectively atomic” is vacuous. Again this is also dependent on the langauge and T. But note a code for $\mathcal{A}$ computes the theory T.

A Preliminary Result

Theorem

Let T be decidable and $\mathcal{A}, \mathcal{B} \models T$ be decidable models. Then either there is a computable isomorphism $h : \mathcal{A} \cong \mathcal{B}$; or there is a decidable $\mathcal{M} \models T$, so that either there is no computable elementary embedding of $\mathcal{A}$ into $\mathcal{M}$, or there is no computable elementary embedding of $\mathcal{B}$ into $\mathcal{M}$.

The Construction

Given $\mathcal{A}$. Build $\mathcal{M}$ via a Henkin construction using a finite priority argument to meet the following:

$\mathcal{R}_\Psi$: $\neg(\mathcal{A} \preccurlyeq \mathcal{M}$ via $\Psi)$ or there is a g witnessing that $\mathcal{A}$ is effectively atomic.

If Ψ is a permutation then we meet $\mathcal{R}_\Psi$ via different types, for some $\vec{a}$, the types of $\vec{a}$ and $\Psi(\vec{a})$ are different. Meeting $\mathcal{R}_\Psi$ is Σ_2. When adding forumlas σ_s for the diagram of $\mathcal{M}$ one looks diagonalize for some $\vec{a}$ and $\Psi(\vec{a})$. We might are have *carefully and systematically* alter the formula we are adding by diagonalize increasing it's logical consequences over T and then repeat our algorithm in case this new formula can be used by a higher priority. The failure to diagonalize for some permutation Ψ produces g.

In RCA_0

Proof.

First at each stage use Σ_1 induction to show that there is no requirement or a least requirement for which we can diagonalize.

Meeting $\mathcal{R}_\Psi$ is Σ_2: either is not total (Σ_2), is not onto (Σ_2), is not $1-1$ (Σ_1), or, for some $\vec{a}$, the types of $\vec{a}$ and $\Psi(\vec{a})$ are different (Σ_1). Assume that Ψ is an embedding of $\mathcal{A}$ into $\mathcal{M}$ then there by Σ_1 induction is a stage where all higher priority requirements stop acting (so no longer impacting how formulas are added to the diagram of $\mathcal{M}$). So after that stage if we can diagonalize to beat Ψ we will. This allows us to show $\mathcal{A}$ is effectively atomic. $\square$