ON THE ELEMENTARY THEORIES OF THE MUCHNIK AND MEDVEDEV LATTICES OF Π^0_1 CLASSES

A Dissertation

Submitted to the Graduate School
of the University of Notre Dame
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy
in
Mathematical Science

by

Joshua A. Cole, B.A., M.S.

Peter Cholak, Director

Graduate Program in Mathematics
Notre Dame, Indiana
November 2009

ON THE ELEMENTARY THEORIES OF THE MUCHNIK AND MEDVEDEV LATTICES OF Π^0_1 CLASSES

Abstract

by

Joshua A. Cole

Recently, researchers in the foundations of mathematics have become interested in the distributive lattices $\mathcal{P}_w$ and $\mathcal{P}_s$, the study of which is a major part of the field of "mass problems." In general, a mass problem $\mathcal{U}$ is a subset of ω^ω (Baire space). We define $\mathcal{U} \leq_s \mathcal{V}$ if there is an index e for a computable functional so that $\forall f \in \mathcal{V}(\Phi_e^f \in \mathcal{U})$. If we do not require the same e for every f, we get a "weak" version: $\mathcal{U} \leq_w \mathcal{V}$ if for all $f \in \mathcal{V}$ there is an e so that $\Phi_e^f \in \mathcal{U}$.

The relations $\leq_s$ and $\leq_w$ naturally induce equivalence relations $\equiv_s$ and $\equiv_w$. We define $\mathcal{P}_s$ to be the collection of $\equiv_s$ degrees of nonempty Π^0_1 subsets of 2^ω (Cantor space); similarly, we define $\mathcal{P}_w$ to be the collection of $\equiv_w$ degrees of nonempty Π^0_1 subsets of 2^ω.

An important open question is whether $\mathcal{P}_w$ is dense. The result of Chapter 2 is that the embedding of the free distributive lattice on countably many generators into $\mathcal{P}_s$ can be done densely. The way it is done gives indirect evidence that the kinds of priority arguments that show the density of $\mathcal{P}_s$ are probably not strong enough to show the density of $\mathcal{P}_w$. The result of Chapter 3 applies these priority arguments to show the decidability of the elementary $\forall\exists$-theory of $\mathcal{P}_s$ as a partial order. The result of Chapter 5 is that certain index sets related to $\mathcal{P}_w$ are

Π^1_1-complete. This leads to a conjecture that the Turing degree of its elementary theory is as high as possible.

CONTENTS

FIGURES

ACKNOWLEDGMENTS

I would like to thank my advisor, Peter Cholak, for his insightful guidance and constant support, especially during this past year in which I have been non-resident. Stephen Simpson introduced the lattices $\mathcal{P}_s$ and $\mathcal{P}_w$ and has been the leading researcher in their study; I thank him for his willingness to discuss them, and for the collaboration that led to Section 5.1. I would also like to thank Takayuki Kihara and Keng Meng Ng for helpful conversations.

I am grateful for the friendly logic group at Notre Dame, and for the many logic seminars with interesting guest speakers. I would also like to thank my friends at Notre Dame (including Troy, Philip and Glenn), and especially my fellow graduate students in the mathematics department. They have helped me learn some mathematics and something about life.

I would also like to acknowledge the support of a fellowship from the Arthur J. Schmitt Foundation, and from National Science Foundation grants NSF-DMS-0652669 and NSF-EMSW21-RTG-0739007.

CHAPTER 1

INTRODUCTION

The reductions $\leq_s$ and $\leq_w$ originally were named Medvedev and Muchnik reducibility; we like to use "s" and "w" to avoid the confusion that results from both names beginning with an "M". An old intuition is that a subset of ω^ω can be thought of as a mathematical problem, and then an element of the subset is thought of as a solution of the problem. If $\mathcal{U}$ and $\mathcal{V}$ are subsets of ω^ω, then $\mathcal{U}$ is *Medvedev* (or *strongly*) reducible to $\mathcal{V}$ ($\mathcal{U} \leq_s \mathcal{V}$) if there is a Turing functional Ψ so that for all $f \in \mathcal{V}$, $\Psi(f) \in \mathcal{U}$. If $\mathcal{U} \leq_s \mathcal{V}$ then a solution to the mass problem $\mathcal{V}$ uniformly yields a solution to the mass problem $\mathcal{U}$. Muchnik reducibility relaxes the uniformity constraint: if $\mathcal{U}$ and $\mathcal{V}$ are subsets of ω^ω, then $\mathcal{U}$ is *Muchnik* (or *weakly*) reducible to $\mathcal{V}$ ($\mathcal{U} \leq_w \mathcal{V}$) if for every $f \in \mathcal{V}$ there is a Turing functional Ψ so that $\Psi(f) \in \mathcal{U}$. Under either reducibility, there is an equivalence relation defined in the usual way: $\mathcal{U} \equiv \mathcal{V}$ just in case $\mathcal{U} \leq \mathcal{V}$ and $\mathcal{V} \leq \mathcal{U}$. We write $deg_s(\mathcal{U})$ ($deg_w(\mathcal{U})$) for the degree of $\mathcal{U}$ under the equivalence relation induced by strong (weak) reducibility.

We work mostly with Π^0_1 mass problems, and this allows for the use of computable approximations. (See Lemma 5.) We take the following as the definition of a Π^0_1 mass problem.

Definition 1. $R \subseteq \omega^\omega \times \omega$ is a *computable relation* if there is an index e for a Turing functional so that for all $X \in \omega^\omega$ and $n \in \omega$, $\Phi_e^X(n) = 1$ if $R(X, n)$ holds and $\Phi_e^X(n) = 0$ if $R(X,n)$ does not hold. We say $\mathcal{U} \subseteq \omega^\omega$ is Π_1^0 if there is a computable relation R so that $\mathcal{U} = \{X : \forall n R(X,n)\}$.

Except in Section 5.1, we work mostly with subsets of 2^ω, and this allows for the use of compactness; often we will use 'Π_1^0 class' as shorthand for 'Π_1^0 subset of 2^ω.' It turns out that each of the collections of Medvedev and Muchnik degrees of nonempty Π_1^0 subsets of 2^ω is a distributive lattice; we will denote them by $\mathcal{P}_s$ and $\mathcal{P}_w$, respectively. For an introduction to the study of $\mathcal{P}_s$ and $\mathcal{P}_w$, see Simpson's [19]. We will summarize some of the basics needed for this thesis.

The join operation, by definition the least upper bound of two degrees, is the same in $\mathcal{P}_w$ and $\mathcal{P}_s$. It is induced by an operation on representative Π_1^0 classes. If $\mathcal{U}$ and $\mathcal{V}$ are Π_1^0 classes, then

$$\mathcal{U} \vee \mathcal{V} = \{f \oplus g : f \in \mathcal{U}, g \in \mathcal{V}\}.$$

The meet operation, by definition the greatest lower bound of two degrees, is also induced by an operation on representative Π_1^0 classes. In $\mathcal{P}_w$ it is simply given by a union:

$$\mathcal{U} \wedge_w \mathcal{V} = \mathcal{U} \cup \mathcal{V}.$$

In $\mathcal{P}_s$, the meet is similar to a union, but we also need to be able to tell which mass problem an element originally came from. If $\mathcal{U}$ and $\mathcal{V}$ are Π_1^0 classes, then

$$\mathcal{U} \wedge_s \mathcal{V} = \{0^\frown f : f \in \mathcal{U}\} \cup \{1^\frown g : g \in \mathcal{V}\},$$

where for a function h and number n, $n^\frown h$ is the function given by $n^\frown h(0) = n$ and $n^\frown h(x) = h(x-1)$ if $x \neq 0$.

It is a straightforward exercise to check that the induced operations on equivalence classes are well-defined and that $\mathcal{P}_s(\mathcal{P}_w)$, together with these induced operations, is a lattice and is distributive. As is usual when working with a collection of equivalence classes, we will often work with representative elements, as we did above in the definitions of the meet and the join.

There is a bottom element $\mathbf{0}$; in both $\mathcal{P}_w$ and $\mathcal{P}_s$ it the degree of the Π^0_1 subsets of 2^ω which contain a computable function. For if $\mathcal{U}$ contains a computable function, then every member of every $\mathcal{V}$ computes a member of $\mathcal{U}$. There is also a top element $\mathbf{1}$; in both $\mathcal{P}_w$ and $\mathcal{P}_s$ it is the degree of $\mathcal{PA}$, the set of characteristic functions of completions of Peano Arithmetic. (This is originally from a paper of Scott [16], which appears in [8]. For another proof, see a result of Simpson [18, Lemma 3.16], which appears in [20].) However, not every Π^0_1 class weakly equivalent to $\mathcal{PA}$ is strongly equivalent to $\mathcal{PA}$. This can be see from the result that every non-zero weak degree contains infinitely many strong degrees. (This latter result was first shown in a preprint by Simpson and Slaman [23]; an alternative proof has been given in a paper by Simpson and me [7, Corollary 5.10].)

Why do we study $\mathcal{P}_s$ and $\mathcal{P}_w$? That $\mathcal{PA}$ has degree $\mathbf{1}$ in both immediately sparks interest. Furthermore, $\mathcal{P}_s$ is a refinement of $\mathcal{P}_w$, as shown by the result quoted in the last paragraph. Thus, it is reasonable to suppose that the study of the local properties of $\mathcal{P}_s$ may shed light on the the local properties of $\mathcal{P}_w$. In turn there are many connections in $\mathcal{P}_w$ to topics in or related to computability theory, such as the c.e. Turing degrees ($\mathcal{R}_T$) [21], almost everywhere domination [22], the diagonally non-recursive functions, randomness, and computational complexity [19]. The theory of mass problems provides a useful context in which to think about problems in these areas. Also, the techniques for studying $\mathcal{P}_s$ are similar to those used in the study of $\mathcal{R}_T$, as we will see in the next chapter.

When not otherwise noted, our notation follows the standard usage in computability theory. Soare's text is a good reference [25].

CHAPTER 2

EMBEDDING $FD(\omega)$ INTO $\mathcal{P}_s$ DENSELY

A version of this chapter has already been published [6]. A way to investigate the local properties of $\mathcal{P}_w$ and $\mathcal{P}_s$ is to ask the questions that were answered in the case of $\mathcal{R}_T$, the c.e. Turing degrees as a partial order. An easy first result is that while $\mathcal{R}_T$ is only an upper semi-lattice, $\mathcal{P}_w$ and $\mathcal{P}_s$ form true lattices. Binns [3] has shown every non-trivial degree splits in both $\mathcal{P}_w$ and $\mathcal{P}_s$, as Sacks showed of $\mathcal{R}_T$. Alfeld [1] has studied the analogous question in the upwards direction, namely which degrees branch. Sacks proved the density of $\mathcal{R}_T$; Cenzer and Hinman [5] proved the density of $\mathcal{P}_s$, but whether $\mathcal{P}_w$ is dense is not known.

Binns and Simpson have studied which lattices embed in $\mathcal{P}_s$ and $\mathcal{P}_w$, and the main theorem of this chapter is an improvement on two of their results in $\mathcal{P}_s$. Binns [3] proved that every finite distributive lattice embeds densely in $\mathcal{P}_s$. Together Binns and Simpson [4] proved that there is a lattice embedding of $FD(\omega)$, the free distributive lattice on countably many generators, below any non-trivial $deg_s(\mathcal{V}) \in \mathcal{P}_s$. Our result makes this embedding dense.

Theorem 2. *If $deg_s(\mathcal{U}) <_s \deg_s(\mathcal{V})$ in $\mathcal{P}_s$, the lattice of degrees of non-empty Π^0_1 subsets of 2^ω under Medvedev reducibility, then there is a lattice-embedding of $FD(\omega)$, the countable free distributive lattice, strictly between $deg_s(\mathcal{U})$ and $deg_s(\mathcal{V})$.*

The method of construction for our extension of the results of Binns and Simpson is derived from the proof of the density of $\mathcal{P}_s$ by Cenzer and Hinman. They used separating classes of c.e. sets to construct Π^0_1 classes. They satisfied requirements with Sacks Coding and Preservation Strategies, the main techniques of Sacks' proof of the density of $\mathcal{R}_T$.

Sacks faced and solved the problem of infinite injury in his proof. Once the proper definitions are made and preliminary lemmas proved, the proof of Cenzer and Hinman follows closely the proof of the Sacks Density Theorem in a style as in, for example, the proof given by Soare [25, 142-145]. Like the proof of the density of $\mathcal{R}_T$, the proof of the density of $\mathcal{P}_s$ has infinite injury. Although the construction in this Chapter is based on the construction of Cenzer and Hinman, with one modification we are able to eliminate the possibility of infinite injury. In fact, the same modification eliminates infinite injury in the construction of Cenzer of Hinman. We will give more details in Sections 2.4 and 2.5.

Another aspect of our dense-embedding result is that it suggests, along with other evidence, that we can often do anything we want in $\mathcal{P}_s$ densely, if we can do it at all. Simpson and Binns [4] showed $FD(\omega)$ embeds in $\mathcal{P}_w$, and then they showed it embeds in $\mathcal{P}_s$. From this point, we make use of known techniques and a finite-injury priority argument to make an embedding of $FD(\omega)$ in $\mathcal{P}_s$ that is dense.

Binns' paper [3] on splitting in $\mathcal{P}_w$ and $\mathcal{P}_s$ is another example of this process. First, splitting is shown in $\mathcal{P}_w$, and in $\mathcal{P}_s$, and then it is shown to occur densely in $\mathcal{P}_s$, still with only a finite-injury priority argument. Moreover, an attempt to use Binns' methods for the dense splitting in $\mathcal{P}_s$ does not directly yield a dense splitting in $\mathcal{P}_w$. Similarly, the proof in this Chapter cannot easily be modified to embed

$FD(\omega)$ densely in $\mathcal{P}_w$. This is because the length of agreement function used in this chapter has no easy, well-behaved analogue in the case of weak reducibility.

The existence of a dense splitting in $\mathcal{P}_w$, or of a dense embedding of $FD(\omega)$ in $\mathcal{P}_w$, would immediately imply the density of $\mathcal{P}_w$. Thus the result of this chapter is a little bit more evidence that new techniques may be needed to answer the question of density for $\mathcal{P}_w$.

2.1 The general plan

Let $\mathcal{U} <_s \mathcal{V}$. (In the rest of this chapter we will often suppress the subscript s on $<, \leq, \equiv$, etc.) With a priority argument, we will construct sequences of c.e. subsets of ω, $\{A_i\}_{i\in\omega}$ and $\{B_i\}_{i\in\omega}$, satisfying certain properties and such that for each i, $A_i \cap B_i = \emptyset$. Then, for each i, we set

$$\mathcal{S}_i = S(A_i, B_i) = \{X \in 2^\omega : (n \in A_i \Rightarrow X(n) = 1)\&(n \in B_i \Rightarrow X(n) = 0)\}.$$

Finally, for each i, we set

$$\mathcal{V}_i = (\mathcal{U} \vee \mathcal{S}_i) \wedge \mathcal{V}.$$

For this to work, we must show that each $\mathcal{S}_i$ is a Π^0_1 class.

Definition 3. If A and B are disjoint c.e. subsets of ω, then

$$\mathcal{S}(A, B) = \{X \in 2^\omega : (n \in A \Rightarrow X(n) = 1)\&(n \in B \Rightarrow X(n) = 0)\}$$

is a *separating class.*

Lemma 4. *Every separating class is a* Π^0_1 *class.*

Proof. Let $\mathcal{S}(A, B)$ be a separating class, and fix effective enumerations of A, B. Define a computable relation $R \subseteq 2^\omega \times \omega$ as follows. If for some $m \leq n$, m has been enumerated into A by stage n, but $X(m) = 0$, then $R(X, n)$ does not hold. Similarly, if for some $m \leq n$, m has been enumerated into B by stage n, but $X(m) = 1$, then $R(X, n)$ does not hold. Otherwise, $R(X, n)$ does hold. Then, $\mathcal{S}(A, B) = \{X : \forall n R(X, n)\}$.

$\square$

We will consider the lattice $\mathcal{L}$ in $\mathcal{P}_s$ generated by $\{deg_s(\mathcal{V}_i)\}_{i\in\omega}$, and show that if certain requirements are satisfied then $\mathcal{L}$ is free and entirely between $deg_s(\mathcal{U})$ and $deg_s(\mathcal{V})$. Note that the free distributive lattice on countably many generators has no maximal or minimal element.

2.2 Requirements

In our priority construction, we will have positive and negative requirements. The positive requirements desire that no join of a finite subset of $\{\mathcal{V}_i\}_{i\in\omega}$ computes the meet of any finite subset of $\{\mathcal{V}_i\}_{i\in\omega}$, if these two finite subsets are disjoint. The negative requirements desire that no join of a finite subset of $\{\mathcal{V}_i\}_{i\in\omega}$ computes $\mathcal{V}$.

To achieve these goals, the following requirements are sufficient, as we will show.

For each pair I, J of finite subsets of ω such that $I \cap J = \emptyset$:

$$P_{I,J} : \mathcal{U} \vee \bigvee_{i\in I} \mathcal{S}_i \not\geq (\mathcal{U} \vee \bigwedge_{j\in J} \mathcal{S}_j) \wedge \mathcal{V}.$$

$$N_I : \mathcal{U} \vee \bigvee_{i\in I} \mathcal{S}_i \not\geq \mathcal{V}.$$

We must verify five facts to show that if we satisfy each instance of the requirements $P_{I,J}$ and N_I, then the lattice $\mathcal{L}$ generated by $\{deg_s(\mathcal{V}_i)\}_{i\in\omega}$ is free and between $deg_s(\mathcal{U})$ and $deg_s(\mathcal{V})$. From here on, we assume that I and J are finite subsets of the natural numbers. We claim the following:

1. Every element of $\mathcal{L}$ is below $\deg_s(\mathcal{V})$.

2. Every element of $\mathcal{L}$ is above $\deg_s(\mathcal{U})$.

   These first two immediately follow from the fact that $\mathcal{U} \leq \mathcal{V}_i \leq \mathcal{V}$ for all i, which is immediate from the definition of the $\mathcal{V}_i$.

3. We show no element of $\mathcal{L}$ is above $deg_s(\mathcal{V})$. By discarding meets, we see it suffices to show

$$\bigvee_{i\in I} \mathcal{V}_i \not\geq \mathcal{V}.$$

   If not, for some Turing functional Φ_0 we have

$$\Phi_0 : \bigvee_{i\in I} \mathcal{V}_i \to \mathcal{V}.$$

   By the distributive laws, we see

$$\bigvee_{i\in I} \mathcal{V}_i \equiv (\mathcal{U} \vee \bigvee_{i\in I} \mathcal{S}_i) \wedge \mathcal{V}.$$

   Combining these last two, there is a Turing functional Φ_1 so that:

$$\Phi_1 : (\mathcal{U} \vee \bigvee_{i\in I} \mathcal{S}_i) \wedge \mathcal{V} \to \mathcal{V}.$$

Then, for each $f \in (\mathcal{U} \vee \bigvee_{i \in I} \mathcal{S}_i)$, $\Phi_1^{0^\frown f} \in \mathcal{V}$. Hence, there is a Turing functional Φ_2 so that $\Phi_2^f \in \mathcal{V}$ for all $f \in (\mathcal{U} \vee \bigvee_{i \in I} \mathcal{S}_i)$. So $(\mathcal{U} \vee \bigvee_{i \in I} \mathcal{S}_i) \geq \mathcal{V}$, contradicting N_I.

4. We show $\mathcal{L}$ is free. Binns and Simpson [4] pointed out that by a lemma from Lattice Theory (for instance, see [9, Theorem II.2.3]), we need only show that if

$$\bigvee_{i \in I} \mathcal{V}_i \geq \bigwedge_{j \in J} \mathcal{V}_j,$$

then $I \cap J \neq \emptyset$.

Substituting and then expanding both sides by the distributive laws, we have

$$(\mathcal{U} \vee \bigvee_{i \in I} \mathcal{S}_i) \wedge \mathcal{V} \geq (\mathcal{U} \vee \bigwedge_{j \in J} \mathcal{S}_j) \wedge \mathcal{V}.$$

Next, by discarding a meet, we have

$$(\mathcal{U} \vee \bigvee_{i \in I} \mathcal{S}_i) \geq (\mathcal{U} \vee \bigwedge_{j \in J} \mathcal{S}_j) \wedge \mathcal{V}.$$

By $P_{I,J}$, we must have $I \cap J \neq \emptyset$.

5. Finally, we must show no element of $\mathcal{L}$ is below $\deg_s(\mathcal{U})$. By discarding joins we see it suffices to show

$$\bigwedge_{j \in J} \mathcal{V}_j \not\leq \mathcal{U}.$$

This follows from the proof of (4) and the fact that for $I \cap J = \emptyset$,

$$\mathcal{U} \leq \bigvee_{i \in I} \mathcal{V}_i.$$

2.3 Further definitions and a key lemma

Lemma 5. *If $\mathcal{U}$ is a Π^0_1 subset of $\omega^\omega(2^\omega)$, then there is a computable tree $T \subseteq \omega^{<\omega}(2^{<\omega})$ such that $\mathcal{U} = [T]$, where $[T]$ is the set of infinite paths through T.*

Proof. Let $R \subseteq \omega^\omega \times \omega$ be a computable relation such that $\mathcal{U} = \{X : \forall n R(X, n)\}$; let e be such that for all $X \in \omega^\omega$ and $n \in \omega$, $\Phi^X_e(n) = 1$ if $R(X, n)$ holds and $\Phi^X_e(n) = 0$ if $R(X, n)$ does not hold. Define $T \subseteq \omega^{<\omega}$ as follows. If $\sigma(n) = 1$ and $\Phi^\sigma_{e,|\sigma|}(n) = 0$, then $\sigma \notin T$. Similarly, if $\sigma(n) = 0$ and $\Phi^\sigma_{e,|\sigma|}(n) = 1$, then $\sigma \notin T$. Otherwise, $\sigma \in T$. $\square$

Sequences of such trees will be our computable approximations of Π^0_1 classes, in this chapter.

We will want to consider sets of strings of fixed length in these trees, and so we have the following notation.

Definition 6. If T is a tree, let $T^s = \{\sigma \in T : |\sigma| = s\}$.

Definition 7. If T is a tree, then $\sigma \in T$ is *extendible* if there is $X \in [T]$ so that $\sigma \subset X$. Otherwise, $\sigma \in T$ is a *dead end*.

Definition 8. If T is a tree, then $\tilde{T} = \{\sigma \in T : \sigma \text{ is extendible in } T\}$.

Proposition 9. *If $\mathcal{Q}$ is a Π^0_1 class with no computable member, and T is a computable tree such that $[T] = \mathcal{Q}$, then T has infinitely many dead ends.*

We worry the dead ends will cause a problem in our priority argument. Worrying about this difficulty may prove pedagogically useful. However, in the end it is not really a difficulty, and so some readers may wish to skim the rest of this section, which ends on page 17, and consult Sections 3.9.1 and 3.10. Next we give a definition, then a lemma to help with this difficulty.

Definition 10. A sequence of sets $\{C_i\}_{i\in\omega}$ is *nested* if for each $j \leq k$, $C_j \supseteq C_k$.

Lemma 11. *If $\mathcal{U}$ is a Π^0_1 subset of $\omega^\omega(2^\omega)$, there is a nested sequence of uniformly computable trees $\{T_{\mathcal{U},s}\}_{s\in\omega}$ so that each $T_{\mathcal{U},s}$ is a subset of $\omega^{<\omega}(2^{<\omega})$, $T_\mathcal{U} = \bigcap T_{\mathcal{U},s}$ contains only extendible nodes, and $[T_\mathcal{U}] = \mathcal{U}$.*

Proof. Let $\widehat{T}_\mathcal{U}$ be a computable tree such that $[\widehat{T}_\mathcal{U}] = \mathcal{U}$.

Set

$$T_{\mathcal{U},s} = \{\sigma \in \widehat{T}_\mathcal{U} : \exists \tau \in \widehat{T}^s_\mathcal{U}(\tau \subseteq \sigma \text{ or } \sigma \subseteq \tau)\}.$$

Set

$$T_\mathcal{U} = \bigcap T_{\mathcal{U},s}.$$

It is straightforward to check $T_\mathcal{U}$ contains only extendible nodes and $[T_\mathcal{U}] = \mathcal{U}$. We will call the sequence $\{T_{\mathcal{U},s}\}$ *the canonical approximation of $\mathcal{U}$ with respect to $\widehat{T}_\mathcal{U}$*. Usually we will take the tree $\widehat{T}_\mathcal{U}$ for granted and simply speak of *the canonical approximation*. Note that while each $T_{\mathcal{U},s}$ is computable, $T_\mathcal{U}$ may not be computable.

□

If we are given two Π^0_1 subsets of ω^ω, and canonical approximations to each, we may want to have a canonical approximation for the join or meet of these subsets, in terms of our given approximations. We may even want to do this with arbitrary finite combinations of Π^0_1 subsets of ω^ω.

In general, we have the following definitions for the join and meet of trees.

Definition 12. Suppose T_0, T_1 are computable trees. Then we define,

$T_0 \vee T_1 = \{\sigma : \sigma(0)^\frown\sigma(2)^\frown\sigma(4)^\frown ... \in T_0 \ \& \ \sigma(1)^\frown\sigma(3)^\frown\sigma(5)^\frown ... \in T_1\}$.

$T_0 \wedge T_1 = \{0^\frown\sigma : \sigma \in T_0\} \cup \{1^\frown\sigma : \sigma \in T_1\}$.

Proposition 13. *Suppose T_0 and T_1 are computable trees.*

1. *$T_0 \vee T_1$ and $T_0 \wedge T_1$ are computable trees.*
2. *If $[T_0] = \mathcal{Q}_0$ and $[T_1] = \mathcal{Q}_1$, then $\mathcal{Q}_0 \vee \mathcal{Q}_1 = [T_0 \vee T_1]$ and $\mathcal{Q}_0 \wedge \mathcal{Q}_1 = [T_0 \wedge T_1]$.*

Definition 14. If $\mathcal{U}$ and $\mathcal{V}$ are Π^0_1 subsets of ω^ω with canonical approximations $\{T_{\mathcal{U},s}\}_{s\in\omega}$ and $\{T_{\mathcal{V},s}\}_{s\in\omega}$, respectively, define the *canonical approximations* $\{T_{(\mathcal{U}\vee\mathcal{V}),s}\}_{s\in\omega}$ and $\{T_{(\mathcal{U}\wedge\mathcal{V}),s}\}_{s\in\omega}$ as follows:

$T_{(\mathcal{U}\vee\mathcal{V}),s} = T_{\mathcal{U},s} \vee T_{\mathcal{V},s}$.

$T_{(\mathcal{U}\wedge\mathcal{V}),s} = T_{\mathcal{U},s} \wedge T_{\mathcal{V},s}$.

The definition can be extended inductively to give a canonical approximation of any Π^0_1 subset of ω^ω built up out of finitely many joins and meets of canonically approximated Π^0_1 subsets of ω^ω.

Lemma 15. $\mathcal{U} \vee \mathcal{V} = [\bigcap_{s\in\omega} T_{(\mathcal{U}\vee\mathcal{V}),s}]$ *and* $\mathcal{U} \wedge \mathcal{V} = [\bigcap_{s\in\omega} T_{(\mathcal{U}\wedge\mathcal{V}),s}]$.

By induction with this lemma and the previous definition, one can show that the canonical approximation of a Π^0_1 subset of ω^ω constructed by finitely many joins and meets of Π^0_1 subsets of ω^ω is a nested uniformly computable sequence such that the set of paths through its intersection is the intended set.

As we build separating classes, which are Π^0_1 subsets of 2^ω, we will want stagewise canonical approximations of them as well.

Definition 16. Suppose $A = \bigcup_{s\in\omega} A_s$ and $B = \bigcup_{s\in\omega} B_s$ are disjoint, c.e., and constructed in stages (i.e. $A_s \subseteq A_{s+1}$ and $B_s \subseteq B_{s+1}$ for all s). Let $\mathcal{S} = S(A, B)$. Then we define as follows a computable tree $\widehat{T}_{\mathcal{S}}$ with respect to which the canonical approximation $\{T_{\mathcal{S},s}\}_{s\in\omega}$ for $\mathcal{S}$ is to be taken via the method in the proof of Lemma 11.

$$\widehat{T}_{\mathcal{S}} = \{\sigma : (\sigma(n) = 1 \Rightarrow n \notin B_{|\sigma|})\&(\sigma(n) = 0 \Rightarrow n \notin A_{|\sigma|})\}.$$

Definition 17. If Ψ is a Turing functional, we will write $\Psi_s^f(x)$ for its output after running s steps with oracle f and input x.

To carry out our Sacks Preservation and Coding Strategies, we will need a way to measure incremental progress toward a final result we want to avoid.

Definition 18. (Cenzer-Hinman [5, Definition 18]: Length of Agreement) If Ψ is a Turing functional and $\mathcal{U}$ and $\mathcal{V}$ are Π_1^0 classes, define:

$$\ell^{\mathcal{U},\mathcal{V}}(\Psi, s) = \mu y[\exists \sigma \in T^s_{\mathcal{U},s}(\Psi_s^\sigma \upharpoonright (y+1) \notin T_{\mathcal{V},s})].$$

If $\Psi_s^\sigma \upharpoonright (y+1)$ is undefined, we say it is not in $T_{\mathcal{V},s}$.

Note that $\ell^{\mathcal{U},\mathcal{V}}(\Psi, s) \geq n$ iff $\forall \sigma \in T^s_{\mathcal{U},s}(\Psi_s^\sigma \upharpoonright n \in T_{\mathcal{V},s})$.

When it is obvious which $\mathcal{U}$ and $\mathcal{V}$ are under consideration, the superscripts on ℓ are sometimes dropped. Similarly Ψ will be dropped from the argument when it is obvious which functional is under consideration.

To make the proof easier to read, we will also be interested in stages at which the length of agreement becomes greater than it ever has been, and so we have the following definitions.

Definition 19. If Ψ is a Turing functional and $\mathcal{U}$ and $\mathcal{V}$ are Π_1^0 classes, we define

$$\bar{\ell}^{\mathcal{U},\mathcal{V}}(\Psi, s) = max\{\ell^{\mathcal{U},\mathcal{V}}(\Psi, t) : t \leq s\}.$$

Or, more simply:

$$\bar{\ell}(s) = max\{\ell(t) : t \leq s\}.$$

Note that $\limsup_s \ell(s) = \lim_s \overline{\ell}(s)$.

Definition 20. If Ψ is a Turing functional and $\mathcal{U}$ and $\mathcal{V}$ are Π^0_1 classes, $s+1$ is an *expansionary stage (for $\mathcal{U}, \mathcal{V}$, and Ψ)* if $\overline{\ell}^{\mathcal{U},\mathcal{V}}(\Psi, s+1) > \overline{\ell}^{\mathcal{U},\mathcal{V}}(\Psi, s)$.

We will act for the sake of a requirement only at an expansionary stage for the the relevant length of agreement function. This way, we easily see that if there is an upper bound on the length of agreement function for a requirement, it will act only finitely often.

The following lemma confirms that our definition of length of agreement is well-behaved, and it will be one of the essential elements for the proof that our construction succeeds.

Lemma 21. *If Ψ is a Turing functional and $\mathcal{U}$ and $\mathcal{V}$ are Π^0_1 classes:*

1. *If $\Psi : \mathcal{U} \to \mathcal{V}$, then $\lim_s \ell^{\mathcal{U},\mathcal{V}}(\Psi, s) = \infty$.*
2. *If $\limsup_s \ell^{\mathcal{U},\mathcal{V}}(\Psi, s) = \lim_s \overline{\ell}^{\mathcal{U},\mathcal{V}}(\Psi, s) = \infty$, then $\Psi : \mathcal{U} \to \mathcal{V}$.*

Proof. (1) Suppose that $\Psi : \mathcal{U} \to \mathcal{V}$. We prove that for every $n \in \omega$ there is a stage t such that for all $t' \geq t$, $\ell(t') \geq n$. Fix n. For each $\sigma \in T^{|\sigma|}_{\mathcal{U},0}$, define $\tau_\sigma = \Psi^\sigma_{|\sigma|} \upharpoonright n$ if it is defined. (If it is not defined, by convention we say that τ_σ is not in any tree.) Define

$$Bad = \{\sigma \in T_{\mathcal{U},0} : \tau_\sigma \notin T_{\mathcal{V},|\sigma|}\}.$$

Define

$$Good = \{\sigma \in T_{\mathcal{U},0} : \tau_\sigma \in T_{\mathcal{V},|\sigma|}\}.$$

Immediately $Bad \cup Good = T_{\mathcal{U},0}$ and $Bad \cap Good = \emptyset$. Also, note that Bad is closed downwards and is therefore a tree.

If Bad were infinite, by Compactness there would be an $X \in [Bad]$. Since $Bad \subseteq T_{\mathcal{U},0}$ and $[T_{\mathcal{U},0}] = \mathcal{U}$, we would have $X \in \mathcal{U}$. By hypothesis $\Psi : \mathcal{U} \to \mathcal{V}$, and so there would be $\sigma \subset X$ so that $\Psi_p^\sigma \upharpoonright n \in T_{\mathcal{V}}$ for some stage p. Letting $q = max\{|\sigma|, p\}$, we have $\Psi_q^{X \upharpoonright q} \upharpoonright n \in T_{\mathcal{V},q}$, whence $X \upharpoonright q \in Good$, a contradiction.

Since Bad is finite, there is stage t so that if $t' \geq t$ and $\sigma \in T_{\mathcal{U},t'}^{t'}$, then $\sigma \in Good$. This means that for all $t' \geq t$, $\ell(t') \geq n$, as desired.

(2) Suppose $\limsup_s \ell(s) = \infty$. Given $X \in \mathcal{U}$ we want to show $Y = \Psi(X) \in \mathcal{V}$. We do this by showing $Y \upharpoonright n \in T_{\mathcal{V}}$ for all n. Since $\limsup_s \ell(s) = \infty$ we can find a stage p so that $\ell(p) \geq n$. Then $\Psi_p^{X \upharpoonright p} \upharpoonright n$ is defined and in $T_{\mathcal{V},p}$. For all stages $p' \geq p$, $\Psi_{p'}^{X \upharpoonright p'} \upharpoonright n = \Psi_p^{X \upharpoonright p} \upharpoonright n$. Set $\tau = \Psi_p^{X \upharpoonright p} \upharpoonright n$. Note that $Y \upharpoonright n = \tau$.

If $Y \upharpoonright n = \tau \notin T_{\mathcal{V}}$, then at some stage q we have $\tau \notin T_{\mathcal{V},q}$. Because $\{T_{\mathcal{V},s}\}_{s \in \omega}$ is nested, for all $q' \geq q$, $\tau \notin T_{\mathcal{V},q'}$. Let $r \geq \max\{q, p\}$. (Actually, it is necessary that $q > p$.) For all $r' \geq r$, $\Psi_{r'}^{X \upharpoonright r'} \upharpoonright n = \Psi_p^{X \upharpoonright p} \upharpoonright n = \tau \notin T_{\mathcal{V},r'}$. Then $\ell(r') < n$ for all $r' \geq r$, contradicting the assumption that $\limsup_s \ell(s) = \infty$.

Hence, $Y \upharpoonright n = \Psi^X \upharpoonright n = \Psi_p^{X \upharpoonright p} \upharpoonright n = \tau \in T_{\mathcal{V}}$, as desired.

□

Binns [3, Lemma 6] proved something similar to part (1) of our lemma here. Part (1) is actually slightly stronger than we will need for the construction. In place of part (1), the weaker condition 'If $\Psi : \mathcal{U} \to \mathcal{V}$, then $\limsup_s \ell(s) = \infty$.' would be sufficient. This weaker form of part (1) follows immediately from Definition 18 and Proposition 19 in the proof of Cenzer and Hinman [5]. Part (2) is actually not true for one of the length of agreement functions used in the construction of Cenzer and Hinman, and this causes the infinite injury that forces the creation of a hatted length of agreement function.

It turns out that a simpler length of agreement function, using a single tree to represent each relevant class, is sufficient. We will implement this later, in Sections 3.9.1 and 3.10.

2.4 Placing restraints on Π^0_1 classes

Suppose we are building a Π^0_1 class $\mathcal{S} = S(A, B)$ by building A and B as c.e. subsets of ω such that $A \cap B = \emptyset$ and certain other requirements are met. As indicated in Definition 16, there is a canonical approximation $\{T_{\mathcal{S},s}\}_{s\in\omega}$ so that $\mathcal{S} = [\bigcap_{s\in\omega} T_{\mathcal{S},s}]$.

For a strategy at stage t to restrain $\mathcal{S}$ up to level m means to attempt to ensure that for all stages $t' \geq t$, if $|\tau| \leq m$ and $\tau \in T_{\mathcal{S},t}$, then $\tau \in T_{\mathcal{S},t'}$.

The strategy will force lower priority strategies to comply with this request. The restraint may fail ('be injured') if a higher priority strategy makes an enumeration that violates this request. In particular, since $\mathcal{S} = S(A, B)$, if $\tau \in T_{\mathcal{S},t}$ but $\tau \notin T_{\mathcal{S},t'}$, it must be that for some $x < |\tau|$, either

1. $\tau(x) = 1$, but $x \in B_{t'} \setminus B_t$ or
2. $\tau(x) = 0$, but $x \in A_{t'} \setminus A_t$.

In other words, restraining $\mathcal{S}$ up to level m at stage t amounts to an attempt to prevent the following situations: $x \in B_{t'} \setminus B_t$ or $x \in A_{t'} \setminus A_t$ for some $t' \geq t$ and some $x < m$. This amounts to restraining $A_t \upharpoonright m$ and $B_t \upharpoonright m$ in the sense standard for c.e. priority arguments.

If we are told that we need not worry about anything beyond ensuring that at expansionary stages t, A is protected from injury by lower priority requirements up to some specified level m (i.e. $A_t \upharpoonright m = A \upharpoonright m$, if there is no injury by higher priority requirements), then the work is even simpler. For if $A_t(x) = 1$ for some

$x < m$, then there is no way we will enumerate $x \in B_{t'}$ at a later stage $t' \geq t$, because we insist $A \cap B = \emptyset$. Therefore, case (1) from above will never be a concern, and we need not restrain B at all. This simplified method for placing restraints is the one we will use in our construction.

2.5 Negative requirements and strategies

To satisfy N_I, we will have for each Turing functional Φ, the requirement

$$N_{I,\Phi} \quad \Phi : \mathcal{U} \vee \bigvee_{i \in I} \mathcal{S}_i \not\twoheadrightarrow \mathcal{V}.$$

A simplified version of the Sacks Preservation Strategy is used for satisfying the negative requirements. At expansionary stages we will restrain each A_i at least up to its use in the relevant computations.

1. Wait for an expansionary stage s.
2. For each $i \in I$, restrain A_i up to its maximum use in all computations used in calculating $\ell^{(\mathcal{U} \vee \bigvee_{i \in I} \mathcal{S}_i), \mathcal{V}}(\Phi, s)$. We may as well take all the restraints to be $s + 1$. Also, initialize all $P_{I',J',\Psi}$ strategies of lower priority than $N_{I,\Phi}$ with markers $m_{\sigma,j'} < s + 1$ for some $\sigma \in 2^{<\omega}$ and $j' \in J'$.
3. Go back to step 1 and wait for another expansionary stage.
4. $N_{I,\Phi}$ is injured if a higher priority positive requirement performs an enumeration that violates an A_i-restraint. No action is taken.

Current Outcome at stage s is $\overline{\ell}(s)$.

Final Outcome is $\lim_s \overline{\ell}(s) = \limsup_s \ell(s)$.

Verification: that $N_{I,\Phi}$ is satisfied and acts only finitely often.

Suppose $N_{I,\Phi}$ is not satisfied. Then $\Phi : \mathcal{U} \vee \bigvee_{i\in I} \mathcal{S}_i \to \mathcal{V}$. By Lemma 21(1) $\lim_s \ell(s) = \infty$. Hence $\lim_s \overline{\ell}(s) = \limsup_s \ell(s) = \infty$, and there were infinitely many expansionary stages.

By induction, assume that higher priority requirements act only finitely often. Then, after some stage q, no A_i-restraints are ever injured. We derive a contradiction to the theorem's hypothesis by giving a uniform procedure to calculate a $Y \in \mathcal{V}$ given $X \in \mathcal{U}$.

Fix n. We describe how to calculate $Y \restriction n$. Look for the first expansionary stage $t \geq q$ so that $\ell^{(\mathcal{U}\vee\bigvee_{i\in I}\mathcal{S}_i),\mathcal{V}}(\Phi, t) \geq n$. Such a t exists because there were infinitely many expansionary stages. Define $\sigma_t = (X \oplus \bigoplus_{i\in I} A_{i,t}) \restriction t$. Note that $\sigma_t \in T^t_{(\mathcal{U}\vee\bigvee_{i\in I}\mathcal{S}_i),t}$. Therefore, by the definition of the length of agreement function, $\Phi^{\sigma_t}_t \restriction n \in T_{\mathcal{V},t}$.

Set $Y \restriction n = \Phi^{\sigma_t}_t \restriction n$. At stage t, step (2) of the strategy directs us to restrain each A_i up to level $t+1$, which is greater than the use of each A_i in the computation showing $\Phi^{\sigma_t}_t \restriction n \in T_{\mathcal{V},t}$. By the choice of q, the restraints up to $t+1$ on each A_i will never be violated. Therefore, $\Phi^{\sigma_t}_t \restriction n = \Phi^{X\oplus\bigoplus_{i\in I} A_i} \restriction n$. Furthermore, $\Phi^{X\oplus\bigoplus_{i\in I} A_i} \restriction n \in T_{\mathcal{V}}$, because $X \oplus \bigoplus_{i\in I} A_i \in \mathcal{U} \vee \bigvee_{i\in I} \mathcal{S}_i$ and $\Phi : \mathcal{U} \vee \bigvee_{i\in I} \mathcal{S}_i \to \mathcal{V}$. Hence $Y \restriction n \in T_{\mathcal{V}}$ for each n, and so $Y \in \mathcal{V}$, as desired.

This contradiction shows that the requirement is in fact satisfied. By the contrapositive of Lemma 21(2), there are only finitely many expansionary stages, and so $N_{I,\Phi}$ acts only finitely often to impose restraints on each A_i.

2.5.1 Contrast with negative requirements in the proof of the density of $\mathcal{P}_s$.

Our negative requirements, met by Sacks Preservation Strategies, were:

$$N_{I,\Phi} \quad \Phi : \mathcal{U} \vee \bigvee_{i \in I} \mathcal{S}_i \nrightarrow \mathcal{V}.$$

Cenzer and Hinman had negative requirements very similar to these [5, p. 590]. In our notation, they began with $\mathcal{U} < \mathcal{V}$ and were building $\mathcal{S} = S(A, B)$, where A and B are disjoint c.e subsets of ω built by the construction. The requirements ensured that $\mathcal{U} < (\mathcal{U} \vee \mathcal{S}) \wedge \mathcal{V} < \mathcal{V}$. For each Turing functional Φ, their negative requirement was (in our notation):

$$N_{\Phi} \quad \Phi : \mathcal{U} \vee \mathcal{S} \nrightarrow \mathcal{V}.$$

These are the same as our negative requirements, except that we have the join of finitely many $\mathcal{S}_i$ in place of a single $\mathcal{S}$. Cenzer and Hinman sought to simplify, by replacing each N_Φ with a requirement:

$$N_{\Phi}^{A} \quad \neg \forall X \in \mathcal{U} (\Phi^{X \oplus A} \in \mathcal{V}).$$

Because $A \in S(A, B) = \mathcal{S}$, the satisfaction of N_Φ^A guarantees the satisfaction of N_Φ. In a way, N_Φ^A is a simpler requirement than N_Φ: there is less to keep track of. However, in general, $\{A\}$ is not a Π_1^0 class. Therefore, the length of agreement function ℓ from Definition 18 cannot be directly adapted to work for N_Φ^A.

This approach of Cenzer and Hinman would correspond for us to requirements of the form

$$N_{I,\Phi}^{\{A_i\}_{i \in I}} \quad \neg \forall X \in \mathcal{U} (\Phi^{X \oplus \bigoplus_{i \in I} A_i} \in \mathcal{V}).$$

Again, because $A_i \in S(A_i, B_i) = \mathcal{S}_i$ for each i, the satisfaction of $N_{I,\Phi}^{\{A_i\}_{i\in I}}$ guarantees the satisfaction of $N_{I,\Phi}$. Of course, the length of agreement function ℓ cannot be directly adapted for $N_{I,\Phi}^{\{A_i\}_{i\in I}}$ either.

Because the original length of agreement function was not directly adaptable to the demands of the new requirement, Cenzer and Hinman defined another length of agreement function [5, pp. 594-595]. This length of agreement function works directly with the c.e. set A:

$$\widehat{\ell}^{(\mathcal{U}\times A),\mathcal{V}}(\Phi, s) = \mu y[(\exists \sigma \in T^s_{\mathcal{U},s})\ \widehat{\Phi}^{\sigma,A_s}_s \upharpoonright (y+1) \notin T_{\mathcal{V},s}],$$

where $\widehat{\Phi}$ is defined via the hat trick adapted for Π^0_1-classes. The hat trick is needed to handle the infinite injury that accompanies this new length of agreement function. For suppose ℓ is defined as $\widehat{\ell}$, except with Φ in place of $\widehat{\Phi}$. It can happen that $\limsup_s \ell^{(\mathcal{U}\times A),\mathcal{V}}(\Phi, s) = \infty$, although N^A_Φ is satisfied. This is precisely where Lemma 21(2) fails, as mentioned after the proof of that lemma.

See Soare [25, Chapter 8] for an explanation of the original hat trick, as used to combat infinite injury in a proof of the Sacks Density Theorem. It seems that the hat trick is necessary for a direct proof of the Sacks Density Theorem by priority argument.

This strategy of Cenzer and Hinman using the hat trick would also work in our case. With only straightforward extensions of definitions, requirements of the form $N_{I,\Phi}^{\{A_i\}_{i\in I}}$ can be satisfied for us without need of any further work.

However, it is interesting to note that infinite injury and the hat trick machinery can be avoided as we do in this chapter, by working with the length of agreement functions for the original requirements, namely $N_{I,\Phi}$ in our case, and N_Φ for Cenzer and Hinman. The construction and verification used in our negative strategy goes through without any extra work when applied to the N_Φ-requirements of Cenzer and Hinman.

2.6 Positive requirements and strategies

To satisfy $P_{I,J}$ we have for each Turing functional Ψ the requirement,

$$P_{I,J,\Psi} \quad \Psi : \mathcal{U} \vee \bigvee_{i \in I} \mathcal{S}_i \nrightarrow (\mathcal{U} \vee \bigwedge_{j \in J} \mathcal{S}_j) \wedge \mathcal{V}$$

for I, J finite, $I \cap J = \emptyset$.

If $P_{I,J,\Psi}$ fails, we ensure there are Turing functionals

$$\Gamma_j : \mathcal{S}_j \to \mathcal{V}$$

for each $j \in J$ and

$$\Delta_i : \mathbf{0} \to \mathcal{S}_i$$

for each $i \in I$.

Then we can put Ψ, Γ_j, and Δ_i together to show $\mathcal{U} \geq \mathcal{V}$, contradicting the hypothesis of the theorem. Here's how: given $X \in \mathcal{U}$, we can effectively produce $X_i \in \mathcal{S}_i$ for each i via the finitely many Δ_i. Then, $\Psi(X \oplus \bigoplus_{i \in I} X_i)$ produces either $1^\frown Z$, where $Z \in \mathcal{V}$, or $0^\frown Z$, where

$$Z \in \mathcal{U} \vee \bigwedge_{j \in J} \mathcal{S}_j.$$

If we see $1^\frown Z$, then $Z \in \mathcal{V}$, and we are done. If we see $0^\frown Z$, then given the finite piece of information $|J|$, we can extract from Z a $W \in \mathcal{S}_j$ for some $j \in J$, and we will know which j it is. Then $\Gamma_j(W)$ gives $Y \in \mathcal{V}$, as desired.

To create the Turing functionals Γ_j we use a Sacks Coding Strategy. To create the Turing functionals Δ_i, in a way similar to a Sacks Preservation Strategy, we make each A_i computable via restraints set at expansionary stages. We proceed as follows:

1. Set $n = 0$.

2. Wait for an expansionary stage s at which $\ell^{(\mathcal{U}\vee\bigvee_{i\in I}\mathcal{S}_i),((\mathcal{U}\vee\bigwedge_{j\in J}\mathcal{S}_j)\wedge\mathcal{V})}(\Psi, s) \geq n$.

3. For each $i \in I$, restrain A_i up to level $s+1$. Note that $s+1$ will be greater than the length of agreement at stage s. Also, initialize all $P_{I',J',\Psi}$-strategies of lower priority with markers $m_{\sigma,j'} < s+1$, where $\sigma \in 2^{<\omega}$ and $j' \in J'$.

4. For each $j \in J$ and each $\sigma \in 2^{<\omega}$ with $|\sigma| = n$, choose large markers $m_{\sigma,j}$. Here "large" means greater than any restraints on A_j and greater than any other markers for $\mathcal{S}_j$ (whether or not at this stage they have been enumerated into A_j or B_j) established for the sake of this or any other P-strategy. Increment n and go to step 2 for this new n. Meanwhile, for the old n, go to the next step.

5. Wait for another expansionary stage $t > s$ at which either

   (i) $\sigma^\frown 0 \notin T_{\mathcal{V},t}$ or

   (ii) $\sigma^\frown 1 \notin T_{\mathcal{V},t}$.

6. If (i) happens for σ at stage t, place $m_{\sigma,j}$ into A_j for each $j \in J$, and do not act again for this $m_{\sigma,j}$.

7. If (ii) happens for σ at stage t, place $m_{\sigma,j}$ into B_j for each $j \in J$, and do not act again for this $m_{\sigma,j}$.

8. $P_{I,J,\Psi}$ is injured if a higher priority P-strategy makes an enumeration that violates an A_i-restraint. No action is taken.

As directed in the strategies, a $P_{I,J,\Psi}$ strategy is initialized if a higher priority N- or P-strategy makes an A_j-restraint greater than a current marker $m_{\sigma,j}$ for $P_{I,J,\Psi}$. If $P_{I,J,\Psi}$ is initialized, all current markers $m_{\sigma,j}$ are discarded and we return to step 1, setting $n = 0$. The $P_{I,J,\Psi}$ outcomes are the same as the outcomes for negative strategies:

Current Outcome at stage s is $\overline{\ell}(s)$.

Final Outcome is $\lim_s \overline{\ell}(s) = \limsup_s \ell(s)$.

<u>Verification</u>: that $P_{I,J,\Psi}$ is satisfied and acts only finitely often.

First note that the actions $P_{I,J}$ takes for the sake of $\{\mathcal{S}_i\}_{i\in I}$ and $\{\mathcal{S}_j\}_{j\in J}$ never conflict because $I \cap J = \emptyset$.

By induction, assume all higher priority requirements have finite outcomes, so that there is a stage q after which $P_{I,J,\Psi}$ is never initialized and is never injured.

If $P_{I,J,\Psi}$ were not satisfied, we would have by Lemma 21(1) that $\lim_s \overline{\ell}(s) = \limsup_s \ell(s) = \infty$. In this case, we need to define the functionals Γ_j and Δ_i which will yield a contradiction. We begin by giving $\Gamma_j : \mathcal{S}_j \to \mathcal{V}$, for each $j \in J$.

Given $X \in \mathcal{S}_j$ and inductively assuming we have $\sigma_n \in T_{\mathcal{V}}$ with $|\sigma_n| = n$, we say how to determine whether $\sigma_n{}^\frown 0 \in T_{\mathcal{V}}$ or $\sigma_n{}^\frown 1 \in T_{\mathcal{V}}$. First determine the value, if any, of $m_{\sigma_n,j}$ after the last initialization before stage q. If $m_{\sigma_n,j}$ was not defined by stage q, simply find its value once it is defined. Note that because $\lim_s \overline{\ell}(s) = \infty$, $m_{\sigma_n,j}$ must eventually be defined. Further, $m_{\sigma_n,j}$ can never change after stage q.

Case 1 If $X(m_{\sigma_n,j}) = 1$, then $m_{\sigma_n,j} \notin B_j$, because $X \in S(A_j, B_j)$. Thus, if (ii) happened from step 4 in the strategy, it happened after (i). Since $T_{\mathcal{V}}$ contains only extendible nodes and $\sigma_n \in T_{\mathcal{V}}$, we cannot have both (i) and (ii) happening. Thus, (ii) never happened. Hence, $\sigma_n{}^\frown 1 \in T_{\mathcal{V}}$. Set $\sigma_{n+1} = \sigma_n{}^\frown 1$.

Case 2 If $X(m_{\sigma_n,j}) = 0$, then $m_{\sigma_n,j} \notin A_j$, because $X \in S(A_j, B_j)$. Thus, if (i) happened from step 4 in the strategy, it happened after (ii). Since $T_{\mathcal{V}}$ contains only extendible nodes and $\sigma_n \in T_{\mathcal{V}}$, we cannot have both (i) and (ii) happening. Thus (i) never happened. Hence $\sigma_n{}^\frown 0 \in T_{\mathcal{V}}$. Set $\sigma_{n+1} = \sigma_n{}^\frown 0$.

Defining $Y = \bigcup_n \sigma_n$, we see that because each of its initial segments is in $T_{\mathcal{V}}$, we have $Y \in \mathcal{V}$, as desired.

We now give $\Delta_i : \mathbf{0} \to \mathcal{S}_i$ for each $i \in I$. We do this by calculating $A_i \in \mathcal{S}_i = S(A_i, B_i)$. To calculate $A_i \upharpoonright x$, wait for an expansionary stage $t > q$ so that $\ell(t) = \overline{\ell}(t) \geq x$. At stage t in step (3) of the strategy we will restrain A_i up to level $t + 1$, which must be greater than x. Recall that after stage q, $P_{I,J,\Psi}$ is never injured. So $A_i \upharpoonright x$ is never injured after stage t. Hence $A_{i,t} \upharpoonright x = A_i \upharpoonright x$. Now, Γ_j and Δ_i, for each $j \in J$ and $i \in I$, together with

$$\Psi : \mathcal{U} \vee \bigvee_{i \in I} \mathcal{S}_i \to (\mathcal{U} \vee \bigwedge_{j \in J} \mathcal{S}_j) \wedge \mathcal{V},$$

will give a uniform reduction from $\mathcal{U}$ to $\mathcal{V}$ as shown above, which is a contradiction.

The contradiction we have arrived at shows that $P_{I,J,\Psi}$ is satisfied. Therefore, by the contrapositive of Lemma 21(2), $P_{I,J,\Psi}$ has finite outcome. Because it acts only at expansionary stages, $P_{I,J,\Psi}$ acts only finitely often.

Because there will be no infinite outcomes, any priority ordering for the requirements will yield a construction in which the strategies satisfy their requirements as described above, thus proving Theorem 2.

2.7 A note on preservation strategies

To accomplish our preservation strategy as part of satisfying a positive requirement $P_{I,J,\Psi}$, we simply ensure that for each $i \in I$, A_i is computable, if it turns out the requirement is violated. This helps lead to a contradiction of the theorem's hypothesis.

On the other hand, in the verification that our preservation strategy satisfies a negative requirement $N_{I,\Phi}$, we do not say that for each $i \in I$, A_i must be computable whenever the requirement is violated. However, it is true, so long as we take the restraints at each expansionary stage s to be $s+1$, as mentioned in step (2) of the strategy for negative requirements. The verification that each A_i is computable is exactly the same as in the verification for the positive requirements. Furthermore, the computability of each A_i would be enough to reach the contradiction of the theorem's hypothesis in the verification of the negative requirement.

We leave the verification of the strategy for the negative requirements as is, because it is then as close as possible to the verification of the negative strategy of Cenzer and Hinman, with the exception of the modification that eliminates infinite injury. This way, it is easier to isolate the exact cause of infinite injury for Cenzer and Hinman.

2.8 More on the length of agreement function.

This section gives a more precise account of exactly how the length of agreement function behaves in this chapter. It also applies to the construction of Cenzer and Hinman once the modification we describe in Section 2.5.1 is made. This section is not necessary for the proof of the main result of this chapter.

We begin by noticing that if $\limsup_s \ell(s) = \infty$, then by two applications of Lemma 21, $\lim_s \ell(s) = \infty$. Then $\liminf_s \ell(s) = \infty$. Hence we have the following proposition.

Proposition 22. *For a length of agreement function ℓ as defined in Definition 18, the following are equivalent.*

1. $\limsup_s \ell(s) = \infty$.
2. $\liminf_s \ell(s) = \infty$.
3. $\lim_s \ell(s) = \infty$.

In particular, it is never the case that $\liminf_s \ell(s)$ *is finite but* $\limsup_s \ell(s) = \infty$.

Proposition 22 leaves open the possibility that $\limsup_s \ell(s)$ and $\liminf_s \ell(s)$ are both finite but $\liminf_s \ell(s) < \limsup_s \ell(s)$. However, Π^0_1 classes are nice enough that this never happens.

Proposition 23. *For the length of agreement function ℓ defined in Definition 18,* $\liminf_s \ell(s) = \limsup_s \ell(s) = \lim_s \ell(s)$. *These equal limits may be finite or infinite.*

For the proof of this proposition, we need a further lemma. As motivation, suppose that at some first stage t, $\ell(t) \geq n$. We want to analyze what could later cause the length of agreement to drop below n after t, and also what could bring the length of agreement back up to at least n, once it drops below n.

Lemma 24. *Let $\mathcal{M}, \mathcal{N}$ be Π^0_1 classes. Let Ψ be a Turing functional. Fix $n \in \omega$. There is a stage r such that either*

1. *for all $r' \geq r$, $\ell^{\mathcal{M},\mathcal{N}}(\Psi, r') \geq n$ or*
2. *for all $r' \geq r$, $\ell^{\mathcal{M},\mathcal{N}}(\Psi, r') < n$.*

Note that Lemma 24 immediately implies Proposition 23.

Proof. In this proof, if $\sigma \in 2^{<\omega}$ and $x \leq |\sigma|$ then $\sigma_x = \sigma \upharpoonright x$.

We begin with an easy fact: if $\{C_i\}_{i \in \omega}$ is a nested sequence and for each $i \in \omega$, C_i is finite, then there is $j \in \omega$ so that for all $j' \geq j$, $C_{j'} = C_j$.

Suppose there is a stage t so that $\ell^{\mathcal{M},\mathcal{N}}(\Psi, t) \geq n$. (Otherwise, for any r, (2) holds). Let $Output_{n,t} = \{\tau \in T_{\mathcal{N},t} : \tau = \Psi^\sigma_t \upharpoonright n \text{ for some } \sigma \in T^t_{\mathcal{M},t}\}$. For $t' \geq t$, define $Output_{n,t'} = \{\tau \in Output_{n,t} : \tau \in T_{\mathcal{N},t'}\}$. By the easy fact, there is a stage q so that for all $q' \geq q$, $Output_{n,q} = Output_{n,q'}$. Also by the easy fact, there is $r \geq q$ so that $T^t_{\mathcal{M},r} = T^t_{\mathcal{M},r'}$ for all $r' \geq r$.

We claim this is the desired r. We verify the claim in two cases.

Suppose $\ell^{\mathcal{M},\mathcal{N}}(\Psi, r) \geq n$.

Let $r' \geq r$ and $\sigma \in T^{r'}_{\mathcal{M},r'}$. Because $\ell^{\mathcal{M},\mathcal{N}}(\Psi, t) \geq n$, $\Psi^{\sigma_t}_t \upharpoonright n \in T_{\mathcal{N},t}$. Since $\Psi^{\sigma_t}_t \upharpoonright n$ converges and $r' \geq r \geq t$, we have $\Psi^\sigma_{r'} \upharpoonright n = \Psi^{\sigma_r}_r \upharpoonright n = \Psi^{\sigma_t}_t \upharpoonright n \in T_{\mathcal{N},t}$. Then $\Psi^{\sigma_r}_r \upharpoonright n \in Output_{n,t}$. Since $\ell^{\mathcal{M},\mathcal{N}}(\Psi, r) \geq n$, we must have $\Psi^{\sigma_r}_r \upharpoonright n \in T_{\mathcal{N},r}$. So

$\Psi_r^{\sigma_r} \restriction n \in Output_{n,r}$. Since $r \geq q$, by the choice of q above we have $Output_{n,r'} = Output_{n,r}$, and so $\Psi_r^{\sigma_r} \restriction n \in T_{\mathcal{N},r'}$. But $\Psi_{r'}^{\sigma} \restriction n = \Psi_r^{\sigma_r} \restriction n$, so $\Psi_{r'}^{\sigma} \restriction n \in T_{\mathcal{N},r'}$.

Hence, $\ell^{\mathcal{M},\mathcal{N}}(\Psi, r') \geq n$.

Suppose $\ell^{\mathcal{M},\mathcal{N}}(\Psi, r) < n$.

Then there is $\sigma \in T^r_{\mathcal{M},r}$ so that $\Psi_r^{\sigma} \restriction n \notin T_{\mathcal{N},r}$. But $\ell^{\mathcal{M},\mathcal{N}}(\Psi, t) \geq n$, so we must have $\Psi_t^{\sigma_t} \restriction n \in T_{\mathcal{N},t}$. Then $\Psi_r^{\sigma} \restriction n$ must converge and $\Psi_r^{\sigma} \restriction n = \Psi_t^{\sigma_t} \restriction n$. So, we must have $\Psi_t^{\sigma_t} \restriction n \notin T_{\mathcal{N},r}$. Because $\sigma \in T^r_{\mathcal{M},r}$, we must have $\sigma_t \in T^t_{\mathcal{M},r}$. By the choice of r, we must have $\sigma_t \in T^t_{\mathcal{M},r'}$ for all $r' \geq r$. Then, by the definition of the canonical approximation (see the proof of Lemma 11), for all $r' \geq r$ there is $\sigma' \in T^{r'}_{\mathcal{M},r'}$ so that $\sigma' \supseteq \sigma_t$. We must have $\Psi_{r'}^{\sigma'} \restriction n = \Psi_t^{\sigma_t} \restriction n \notin T_{\mathcal{N},r}$. Because $\{T_{\mathcal{N},s}\}_{s \in \omega}$ is nested, $\Psi_{r'}^{\sigma'} \restriction n \notin T_{\mathcal{N},r'}$.

Hence, $\ell^{\mathcal{M},\mathcal{N}}(\Psi, r') < n$, for all $r' \geq r$.

$\square$

CHAPTER 3

THE DECIDABILITY OF THE $\forall\exists$-THEORY OF $(\mathcal{P}_s, \leq_s)$.

3.1 Introduction to the problem

It is natural to ask about the Turing degree of the elementary theory of a mathematical structure; it is not known whether the elementary theory of $(\mathcal{P}_s, \leq_s)$ is decidable. Binns [3] has shown that the $\exists$-theory is decidable; the main result of this section is that the $\forall\exists$-theory is also decidable. This has been independently shown by Takayuki Kihara.[1]

In this chapter, we assume some very basic model theory. Marker's text is a good reference [14]. We also assume basic knowledge of ordered lattices; in particular, one must always remember that in an ordered lattice, if $x \leq y$, then $x \vee y = y$ and $x \wedge y = x$.

3.1.1 Conventions

Let L be the language $\{0, 1, \leq\}$. Let $\widehat{L}$ be the language $\{0, 1, \leq, \wedge, \vee\}$.

As we proceed, we will use some assumptions about L- and $\widehat{L}$-structures.

If $\mathcal{M}$ is a structure with underlying domain M, then $x \in \mathcal{M}$ means $x \in M$. Similarly, $\mathcal{M} \cup \{n\}$ is the structure with domain $M \cup \{n\}$, assuming we are already

[1] Kihara has a different and shorter proof in his thesis for the master's degree he is working on for Takeshi Yamazaki at the Mathematical Institute, Tohoku University. He and I are currently collaborating on a paper that will feature this result.

given how all functions and relations act with respect to n. The meaning of similar conventions should be clear from the context.

We will assume that all L-structures are in fact partial orders with least and greatest elements, given by $\mathbf{0}$ and $\mathbf{1}$, respectively. In $\widehat{L}$-structures, we assume these things as well, and in addition that the lattice operations are defined by the partial order and that the lattice operations are distributive. In other words, we assume that all L- and $\widehat{L}$-structures satisfy the following axioms:

1. $\forall x \forall y((x \leq y \ \& \ y \leq x) \iff x = y)$.

2. $\forall x \forall y \forall z((x \leq y \ \& \ y \leq z) \Rightarrow x \leq z)$.

3. $\forall x(x \geq 0)$.

4. $\forall x(x \leq 1)$.

In addition we assume the following axioms of all $\widehat{L}$-structures:

1. $\forall x \forall y \forall w[x \vee y = w \iff \forall z(z \geq x \ \& \ z \geq y \Rightarrow z \geq w)]$.

2. $\forall x \forall y \forall w[x \wedge y = w \iff \forall z(z \leq x \ \& \ z \leq y) \Rightarrow z \leq w)]$.

3. $\forall x \forall y \forall w((x \vee y) \wedge w = (x \wedge w) \vee (y \wedge w))$.

4. $\forall x \forall y \forall w((x \wedge y) \vee w = (x \vee w) \wedge (y \vee w))$.

If an $L(\widehat{L})$-structure does not satisfy these axioms, we know for sure that it does not $L(\widehat{L})$-embed into $\mathcal{P}_s$, because these are universal sentences true in $\mathcal{P}_s$. Hence, we may assume these axioms while studying which finite structures embed into $\mathcal{P}_s$.

3.1.2 Partial orders, lattices, and $\mathcal{P}_s$

If $\mathcal{U}$ is a Π^0_1 class, then $deg_s(\mathcal{U})$ is the equivalence class of $\mathcal{U}$ under the equivalence relation induced by $\leq_s$.

Theorem 25. *(Binns) Given Π^0_1 classes $\mathcal{U} <_s \mathcal{V}$ and a finite distributive lattice $\mathcal{L}$, there is a lattice-embedding of $\mathcal{L}$ into $\mathcal{P}_s$ so that the maximum element of $\mathcal{L}$ is mapped to $deg_s(\mathcal{V})$ and each element of $\mathcal{L}$ is mapped to an element of $\mathcal{P}_s$ above $deg_s(\mathcal{U})$.*

Proof. See [3]. □

Corollary 26. *(Binns) In $\widehat{L}$, the one-quantifier theory of $\mathcal{P}_s$ is decidable.*

Proof. See [3]. □

Corollary 27. *Every finite $\widehat{L}$-structure $\widehat{\mathcal{M}}$, in which $\mathbf{0}$ is non-branching, $\widehat{L}$-embeds into $\mathcal{P}_s$.*

Proof. First, let us recall that in $\mathcal{P}_s$, $\mathbf{0}$ is non-branching. This is because a Π^0_1 class $\mathcal{Q}$ is in $\mathbf{0}$ just in case it has no computable member, and the meet of two Π^0_1 classes contains a computable member just in case one of the two classes does.

Let $\widehat{\mathcal{M}}$ be a finite $\widehat{L}$-structure in which $\mathbf{0}$ is non-branching. Consider the structure $\widehat{\mathcal{M}}$ - $\{\mathbf{0}\}$. Because, in $\widehat{\mathcal{M}}$, $\mathbf{0}$ is non-branching, $\widehat{\mathcal{M}}$ - $\{\mathbf{0}\}$ is a finite distributive lattice with a greatest element named $\mathbf{1}$. By Theorem 25, $\widehat{\mathcal{M}}$ - $\{\mathbf{0}\}$ lattice-embeds into $\mathcal{P}_s$ so that $\mathbf{1}$ is mapped to the greatest degree in $\mathcal{P}_s$, and every other element is mapped to a degree above the lowest degree in $\mathcal{P}_s$. If we extend this embedding by mapping $\mathbf{0}$ to the lowest degree in $\mathcal{P}_s$, we have the desired $\widehat{L}$-embedding. □

Our goal is to study the two-quantifier theory of $\mathcal{P}_s$. Dropping the lattice operations, we are able to show that the two-quantifier theory in L is decidable. Interestingly, the proof of this result about the L-theory makes use of $\widehat{L}$ structures. So we take a moment to lay out some facts about L and $\widehat{L}$ structures.

Lemma 28. *Suppose $\mathcal{M}$ is an L-structure and $\widehat{\mathcal{M}}$ is an $\widehat{L}$-structure that extends $\mathcal{M}$, as an L-structure. Further, suppose $\mathcal{P}$ is an $\widehat{L}$-structure. If $\widehat{f} : \widehat{\mathcal{M}} \hookrightarrow \mathcal{P}$ is an $\widehat{L}$-embedding, then $f : \mathcal{M} \hookrightarrow \mathcal{P}$ is an L-embedding, where $f = \widehat{f} \restriction \mathcal{M}$.*

Proof. Note that $L \subset \widehat{L}$. □

Definition 29. Suppose $\mathcal{M}$ is an L-structure and $\widehat{\mathcal{M}}$ is an $\widehat{L}$-structure. Then, $\widehat{\mathcal{M}}$ *minimally extends* $\mathcal{M}$ if $\widehat{\mathcal{M}}$ extends $\mathcal{M}$ over the language L and is the closure of $\mathcal{M}$ in $\widehat{\mathcal{M}}$ under the operations $\wedge$ and $\vee$.

Example 30. Suppose $\widehat{\mathcal{M}}$ is the $\widehat{L}$-structure consisting only of $\mathbf{0}$ and $\mathbf{1}$. Trivially, $\widehat{\mathcal{M}}$ minimally extends the L-structure $\mathcal{M}$ consisting of $\mathbf{0} <_{\mathcal{M}} \mathbf{1}$.

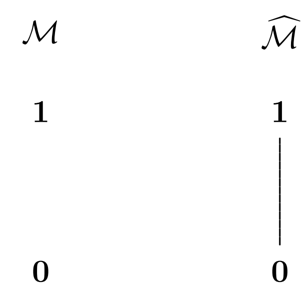

Figure 3.1. Sketch of Example 30.

Example 31. Suppose $\mathcal{M}$ is the L-structure with four linearly ordered elements: $\mathbf{0} <_{\mathcal{M}} x <_{\mathcal{M}} y <_{\mathcal{M}} \mathbf{1}$. Then the unique minimal extension of $\mathcal{M}$, up to isomorphism, is $\widehat{\mathcal{M}}$, the $\widehat{L}$-structure with the same universe and with $\vee$ and $\wedge$ structure entirely determined by the linear order, in accord with our convention: $x \vee y = y$ and $x \wedge y = x$.

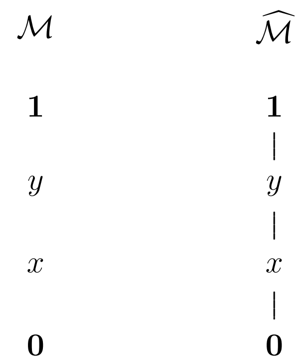

Figure 3.2. Sketch of Example 31.

Example 32. Suppose $\mathcal{M}$ is the L-structure with universe $M = \{\mathbf{0}, x, y, \mathbf{1}\}$ such that x and y are incomparable. Then $\mathcal{M}$ has exactly four minimal extensions, up to isomorphism. We may have $x \vee y = \mathbf{1}$ or w, where w is a new element. And, independently, we may have $x \wedge y = \mathbf{0}$ or z, where z is a new element.

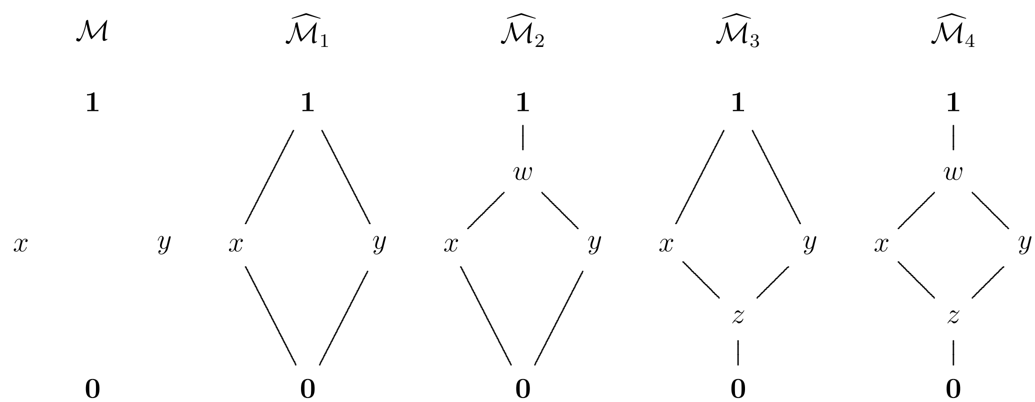

Figure 3.3. Sketch of Example 32.

Example 33. Let $\mathcal{M}$ be the L-structure with universe $\{\mathbf{0}, \mathbf{1}, x, x', y\}$ such that $0 <_{\mathcal{M}} x <_{\mathcal{M}} x' <_{\mathcal{M}} \mathbf{1}$, $x \perp_{\mathcal{M}} y$, and $x' \perp_{\mathcal{M}} y$, where $\perp$ signifies incomparability. (The case where $x' \perp_{\mathcal{M}} y$ is changed to $x' >_{\mathcal{M}} y$ leads to a situation very similar to Example 32.)

We will show three minimal extensions of $\mathcal{M}$, which differ according to whether $x \vee y \perp x'$, $x' < x \vee y < \mathbf{1}$, or $x \vee y = \mathbf{1}$. Note that it is impossible for $x' \geq x \vee y$, because $x' \perp_{\mathcal{M}} y$.

In the following diagrams of these three minimal extensions, we leave out much additional structure, so that we can focus on what happens with $x \vee y$. For example, we leave out $x' \vee y$ and $x' \wedge (x \vee y)$; there are many possibilities for the placement of these two elements.

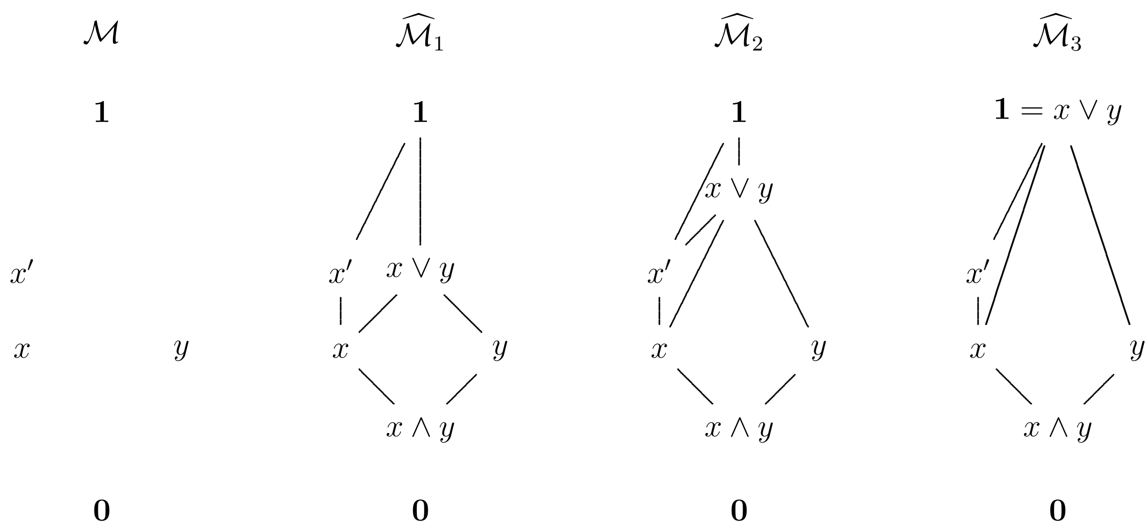

Figure 3.4. Sketch of Example 33.

Proposition 34.

1. *Every L-structure has a minimal extension; moreover, if we desire, we may ensure* $\boldsymbol{0}$ *is non-branching.*

2. *If $\mathcal{M}$ is a finite L-structures that contains elements $x \perp_{\mathcal{M}} y$, then $\mathcal{M}$ has more than one minimal extension.*

3. *Every minimal extension of a finite L-structure is finite.*

4. *If $\mathcal{M}$ is a finite L-structure, it has finitely-many minimal extensions, up to isomorphism.*

5. *There is a function f so that if $\mathcal{M}$ is an L-structure with at most n elements, then every minimal extension of $\mathcal{M}$ has at most $f(n)$ elements.*

6. *f is computable.*

Corollary 35. *Given any finite L-structure $\mathcal{M}$, we can effectively compute each of its finitely-many minimal $\widehat{L}$-extensions.*

The following Corollary is the form in which we will use Binns' main result.

Corollary 36. *Every finite L-structure L-embeds into $\mathcal{P}_s$.*

Proof. By Proposition 34(1), every finite L-structure $\mathcal{M}$ extends to a finite $\widehat{L}$-structure $\widehat{\mathcal{M}}$ in which $\mathbf{0}$ is non-branching. By Corollary 27, there is an $\widehat{L}$-embedding of $\widehat{\mathcal{M}}$ into $\mathcal{P}_s$. This is also an L-embedding, and its reduct to $\mathcal{M}$ (see Lemma 28) gives the desired embedding.

$\square$

Proof of Proposition 34.

(1): By straightforward algebraic technique.

(2): Since $\mathcal{M}$ is finite and $\mathbf{1} \in \mathcal{M}$, there is a minimal $z \in \mathcal{M}$ so that $x \leq z$ and $y \leq z$. We claim there is a minimal extension of $\mathcal{M}$ in which $x \vee y = z$, and a minimal extension in which $x \vee y < z$. This claim is verified by straightforward algebraic technique.

(3), (5), and (6): We explicitly give the computable function $f(n)$.

Let $\mathcal{M}$ be an L-structure with n-elements; $\mathcal{M} = \{x_0, \ldots, x_{n-1}\}$. Let $\widehat{\mathcal{M}}$ be an $\widehat{L}$-structure minimally extending $\mathcal{M}$. Every element of $\widehat{\mathcal{M}}$ can be written as a finite combination of meets and joins of elements of $\mathcal{M}$, by the definition of a minimal extension. Now, since by our conventions, every $\widehat{L}$-structure is a distributive lattice, every such finite combination of meets and joins of elements of $\mathcal{M}$ may be put in disjunctive normal form. In other words, for every $x \in \widehat{\mathcal{M}}$, $x = \bigvee_{i \in I} y_i$, where I is finite and each $y_i = \bigwedge_{j \in J_i} x_j$, where each $J_i \subseteq n$.

Now, we may assume that $y_i \neq y_j$ for all $i \neq j$. Furthermore, we may identify each y_i with the set $J_i \subseteq n$. So each $x \in \widehat{\mathcal{M}}$ may be associated with a unique set of subsets of n. Thus the number of elements of $\widehat{\mathcal{M}}$ is bounded by the cardinality of the power set of the powerset of n; so we may set $f(n) = 2^{2^n}$.

(4): Let $\mathcal{M}$ be an L-structure with at most n elements. Up to isomorphism, there are only finitely many $\widehat{L}$-structures with at most $f(n)$ many elements; for there is a correspondence between such structures and the collection of valid tables for the functions $\vee$ and $\wedge$.

$\square$

Definition 37. Suppose $f : \mathcal{M} \hookrightarrow \mathcal{P}$ is an L-embedding, and $\mathcal{P}$ is also an $\widehat{L}$-structure. We define *the $\widehat{L}$-closure of $\mathcal{M}$ under f* to be an $\widehat{L}$-structure $\widehat{\mathcal{M}}$ that

L-extends $\mathcal{M}$ and is $\widehat{L}$-isomorphic to the closure of $f(\mathcal{M})$ in $\mathcal{P}$ under $\vee$ and $\wedge$.

Remark 38. It is clear that *the $\widehat{L}$-closure of $\mathcal{M}$ under f* is well-defined up to isomorphism, and that if $\widehat{\mathcal{M}}$ is the $\widehat{L}$-closure of $\mathcal{M}$ under f, then $\widehat{\mathcal{M}}$ minimally extends $\mathcal{M}$.

Proposition 39. *Suppose $f : \mathcal{M} \hookrightarrow \mathcal{P}$ is an L-embedding, and that $\mathcal{P}$ is also an $\widehat{L}$-structure. Let $\widehat{\mathcal{M}}$ be the $\widehat{L}$-closure of $\mathcal{M}$ under f. Then, there is a unique $\widehat{L}$-embedding $\widehat{f} : \widehat{\mathcal{M}} \hookrightarrow \mathcal{P}$ that is an extension of f.*

Proof. Straightforward. ☐

We can think of

$$\mathcal{M} \xhookrightarrow{f} \mathcal{P}_s$$

as extended to

$$\mathcal{M} \xhookrightarrow{i} \widehat{\mathcal{M}} \xhookrightarrow{\widehat{f}} \mathcal{P}_s.$$

Note that we think of i as the identity map, as indeed it is literally, according to Definition 37.

3.1.3 The multi-extension of embeddings problem

In the study of degree structures, the two-quantifier theory is sometimes shown to be decidable by means of solving the multi-extension of embeddings problem for that structure. Furthermore, it is even easier to first study the simpler extension of embeddings problem, and then attempt to generalize to the multi-extension of embeddings problem, if possible. Our little explanation of these borrows from Montalbán's nice presentation [15].

Definition 40. Given a structure $\mathcal{P}$ in a computable language $\mathcal{L}$, the *extension of embeddings problem* is the set of pairs $(\mathcal{M}, \mathcal{N})$ such that $\mathcal{M}, \mathcal{N}$ are finite $\mathcal{L}$-structures and every embedding of $\mathcal{M}$ into $\mathcal{P}$ extends to an embedding of $\mathcal{N}$ into $\mathcal{P}$.

$$\begin{tikzcd}
\mathcal{M} \arrow[r, hook] \arrow[d, hook] & \mathcal{P} \\
\mathcal{N} \arrow[ur, hook, "?"'] &
\end{tikzcd}$$

Figure 3.5. Sketch for Definition 40.

Definition 41. Given a structure $\mathcal{P}$ in a computable language $\mathcal{L}$, the *multi-extension of embeddings problem* is the set of finite tuples $(\mathcal{M}, \mathcal{N}_0, \ldots, \mathcal{N}_t)$ such that $\mathcal{M}, \mathcal{N}_0, \ldots, \mathcal{N}_t$ are finite $\mathcal{L}$-structures, and for every embedding of $\mathcal{M}$ into $\mathcal{P}$, there is an i so that the embedding extends to an embedding of $\mathcal{N}_i$ into $\mathcal{P}$.

$$\begin{tikzcd}
 & & & \mathcal{M} \arrow[rrr, hook] \arrow[dlll, hook] \arrow[dll, hook] \arrow[d, hook] \arrow[drr, hook] & & & \mathcal{P} \\
\mathcal{N}_0 & \mathcal{N}_1 & \cdots & \mathcal{N}_i \arrow[urrr, hook, "?"] & \cdots & \mathcal{N}_t &
\end{tikzcd}$$

Figure 3.6. Sketch for Definition 41.

There is useful way to state what it means for an extension of embeddings or multi-extension of embeddings problem to be computable (or "decidable").

Proposition 42. *Suppose $\mathcal{L}$ is a computable language, and $\mathcal{P}$ is an $\mathcal{L}$-structure. Then the following two statements are equivalent.*

1. *In $\mathcal{L}$ the extension of embeddings problem for $\mathcal{P}$ is decidable.*

2. *There is an algorithm, which, given a pair of finite $\mathcal{L}$-structures $\mathcal{M} \subseteq \mathcal{N}$, decides whether for every $\mathcal{L}$-embedding $f : \mathcal{M} \hookrightarrow \mathcal{P}$, there is an extension of f to an $\mathcal{L}$-embedding $f' : \mathcal{N} \hookrightarrow \mathcal{P}$.*

Proof. Straightforward. □

Proposition 43. *Suppose $\mathcal{L}$ is a computable language, and $\mathcal{P}$ is an $\mathcal{L}$-structure. Then the following two statements are equivalent.*

1. *In $\mathcal{L}$ the multi-extension of embeddings problem for $\mathcal{P}$ is decidable.*

2. *There is an algorithm, which, given a finite $\mathcal{L}$-structures $\mathcal{M}$, and finitely-many finite $\mathcal{L}$-structures $\mathcal{N}_0, \ldots, \mathcal{N}_t$ such that $\mathcal{M} \subseteq \mathcal{N}_i$ for each $0 \leq i \leq t$, decides whether for every $\mathcal{L}$-embedding $f : \mathcal{M} \hookrightarrow \mathcal{P}$, there is $0 \leq i \leq t$ such that there is an extension of f to an $\mathcal{L}$-embedding $f' : \mathcal{N}_i \hookrightarrow \mathcal{P}$.*

Proof. Straightforward. □

Proposition 44. *Suppose $\mathcal{P}$ is a structure in a finite relational language $\mathcal{L}$. If the multi-extension of embeddings problem for $\mathcal{P}$ is decidable, then so is the elementary two-quantifier theory of $\mathcal{P}$.*

This seems to be a bit of folklore. Lerman [13, Theorem VII.4.4] proves something a bit different, but anyone who understands his proof ought to know this result. We include the proof for completeness.

Proof. Note that in this proof $\wedge$ is the logical symbol for "and" and $\vee$ is the logical symbol for "or".

Definition 45. Suppose $\theta(\bar{x})$ is a quantifier-free formula in $\mathcal{L}$. $\theta(\bar{x})$ is a *complete atomic diagram for* $\bar{x}$ if $\theta(\bar{x}) \equiv \bigwedge_{i \leq n} \theta_i(\bar{x})$, where the following two conditions hold:

1. Each formula $\theta_i(\bar{x})$ is of the form $R_j(\bar{x}_0)$ or $\neg R_j(\bar{x}_0)$ for some relation R_j in $\mathcal{L}$ and $\bar{x}_0 \subseteq \bar{x}$ of appropriate size, and
2. For each relation R_j in $\mathcal{L}$ and each $\bar{x}_0 \subseteq \bar{x}$ of appropriate size, exactly one of $R_j(\bar{x}_0), \neg R_j(\bar{x}_0)$ is equivalent to $\theta_i(\bar{x})$ for some $i \leq n$.

Let ψ be a sentence in $\mathcal{L}$ with at most two alternations of quantifiers. Say $\psi \equiv \forall \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y})$, where $\varphi(\bar{x}, \bar{y})$ is a quantifier-free formula in $\mathcal{L}$. Replace $\varphi(\bar{x}, \bar{y})$ with its disjunctive normal form. Then $\psi \equiv \forall \bar{x} \exists \bar{y} [\bigvee_{i \leq n} \varphi_i(\bar{x}, \bar{y})]$, where each $\varphi_i(\bar{x}, \bar{y})$ is a conjunction of atomic and negative atomic sentences. For each $\varphi_i(\bar{x}, \bar{y})$ let $\{\varphi_{ij}(\bar{x}, \bar{y})\}_{j \leq m_i}$ be a listing of all complete atomic diagrams in $(\bar{x}, \bar{y})$ that are consistent with the conjuncts that make up $\varphi_i(\bar{x}, \bar{y})$. If there are no such $\varphi_{ij}(\bar{x}, \bar{y})$, let $\varphi_{i0}(\bar{x}, \bar{y})$ be $\perp$, a logically false sentence. Note that m_i really is a finite number, because $\mathcal{L}$ is a finite language and $(\bar{x}, \bar{y})$ is finite.

Now, note that for each i, $\varphi_i(\bar{x}, \bar{y}) \equiv \bigvee_{j \leq m_i} \varphi_{ij}(\bar{x}, \bar{y})$. Hence, $\psi \equiv \forall \bar{x} \exists \bar{y} [\bigvee_{i \leq n} \bigvee_{j \leq m_i} \varphi_{ij}(\bar{x}, \bar{y})]$. By reindexing and renaming, we may turn the nested disjunctions into a single disjunction: $\psi \equiv \forall \bar{x} \exists \bar{y} [\bigvee_{i \leq t} \varphi_i(\bar{x}, \bar{y})]$, where, we recall, each $\varphi_i(\bar{x}, \bar{y})$ is some complete atomic diagram for $(\bar{x}, \bar{y})$, or is $\perp$.

Now, let $\{\theta_\ell(\bar{x})\}_{\ell \leq u}$ be a listing of all complete atomic diagrams for $\bar{x}$. Again, note that u is a finite number. It is straightforward to check that $\psi \equiv \forall \bar{x} \exists \bar{y} [\bigwedge_{\ell \leq u} (\theta_\ell(\bar{x}) \Rightarrow \bigvee_{i \leq t} \varphi_i(\bar{x}, \bar{y}))]$.

Here, we can notice that to decide the $\forall\exists$-theory it suffices to decide all sentences of the form $\psi_\ell \equiv \forall \bar{x} \exists \bar{y} [\theta_\ell(\bar{x}) \Rightarrow \bigvee_{i \leq t} \varphi_i(\bar{x}, \bar{y})]$, where $\theta_\ell(\bar{x})$ is a complete atomic diagram for $\bar{x}$, and each $\varphi_i(\bar{x}, \bar{y})$ is a complete atomic diagram for $(\bar{x}, \bar{y})$ or is $\perp$. Let $\mathcal{M}$ be the $\mathcal{L}$ structure defined by $\theta_\ell(\bar{x})$. For each $i \leq t$, let $\mathcal{N}_i$ be the $\mathcal{L}$-structure defined by $\varphi_i(\bar{x}, \bar{y})$; if some $\varphi_i(\bar{x}, \bar{y})$ is $\perp$, then let $\mathcal{N}_i$ be the empty structure. Note that, literally speaking, it is very possible for some $\mathcal{N}_i$ to extend $\mathcal{M}$, because they share the same names for $\bar{x}$. We can see now that $\mathcal{P} \models \psi_\ell$ just in case $(\mathcal{M}, \mathcal{N}_0, \ldots, \mathcal{N}_t)$ is an element of the multi-extension of embeddings problem for $\mathcal{P}$. (Note that if $\theta_\ell(\bar{x})$ is false for every possible substitution for $\bar{x}$ from elements of $\mathcal{P}$, there is no embedding of $\mathcal{M}$ into $\mathcal{P}$, so $(\mathcal{M}, \mathcal{N}_0, \ldots, \mathcal{N}_t)$ is trivially a member of the multi-extension of embeddings problem for $\mathcal{P}$.)

$\square$

It is not to hard to see that the converse of the proposition is also true.

3.2 Algebraic considerations

We begin our investigation with the extension of embeddings problem for $\mathcal{P}_s$. To get a handle on the problem, we assume that the extension we are inquiring into is only a one-element extension. So, we imagine we are given finite L-structures $\mathcal{M}$ and $\mathcal{M} \cup \{n\}$. Then, we must be able to answer, according to some effective procedure, whether every $f : \mathcal{M} \hookrightarrow \mathcal{P}_s$ extends to some $f' : \mathcal{M} \cup \{n\} \hookrightarrow \mathcal{P}_s$.

With previous results for embeddings into $\mathcal{P}_s$ in mind (as in Chapter 2, for example), we would expect that if there is "space" for the embedding to even have a chance to extend to n, then we should be able to do so.

We give the following definitions to make precise this idea.

Definition 46. Suppose $\mathcal{N} = (N, \mathbf{0}_{\mathcal{N}}, \mathbf{1}_{\mathcal{N}}, \leq_{\mathcal{N}})$ is a finite L-structure and $M_0 \subseteq N$. Then, for each $n \in N$, define the following subsets of M_0.

$$A_n(M_0, \leq_{\mathcal{N}}) = \{x \in M_0 : x >_{\mathcal{N}} n\} \cup \{\mathbf{1}\}.$$

$$B_n(M_0, \leq_{\mathcal{N}}) = \{x \in M_0 : x <_{\mathcal{N}} n\} \cup \{\mathbf{0}\}.$$

$$I_n(M_0, \leq_{\mathcal{N}}) = \{x \in M_0 : x \perp_{\mathcal{N}} n\},$$

where $x \perp_{\mathcal{N}} n$ means $x \nleq_{\mathcal{N}} n$ and $x \ngeq_{\mathcal{N}} n$.

If M_0 is the domain of some L-structure $\mathcal{M} \subseteq \mathcal{N}$, we may use the notation $A_n(\mathcal{M}, \leq_{\mathcal{N}})$, $B_n(\mathcal{M}, \leq_{\mathcal{N}})$, and $I_n(\mathcal{M}, \leq_{\mathcal{N}})$.

We make a further definition: if $\widehat{\mathcal{M}} = (M, \mathbf{0}_{\widehat{\mathcal{M}}}, \mathbf{1}_{\widehat{\mathcal{M}}}, \leq_{\widehat{\mathcal{M}}}, \wedge_{\widehat{\mathcal{M}}}, \vee_{\widehat{\mathcal{M}}})$ is an $\widehat{L}$-structure such that $M_0 \subseteq M$ and $\leq_{\widehat{\mathcal{M}}}$ agrees with $\leq_{\mathcal{N}}$, we define, for each $n \in N$,

$$a_n(\widehat{\mathcal{M}}, M_0, \leq_{\mathcal{N}}) = \bigwedge_{x \in A_n(M_0, \leq_{\mathcal{N}})}^{\widehat{\mathcal{M}}} x,$$

$$b_n(\widehat{\mathcal{M}}, M_0, \leq_{\mathcal{N}}) = \bigvee_{x \in B_n(M_0, \leq_{\mathcal{N}})}^{\widehat{\mathcal{M}}} x,$$

where the $\widehat{\mathcal{M}}$ over the meet and the join mean to take these operations in the structure $\widehat{\mathcal{M}}$.

Again, if M_0 is the domain of some L-structure $\mathcal{M} \subseteq \mathcal{N}$, we may use the notation $a_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})$ for $a_n(\widehat{\mathcal{M}}, M_0, \leq_{\mathcal{N}})$ and $b_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})$ for $b_n(\widehat{\mathcal{M}}, M_0, \leq_{\mathcal{N}})$.

See [24] for a similar definition, used while working on the extension of embeddings problem for the c.e. Turing degrees.

A is meant to recall "Above". B is meant to recall "Below". I is meant to recall "Independent."

Proposition 47. *Suppose, as in the previous definition, that $\mathcal{M} \subseteq \mathcal{N}$ are L-structures, and that $\widehat{\mathcal{M}}$ is an $\widehat{L}$-structure extending $\mathcal{M}$ such that $\leq_{\widehat{\mathcal{M}}}$ agrees with $\leq_{\mathcal{N}}$. (Note that $\widehat{\mathcal{M}}$ may contain elements of $\mathcal{N} - \mathcal{M}$). Then, for each $n \in \mathcal{N}$,*

1. *$A_n(\mathcal{M}, \leq_{\mathcal{N}}) \neq \emptyset$, since $\mathbf{1}$, at least, is a member, even if $n = \mathbf{1}$.*
2. *$B_n(\mathcal{M}, \leq_{\mathcal{N}}) \neq \emptyset$, since $\mathbf{0}$, at least, is a member, even if $n = \mathbf{0}$.*
3. *$a_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}) \geq_{\widehat{\mathcal{M}}} b_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})$.*

   For all $i \in I_n(\mathcal{M}, \leq_{\mathcal{N}})$,

4. *$i \not\leq_{\widehat{\mathcal{M}}} b_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})$.*
5. *$a_i(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}) \not\leq_{\widehat{\mathcal{M}}} b_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})$.*
6. *$i \not\geq_{\widehat{\mathcal{M}}} a_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})$.*
7. *$b_i(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}) \not\geq_{\widehat{\mathcal{M}}} a_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})$.*

   For all $n' \in \mathcal{N}$,

8. *If $n >_{\mathcal{N}} n'$, then $A_{n'}(\mathcal{M}, \leq_{\mathcal{N}}) \subseteq A_n(\mathcal{M}, \leq_{\mathcal{N}})$, hence $a_{n'}(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}) \geq_{\widehat{\mathcal{M}}} a_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})$.*
9. *If $n <_{\mathcal{N}} n'$, then $B_{n'}(\mathcal{M}, \leq_{\mathcal{N}}) \supseteq B_n(\mathcal{M}, \leq_{\mathcal{N}})$, hence $b_{n'}(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}) \geq_{\widehat{\mathcal{M}}} b_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})$.*

Note that in this proposition, i is a variable for an element independent of n, not for an index.

Proof. Straightforward. For example, consider (4). If $i \in I_n(\mathcal{M}, \leq_{\mathcal{N}})$ is such that $i \leq_{\widehat{\mathcal{M}}} b_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})$, then for all $x \in B_n(\mathcal{M}, \leq_{\mathcal{N}})$, $i \leq_{\widehat{\mathcal{M}}} x$. Since $\widehat{\mathcal{M}}$ respects $\leq_{\mathcal{N}}$, we have $i \leq_{\mathcal{N}} x \leq_{\mathcal{N}} n$, contradicting $i \perp_{\mathcal{N}} n$.

□

These basic facts will be enough to decide the extension of embeddings problem in $\mathcal{P}_s$ when it is restricted to the cases of one-element extensions. In order to decide the full extension of embeddings problem, we will proceed inductively, and will need the following Proposition.

Proposition 48. *Suppose $\widehat{\mathcal{N}}_0 \subseteq \widehat{\mathcal{N}}_1$ are finite $\widehat{L}$-structures, respectively extending finite L-structures $\mathcal{N}_0$ and $\mathcal{N}_1$. Suppose also that $\mathcal{N}_i \subseteq \mathcal{N}$ for $i = 0, 1$, where $\mathcal{N}$ is an L-structure. For each $n \in \mathcal{N}$:*

1. $a_n(\widehat{\mathcal{N}}_1, \mathcal{N}_1, \leq_{\mathcal{N}}) \leq_{\widehat{\mathcal{N}}_1} a_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}})$.
2. $b_n(\widehat{\mathcal{N}}_1, \mathcal{N}_1, \leq_{\mathcal{N}}) \geq_{\widehat{\mathcal{N}}_1} b_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}})$.

Proof. Note that $A_n(\mathcal{N}_1, \leq_{\mathcal{N}}) \supseteq A_n(\mathcal{N}_0, \leq_{\mathcal{N}})$ and $B_n(\mathcal{N}_1, \leq_{\mathcal{N}}) \supseteq B_n(\mathcal{N}_0, \leq_{\mathcal{N}})$. □

Our goal as we proceed inductively is to extend the embedding to a new element $n \in \mathcal{N}$ in such a way as to avoid making more difficult the future embedding of those elements $n' \in \mathcal{N}$ yet to be embedded. To accomplish this, we need some freedom in where we place the new element n. Unfortunately, it turns out that in some cases we will have no freedom.

Definition 49. Suppose $\mathcal{M} \subseteq \mathcal{N}_* \subseteq \mathcal{N}$ are finite L-structures, and $\widehat{\mathcal{N}}_*$ is an $\widehat{L}$-structure that is a minimal extension of $\mathcal{N}_*$. For each $n \in \mathcal{N} - \mathcal{M}$, n is *degenerate for* $\widehat{\mathcal{N}}_*$ if $a_n(\widehat{\mathcal{N}}_*, \mathcal{N}_*, \leq_{\mathcal{N}}) = b_n(\widehat{\mathcal{N}}_*, \mathcal{N}_*, \leq_{\mathcal{N}})$. For $n_1, n_2 \in \mathcal{N} - \mathcal{M}$ such that $n_1 \neq n_2$, let (n_1, n_2) be a *degenerate pair for* $\widehat{\mathcal{N}}_*$ if each of n_1 and n_2 is degenerate in $\widehat{\mathcal{N}}_*$ and $a_{n_1}(\widehat{\mathcal{N}}_*, \mathcal{N}_*, \leq_{\mathcal{N}}) = b_{n_2}(\widehat{\mathcal{N}}_*, \mathcal{N}_*, \leq_{\mathcal{N}})$.

We need some straightforward results before we put this definition to use.

Lemma 50. *Suppose $\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{N}$ are finite L-structures, $\widehat{\mathcal{N}}_0 \subseteq \widehat{\mathcal{N}}_1$ are $\widehat{L}$-structures, and $h_0 : \mathcal{N}_0 \hookrightarrow \widehat{\mathcal{N}}_0$ is an L-embedding extended by the L-embedding $h_1 : \mathcal{N}_1 \hookrightarrow \widehat{\mathcal{N}}_1$.*

Then, for each $n \in \mathcal{N}_1$,

$$h_1(n) \leq_{\widehat{\mathcal{N}}_1} a_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}})$$

and

$$h_1(n) \geq_{\widehat{\mathcal{N}}_1} b_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}}).$$

(In the calculation of $a_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}})$ and $b_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}})$ here, we think of $\mathcal{N}_0$ as a substructure of $\widehat{\mathcal{N}}_0$ given by h_0.)

$$\begin{tikzcd}
\mathcal{N}_0 \arrow[r, "h_0"] \arrow[d, "i"] & \widehat{\mathcal{N}}_0 \arrow[d, "i"] \\
\mathcal{N}_1 \arrow[r, "h_1"] & \widehat{\mathcal{N}}_1
\end{tikzcd}$$

Figure 3.7. Sketch for Lemma 50.

Proof. For notational ease in this proof, let $B = B_n(\mathcal{N}_0, \leq_{\mathcal{N}})$ and let $A = A_n(\mathcal{N}_0, \leq_{\mathcal{N}})$. By the definition of an embedding, for all $x \in B$, $h_1(n) \geq_{\widehat{\mathcal{N}}_1} h_1(x) = h_0(x)$. So,

$$h_1(n) \geq_{\widehat{\mathcal{N}}_1} \bigvee_{x \in B} h_0(x) = b_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}}).$$

Similarly, by definition, for all $x \in A$, $h_1(n) \leq_{\widehat{\mathcal{N}}_1} h_1(x) = h_0(x)$. So,

$$h_1(n) \leq_{\widehat{\mathcal{N}}_1} \bigwedge_{x \in A} h_0(x) = a_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}}).$$

□

Note that the following slight variation of Lemma 50 is true by a proof only slightly more involved; this form will be useful for us later.

Corollary 51. *Suppose $\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{N}$ are finite L-structures and $g : \mathcal{N}_0 \hookrightarrow \mathcal{P}_s$ is an L-embedding extended by the L-embedding $h : \mathcal{N}_1 \hookrightarrow \mathcal{P}_s$. Let $\widehat{\mathcal{N}}_0$ be the $\widehat{L}$-closure of $\mathcal{N}_0$ induced by g. Let $\widehat{g} : \widehat{\mathcal{N}}_0 \hookrightarrow \mathcal{P}_s$ be the $\widehat{L}$-embedding that is the unique extension of g to $\widehat{\mathcal{N}}_0$, as given by Proposition 39. The following diagram illustrates the situation.*

$$\begin{tikzcd}
\mathcal{N}_0 \arrow[r, "i"] \arrow[d, "i"] & \widehat{\mathcal{N}}_0 \arrow[r, "\widehat{g}"] & \mathcal{P}_s \arrow[d, "i"] \\
\mathcal{N}_1 \arrow[rr, "h"] & & \mathcal{P}_s
\end{tikzcd}$$

Figure 3.8. Sketch for Corollary 51.

***Then**, for each $n \in \mathcal{N}_1$, $h(n) \leq_s \widehat{g}(a_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}}))$ and $h(n) \geq_s \widehat{g}(b_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}}))$.*

If $n \in \mathcal{N} - \mathcal{N}_0$ is degenerate for $\widehat{\mathcal{N}_0}$ then n's place is determined in any extension of $\widehat{\mathcal{N}_0}$ that contains n and respects $\leq_{\mathcal{N}}$:

Corollary 52. *Suppose $\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{N}$ are finite L-structures and $h_0 : \mathcal{N}_0 \hookrightarrow \widehat{\mathcal{N}_0}$ is an L-embedding extended by the L-embedding $h_1 : \mathcal{N}_1 \hookrightarrow \widehat{\mathcal{N}}_1$. Further, suppose $n \in \mathcal{N}_1$ is degenerate for $\widehat{\mathcal{N}}_0$. (Here we think of $\widehat{\mathcal{N}}_0$ as an L-extension of $\mathcal{N}_0$ given by h_0.)*

Then, $h_1(n) = a_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}}, n) = b_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}}, n)$.

Proof. Apply Lemma 50. $\square$

Hence, if there is a degenerate pair (n_1, n_2) for $\widehat{\mathcal{N}_0}$, there is no extension of $\widehat{\mathcal{N}_0}$ that contains both n_1 and n_2 and respects $\leq_{\mathcal{N}}$. For n_1 and n_2 would have to be in exactly the same spot. So an $\widehat{L}$-structure can contain at most one of the two elements of a pair degenerate for it.

This is a fact we will use later, so we stop to formally state it (in a slightly purer form).

Proposition 53. *Suppose $\mathcal{N}_0$ is a finite L-structure, and $\widehat{\mathcal{N}}_0$ is an $\widehat{L}$-structure that is an L-extension of $\mathcal{N}_0$. Then, there do not exist $n_1, n_2 \in \mathcal{N}_0$ so that (n_1, n_2) is a degenerate pair for $\widehat{\mathcal{N}}_0$.*

Lemma 54. *Suppose $\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{N}$ are finite L-structures, $\widehat{\mathcal{N}}_0 \subseteq \widehat{\mathcal{N}}_1$ are finite $\widehat{L}$-structures, and $\widehat{\mathcal{N}}_0$ and $\widehat{\mathcal{N}}_1$ extend $\mathcal{N}_0$ and $\mathcal{N}_1$ as L-structures, respectively. If $n \in \mathcal{N}$ is degenerate for $\widehat{\mathcal{N}}_0$, then $b_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}}) = a_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}}) =$ $a_n(\widehat{\mathcal{N}}_1, \mathcal{N}_1, \leq_{\mathcal{N}}) = b_n(\widehat{\mathcal{N}}_1, \mathcal{N}_1, \leq_{\mathcal{N}})$.*

Proof. The first equality is what it means for n to be degenerate for $\widehat{\mathcal{N}}_0$. The second and third follow from Proposition 48, the first equality, and Proposition 47(3):

$a_n(\widehat{\mathcal{N}}_1, \mathcal{N}_1, \leq_{\mathcal{N}}) \leq_{\mathcal{N}_1} a_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}}) \leq_{\mathcal{N}_1} b_n(\widehat{\mathcal{N}}_0, \mathcal{N}_0, \leq_{\mathcal{N}}) \leq_{\mathcal{N}_1} b_n(\widehat{\mathcal{N}}_1, \mathcal{N}_1, \leq_{\mathcal{N}}) \leq_{\mathcal{N}_1} a_n(\widehat{\mathcal{N}}_1, \mathcal{N}_1, \leq_{\mathcal{N}})$.

$\square$

At this point, we are prepared to give the condition that will guarantee an element n yet to be embedded will pose no obstacle to accomplishing the desired embedding. Then, in the next section, we will be able to solve the extension of embeddings and multi-extension of embeddings problems, based almost entirely on whether this condition holds for all n.

Definition 55. Suppose $\mathcal{M} \subseteq \mathcal{N}_* \subseteq \mathcal{N}$ are finite L-structures and $\widehat{\mathcal{N}_*}$ is an $\widehat{L}$-structure that minimally extends $\mathcal{N}_*$. For $n \in \mathcal{N} - \mathcal{N}_*$, we define $C(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}}, n)$ as the conjunction of the following conditions.

1. If n is degenerate for $\widehat{\mathcal{N}_*}$, then $a_n(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}}) = b_n(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}}) \notin \mathcal{N}_*$.

2. $i \not\geq_{\widehat{\mathcal{N}_*}} a_n(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}})$, for all $i \in I_n(\mathcal{N}_*, \leq_{\mathcal{N}})$.

3. $i \not\leq_{\widehat{\mathcal{N}_*}} b_n(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}})$, for all $i \in I_n(\mathcal{N}_*, \leq_{\mathcal{N}})$.

   If $n \perp n'$ for some $n' \in \mathcal{N} - \mathcal{N}_*$, then

4. $a_n(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}}) \not\leq_{\mathcal{N}_*} b_{n'}(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}})$ &

5. $b_n(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}}) \not\geq_{\mathcal{N}_*} a_{n'}(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}})$.

Remark 56. Suppose n is degenerate in $\widehat{\mathcal{N}}_*$, and therefore $a_n(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}}) = b_n(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}})$. If $C(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}}, n)$ holds, then by parts (4) and (5):

4. if $n \perp_{\mathcal{N}} n'$ for some $n' \in \mathcal{N} - \mathcal{N}_*$, then
   $b_n(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}}) \not\leq_{\widehat{\mathcal{N}_*}} b_{n'}(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}})$, and

5. if $n \perp_{\mathcal{N}} n'$ for some $n' \in \mathcal{N} - \mathcal{N}_*$, then
   $a_n(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}}) \not\geq_{\widehat{\mathcal{N}_*}} a_{n'}(\widehat{\mathcal{N}_*}, \mathcal{N}_*, \leq_{\mathcal{N}})$.

Remark 57. Note that if clause (2) or (3) in the definition fails, we will not be able to extend $\widehat{\mathcal{N}}_*$ to a structure that includes n and respects $\leq_{\mathcal{N}}$. Also, by Corollary 52, (1) is necessary for such an extension to be possible. If (4) or (5) were to fail, we might be able to extend $\widehat{\mathcal{N}}_*$ to a structure $\widehat{\mathcal{N}}_{**}$ including n. However, we would not then be able to extend $\widehat{\mathcal{N}}_{**}$ to a structure including n', because (2) or (3), respectively, would fail to hold for n' with respect to $\widehat{\mathcal{N}}_{**}$.

3.3 The decision procedure for the $\forall\exists$-theory of $\mathcal{P}_s$

Recall the languages $L = \{0, 1, \leq\}$ and $\widehat{L} = \{0, 1, \leq, \vee, \wedge\}$ and our conventions as in Section 3.1.1.

Theorem 58. *Suppose $\mathcal{M} \subseteq \mathcal{N}$ are finite L-structures. The following are equivalent.*

1. *For every L-embedding $f : \mathcal{M} \hookrightarrow \mathcal{P}_s$, there is an L-embedding $f' : \mathcal{N} \hookrightarrow \mathcal{P}_s$ that extends f.*

2. *For each $\widehat{L}$-structure $\widehat{\mathcal{M}}$ minimally extending $\mathcal{M}$, and such that $\boldsymbol{0}$ is non-branching in $\widehat{\mathcal{M}}$, the following two hold:*

   (a) There are no $n_1, n_2 \in \mathcal{N}$ so that (n_1, n_2) is a degenerate pair for $\widehat{\mathcal{M}}$.

   (b) For each $n \in \mathcal{N} - \mathcal{M}$, $C(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}, n)$ holds, where C is as in Definition 55.

Corollary 59. *The extension of embeddings problem is decidable for $\mathcal{P}_s$ in the language $L = \{0, 1, \leq\}$.*

Proof. Use the notion of the extension of embeddings problem given by Proposition 42. Corollary 35 tells us that for each L-structure $\mathcal{M}$, we effectively know all its minimal extensions. Checking condition (2) of Theorem 58 is an effective procedure. So, given finite L-structures $\mathcal{M} \subseteq \mathcal{N}$, we simply calculate the minimal extensions of $\mathcal{M}$, and then check to see if condition (2) holds for each one of them with respect to $\mathcal{N}$. If so, $(\mathcal{M}, \mathcal{N})$ is an element of the extension of embeddings problem for $\mathcal{P}_s$ in L. Otherwise, not. □

Theorem 60. *Suppose $\mathcal{M}, \mathcal{N}_0, \ldots, \mathcal{N}_t$ are finitely many L-structures such that $\mathcal{M} \subseteq \mathcal{N}_i$ for each $0 \leq i \leq t$. Then the following are equivalent.*

1. *For every L-embedding $f : \mathcal{M} \hookrightarrow \mathcal{P}_s$, there is an i such that $0 \leq i \leq t$ and an L-embedding $f' : \mathcal{N}_i \hookrightarrow \mathcal{P}_s$ that extends f.*

2. *For each $\widehat{L}$-structure $\widehat{\mathcal{M}}$ minimally extending $\mathcal{M}$, and such that $\boldsymbol{0}$ is non-branching in $\widehat{\mathcal{M}}$, there is $0 \leq i \leq t$ such that the following two hold:*

   (a) *There are no $n_1, n_2 \in \mathcal{N}_i$ so that (n_1, n_2) is a degenerate pair for $\widehat{\mathcal{M}}$.*

   (b) *For each $n \in \mathcal{N}_i - \mathcal{M}$, $C(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}_i}, n)$ holds, where C is as in Definition 55.*

Proof. Apply Theorem 58.

□

Corollary 61. *The $\forall\exists$-theory of $\mathcal{P}_s$ is decidable in the language $(0, 1, \leq)$.*

Proof. Apply Proposition 43, then mimic the proof of Corollary 59. In particular, $(\mathcal{M}, \mathcal{N}_0, \ldots, \mathcal{N}_t)$ is an element of the multi-extension of embeddings problem just in case, for every $\widehat{\mathcal{M}}$ minimally extending $\mathcal{M}$, there is an i so that (2a) and (2b) of Theorem 60 hold for $\widehat{\mathcal{M}}$ and $\mathcal{N}_i$. Finally, apply Proposition 44.

□

3.4 When there is not always an extension

Proof of (1) ⇒ (2) in Theorem 58, by establishing the contrapositive.

Assume $\mathcal{M} \subseteq \mathcal{N}$ are finite L-structures. The negation of (2) is that there is a finite $\widehat{L}$-structure $\widehat{\mathcal{M}}$ minimally extending $\mathcal{M}$ with $\mathbf{0}$ non-branching in $\widehat{\mathcal{M}}$, such that either

$$\text{There is a pair } (n_1, n_2) \text{ in } \mathcal{N} \text{ degenerate for } \widehat{\mathcal{M}} \text{ OR}$$
$$\text{there is } n \in \mathcal{N} \text{ so that } C(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}, n) \text{ does not hold.}$$

Either way, by Corollary 27, there is an $\widehat{L}$-embedding $\widehat{f} : \widehat{\mathcal{M}} \hookrightarrow \mathcal{P}_s$. By Lemma 28, $f : \mathcal{M} \hookrightarrow \mathcal{P}_s$ is an L-embedding, where $f = \widehat{f} \restriction \widehat{\mathcal{M}}$. This establishes the hypothesis of (1).

If there is a pair (n_1, n_2) degenerate for $\widehat{\mathcal{M}}$, then, by Proposition 53, there is no $\widehat{L}$-structure $\widehat{\mathcal{N}}$ that extends $\widehat{\mathcal{M}}$ and contains n_1 and n_2. Hence, by Proposition 39, there is no L-embedding $f' : \mathcal{N} \hookrightarrow \mathcal{P}_s$ that extends f. This is the negation of the conclusion of (1), so we are done with the case of a degenerate pair.

The other case is that there is $n \in \mathcal{N}$ so that $C(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}, n)$ does not hold. We consider the negation of each of the clauses in the definition of C. If part (1) of C fails for n, then it is clear the conclusion of (1) in Theorem 58 fails;

for Corollary 52 requires n to be mapped to a spot that is already the image of another element of $\mathcal{N}$.

If part (2) of C fails for n, then $i \geq_{\widehat{\mathcal{M}}} a_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})$ for some $i \in I_n(\mathcal{M}, \leq_{\mathcal{N}})$. Then, if $f' : \mathcal{N} \to \mathcal{P}_s$ were a map extending f and an L-embedding, we would have $f'(i) = f(i) \geq_s \widehat{f}(a_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})) \geq_s f'(n)$. (The first inequality comes from an application of Corollary 51.) On the other hand, $i \perp_{\mathcal{N}} n$, contradicting that f' is an L-embedding. A similar argument, replacing $\geq$ by $\leq$ in the appropriate places, shows that if clause (3) of C fails for n, then the conclusion of (1) in Theorem 58 fails.

Suppose part (4) of C for n fails: $n \perp_{\mathcal{N}} n'$ for some $n' \in \mathcal{N} - \mathcal{M}$ and

$$a_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}) \leq_{\widehat{\mathcal{M}}} b_{n'}(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}).$$

Our argument is similar to the cases just discussed. If $f' : \mathcal{N} \to \mathcal{P}_s$ were a map extending f and an L-embedding, we would have $f'(n) \leq_s \widehat{f}(a_n(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})) \leq_s \widehat{f}(b_{n'}(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}})) \leq_s f'(n')$. (The first inequality comes from an application of Corollary 51.) On the other hand, $n \perp_{\mathcal{N}} n'$, contradicting that f' is an L-embedding. A similar argument, replacing $\leq$ by $\geq$ in the appropriate places, shows that if clause (5) of C fails for n, then the conclusion of (1) in Theorem 58 fails.

$\square$

3.4.1 Examples

We give examples of structures $\mathcal{M}$, $\mathcal{N}$, and $\widehat{\mathcal{M}}$, the $\widehat{L}$-closure of an embedding of $\mathcal{M}$ into $\mathcal{P}_s$ that does not extend to an embedding of $\mathcal{N}$.

In the diagrams for this subsection, lines will indicate a $\leq$ relationship; we will omit lines to **1** and **0**.

Example 62. A degenerate pair.

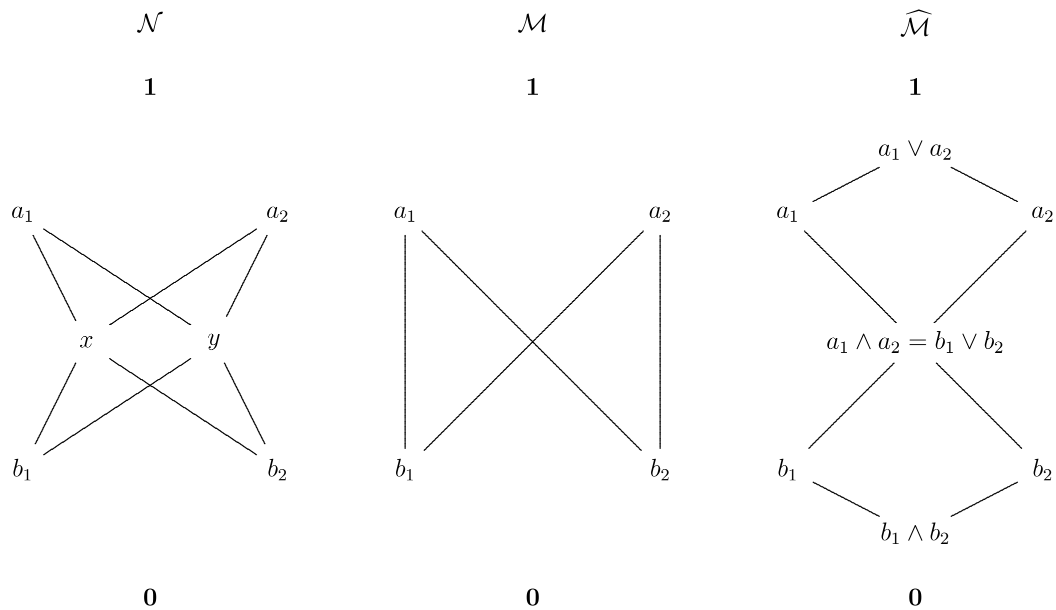

Figure 3.9. Sketch of Example 62.

By Corollary 27 and Lemma 28, there is an $f : \mathcal{M} \hookrightarrow \mathcal{P}_s$ with $\widehat{\mathcal{M}}$ as its $\widehat{L}$-closure.

Here, x and y are degenerate for $\widehat{\mathcal{M}}$; there is no extension of f to $\mathcal{N}$. The reason is that if we extend to $f' : \mathcal{M} \cup \{x\} \hookrightarrow \mathcal{P}_s$, we must have $f(x) = f(a_1) \wedge f(a_2) = f(b_1) \vee f(b_2)$. But then if f'' extends f' and includes y in its domain, there is nowhere for y to go. Thus, this particular pair $(\mathcal{M}, \mathcal{N})$ is not an element of the extension of embeddings problem for $(\mathcal{P}_s, \leq_s)$.

Example 63. Failure of part (1) of C.

If we take $\mathcal{M} \cup \{x\}$ as our base structure, and f' as as our original embedding into $\mathcal{P}_s$, where $\mathcal{M}$, x, and f' are as in Example 62, then part 1 of the corresponding condition C fails for y. Switching back to the usual notation for these examples and C, the situation is as follows.

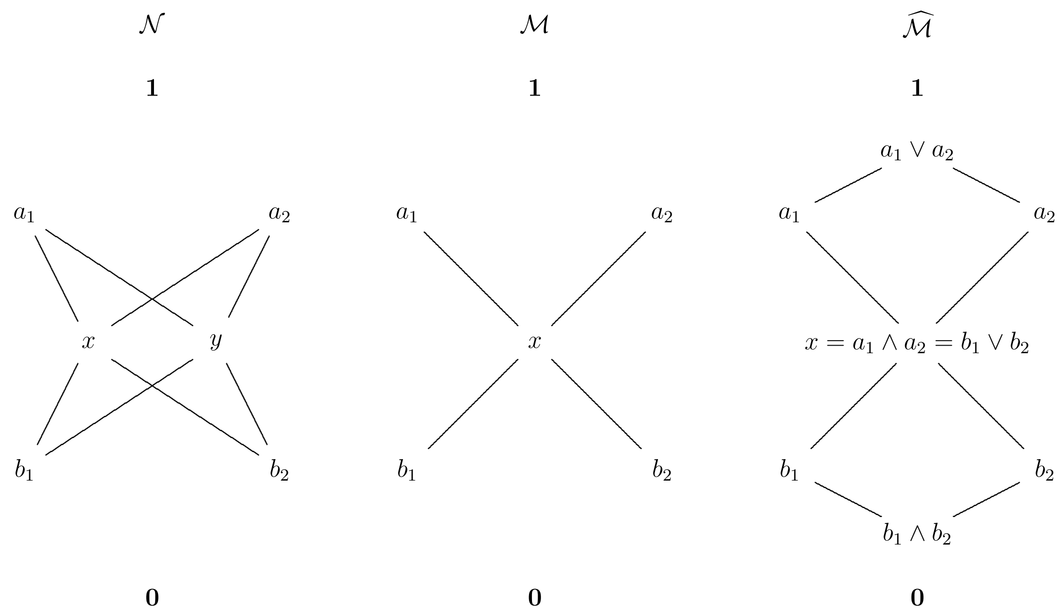

Figure 3.10. Sketch for Example 63.

Part 1 of $C(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}, y)$ fails: y is degenerate for $\widehat{\mathcal{M}}$, but y's spot in $\widehat{\mathcal{M}}$ is already taken by x. So this pair $(\mathcal{M}, \mathcal{N})$ is not an element of the extension of embeddings problem for $(\mathcal{P}_s, \leq_s)$.

Example 64. Failure of part (2) of C.

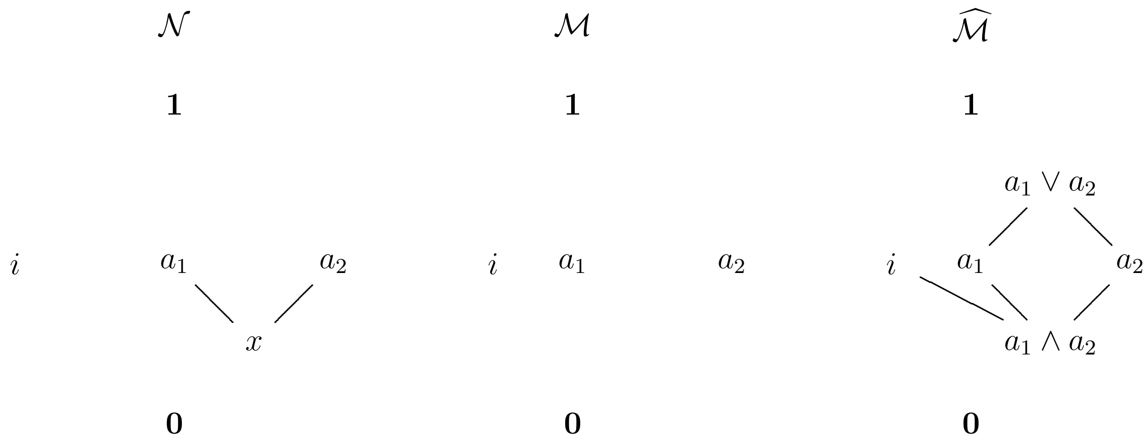

Figure 3.11. Sketch for Example 64.

Part (2) of $C(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}, x)$ fails. The $\widehat{L}$-closure of an embedding of $\mathcal{N}$ would have to place x below $a_1 \wedge a_2$, which is below i, contradicting that in $\mathcal{N}$, x is incomparable with i. Again, by Corollary 27 and Lemma 28, there is an $f : \mathcal{M} \hookrightarrow \mathcal{P}_s$ with $\widehat{\mathcal{M}}$ as its $\widehat{L}$-closure. This shows that $(\mathcal{M}, \mathcal{N})$ is not an element of the extension of embeddings problem for $(\mathcal{P}_s, \leq_s)$.

In the diagram of $\widehat{\mathcal{M}}$, we left out $i \vee a_1$, $i \vee a_2$, $i \vee a_1 \vee a_2$, $i \wedge (a_1 \vee a_2), i \wedge a_1$, $i \wedge a_2$, $(i \wedge a_1) \vee a_2$, $(i \wedge a_2) \vee a_1$, $(i \vee a_1) \wedge a_2$, and $(i \vee a_2) \wedge a_1$, which were not relevant. (Really, we could have left out $a_1 \vee a_2$ as well.)

A dual example shows the failure of part (3) of C.

Example 65. Failure of parts (4) and (5) of C.

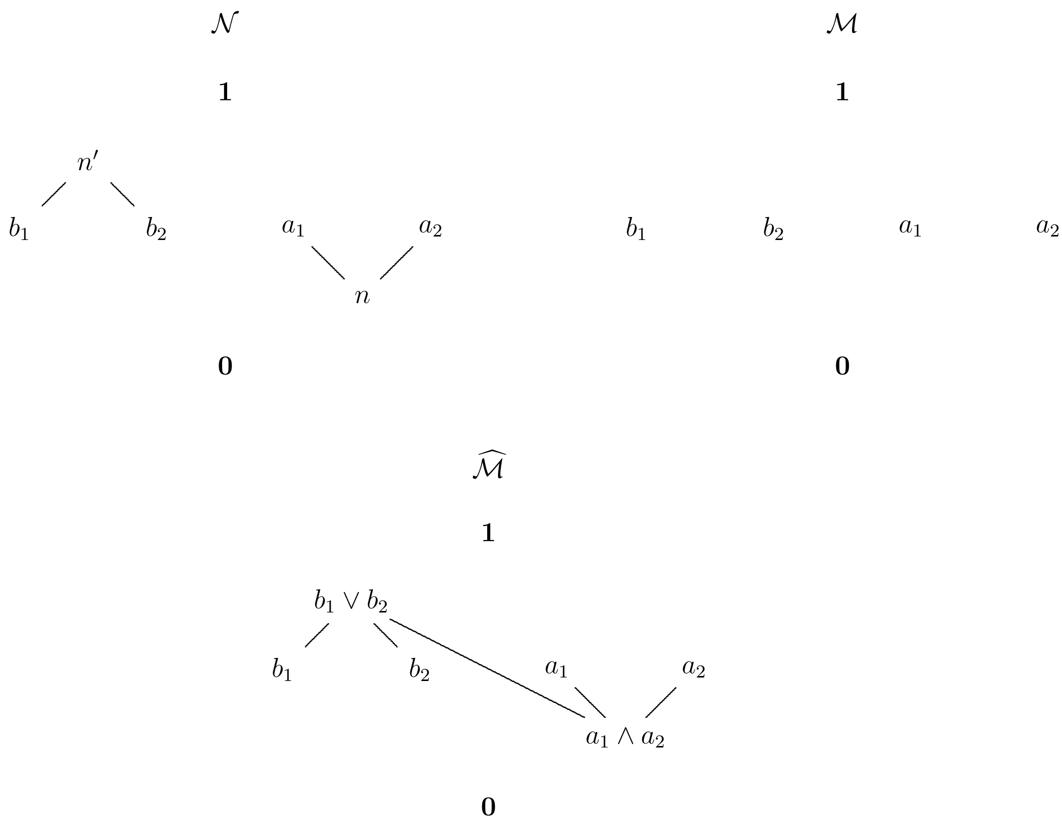

Figure 3.12. Sketch for Example 65.

In this diagram, we again leave out all of $\widehat{\mathcal{M}}$ except what interests us. In this situation, $C(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}, n')$ fails in part (4) and $C(\widehat{\mathcal{M}}, \mathcal{M}, \leq_{\mathcal{N}}, n)$ fails in part (5). In $\widehat{\mathcal{M}}$, if we are to add both n' and n, they will have to go above $b_1 \vee b_2$, and below $a_1 \wedge a_2$, respectively. As a result, we will have $n' \geq_{\widehat{\mathcal{M}}} n$ contrary to n' and n being incomparable in $\mathcal{N}$. As before, $\widehat{\mathcal{M}}$ is the $\widehat{L}$-closure of some $f : \mathcal{M} \hookrightarrow \mathcal{P}_s$ by Corollary 27 and Lemma 28. Hence, $(\mathcal{M}, \mathcal{N})$ is not an element of the extension of embeddings problem for $\mathcal{P}_s$.

3.5 The other half of the proof of Theorem 58

The following Lemma will inductively prove (2) $\Rightarrow$ (1) in Theorem 58.

Lemma 66. *Suppose $\mathcal{M} \subseteq \mathcal{N}_k \subseteq \mathcal{N}_{k+1} \subseteq \mathcal{N}$ are finite L-structures such that $\mathcal{N}_{k+1} = \mathcal{N}_k \cup \{n_{k+1}\}$, $\widehat{\mathcal{N}_k}$ is an $\widehat{L}$-structure that minimally extends $\mathcal{N}_k$, and $C(\widehat{\mathcal{N}_k}, \mathcal{N}_k, \leq_{\mathcal{N}}, n)$ holds for each $n \in \mathcal{N} - \mathcal{N}_k$. Finally, suppose there are no pairs degenerate for $\widehat{\mathcal{N}_k}$.* Then, *given any diagram of embeddings in the following form (where $\widehat{f}_k$ is an $\widehat{L}$-embedding and i is the identity map):*

$$\begin{tikzcd}
\mathcal{N}_k \arrow[r, "i"'] \arrow[d, "i"'] & \widehat{\mathcal{N}}_k \arrow[r, "\widehat{f}_k"] & \mathcal{P}_s \arrow[d, "i"] \\
\mathcal{N}_{k+1} & & \mathcal{P}_s
\end{tikzcd}$$

Figure 3.13. Sketch for Lemma 66.

there is an L-embedding $f_{k+1} : \mathcal{N}_{k+1} \hookrightarrow \mathcal{P}_s$ so that the resulting diagram commutes, and for every $n \in \mathcal{N} - \mathcal{N}_{k+1}$, $C(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}, n)$ holds, where $\widehat{\mathcal{N}}_{k+1}$ is the $\widehat{L}$-closure of $\mathcal{N}_{k+1}$ under f_{k+1}. Finally, we can also ensure there are no pairs from $\mathcal{N}$ that are degenerate for $\widehat{\mathcal{N}}_{k+1}$. The new diagram is as follows, where $\widehat{f}_{k+1}$ is the $\widehat{L}$-embedding such that $i \circ \widehat{f}_{k+1} = f_{k+1}$, which is justified by Proposition 39.

$$\begin{tikzcd}
\mathcal{N}_k \arrow[r, "i"'] \arrow[d, "i"'] & \widehat{\mathcal{N}}_k \arrow[r, "\widehat{f}_k"] \arrow[d, "i"] & \mathcal{P}_s \arrow[d, "i"] \\
\mathcal{N}_{k+1} \arrow[r, "i"] & \widehat{\mathcal{N}}_{k+1} \arrow[r, "\widehat{f}_{k+1}"] & \mathcal{P}_s
\end{tikzcd}$$

Figure 3.14. Sketch for Lemma 66.

Proof of (2) $\Rightarrow$ (1) in Theorem 58 from Lemma 66. Fix $\mathcal{M} \subseteq \mathcal{N}$, finite L-structures. We assume condition (2) from Theorem 58 about $\mathcal{M}$ and $\mathcal{N}$. Let $f : \mathcal{M} \hookrightarrow \mathcal{P}_s$ be an L-embedding. We must show there is an L-embedding $f' : \mathcal{N} \hookrightarrow \mathcal{P}_s$ that extends f.

Let $\{n_1, \ldots, n_\ell\} = \mathcal{N} - \mathcal{M}$ be such that if $k' \neq k$, then $n_{k'} \neq n_k$. Let $\mathcal{N}_k = \mathcal{M} \cup \{n_1, \ldots, n_k\}$ for each $1 \leq k \leq \ell$. Let $\mathcal{N}_0 = \mathcal{M}$.

We will show by induction that for each $0 \leq k \leq \ell$, the following three conditions hold, whose conjunction is named $\Psi(k)$.

1. There is an L-embedding $f_k : \mathcal{N}_k \hookrightarrow \mathcal{P}_s$ that extends f.
2. There are no pairs from $\mathcal{N}$ degenerate for $\widehat{\mathcal{N}}_k$, where $\widehat{\mathcal{N}}_k$ is the $\widehat{L}$ closure of $\mathcal{N}_k$ under f_k.
3. $C(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}, n)$ holds for each $n \in \mathcal{N} - \mathcal{N}_k$.

<u>Base Case: k = 0.</u> Recall that $\mathcal{N}_0 = \mathcal{M}$ and, so, also $\widehat{\mathcal{N}}_0 = \widehat{\mathcal{M}}$, the $\widehat{L}$-closure of $\mathcal{M} = \mathcal{N}_0$ under f. Set $f_0 = f$. Trivially, f_0 is an L-embedding extending f as required by the first condition of $\Psi(0)$. By Remark 38, $\widehat{\mathcal{N}}_0$ is a minimal extension of $\mathcal{N}_0 = \mathcal{M}$. Note that $\mathbf{0}$ is non-branching in $\widehat{\mathcal{N}}_0$, because $\mathbf{0}$ is non-branching in $\mathcal{P}_s$. We may apply (2) from Theorem 58, which gives the second and third conditions of $\Psi(0)$.

<u>Inductive Case.</u> We assume $\Psi(k)$ holds and then see that $\Psi(k+1)$ holds, if $k < \ell$. It is trivial to check that the hypothesis of Lemma 66 follows from our inductive assumption, if $k < \ell$. Then, the conclusion of Lemma 66 trivially implies $\Psi(k+1)$.

Setting $f' = f_\ell$ gives an L-embedding of $\mathcal{N}$ into $\mathcal{P}_s$ that extends f, as desired.

$\square$

3.6 Beginning the proof of Lemma 66

Let $f_k : \mathcal{N}_k \hookrightarrow \mathcal{P}_s$ be given by $\widehat{f}_k \circ i$.

There are two main cases: either n_{k+1} is degenerate in $\mathcal{N}_k$ or it is not. We take the former (and easier) case first. The latter will involve a priority argument.

Case 1: n_{k+1} is degenerate in $\widehat{\mathcal{N}_k}$. The intuition is that all we have to do to obtain f_{k+1} is put n_{k+1} where it must go. Then, proving the rest amounts to showing that nothing much really changes since we only put n_{k+1} in an already existing spot in $\widehat{\mathcal{N}_k}$.

By part (1) of $C(\widehat{\mathcal{N}_k}, \mathcal{N}_k, \leq_{\mathcal{N}}, n_{k+1})$, we have that $a_{n_{k+1}}(\widehat{\mathcal{N}_k}, \mathcal{N}_k, \leq_{\mathcal{N}}) \notin \widehat{\mathcal{N}_k}$. So we define $f_{k+1} : \mathcal{N}_{k+1} \hookrightarrow \mathcal{P}_s$ by letting f_{k+1} agree with f_k on $\mathcal{N}_k$ and defining $f_{k+1}(n_{k+1}) = \widehat{f}_k(a_{n_{k+1}}(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}))$.

By parts (1), (2), and (3) of $C(\widehat{\mathcal{N}_k}, \mathcal{N}_k, \leq_{\mathcal{N}}, n_{k+1})$, it is clear that f_{k+1} is an L-embedding that extends f_k and makes the diagram commute. We can also immediately note that $\widehat{\mathcal{N}}_{k+1}$, as defined by Proposition 39, is the same as $\widehat{\mathcal{N}}_k$. The following Lemma makes precise the intuition that nothing much changes.

Lemma 67. *Suppose we are in case 1 of the proof of Lemma 66, where n_{k+1} is degenerate, and we define f_{k+1} as above, by setting*

$$f_{k+1}(n_{k+1}) = \widehat{f}_k(a_{n_{k+1}}(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}})).$$

Then for all $n \in \mathcal{N} - \mathcal{N}_{k+1}$, the following hold:

$$a_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}) = a_n(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}).$$

$$b_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}) = b_n(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}).$$

Proof. If $n_{k+1} \not\geq_{\mathcal{N}} n$ we trivially have $a_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}) = a_n(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}})$. If $n_{k+1} >_{\mathcal{N}} n$, then

$$a_n(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}) = \bigwedge_{A_n(\mathcal{N}_{k+1}, \leq_{\mathcal{N}})}^{\widehat{\mathcal{N}}_{k+1}} x$$

$$= n_{k+1} \wedge \bigwedge_{A_n(\mathcal{N}_k, \leq_{\mathcal{N}})}^{\widehat{\mathcal{N}}_{k+1}} x$$

$$= n_{k+1} \wedge a_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}).$$

By our definition of f_{k+1} and by Proposition 47(8), we have $n_{k+1} =_{\widehat{\mathcal{N}}_{k+1}} a_{n_{k+1}}(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}) \geq_{\widehat{\mathcal{N}}_{k+1}} a_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}})$. So in fact we can finish by writing

$$= a_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}),$$

and we have shown the first part of the Lemma's conclusion. The second part is analogous, using Proposition 47(9) in place of Proposition 47(8).

□

From Lemma 67, it follows that for every $n \in \mathcal{N}$, if n is not degenerate in $\widehat{\mathcal{N}}_k$, then n is not degenerate in $\widehat{\mathcal{N}}_{k+1}$. So there are no new degenerate elements for $\widehat{\mathcal{N}}_{k+1}$. Hence, by an application of Lemma 54, there are no new degenerate pairs in $\widehat{\mathcal{N}}_{k+1}$. Finally, there no pairs degenerate for $\widehat{\mathcal{N}}_{k+1}$ at all, since by assumption there were none for $\widehat{\mathcal{N}}_k$.

It remains to show that $C(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}, n)$ holds for each $n \in \mathcal{N} - \mathcal{N}_{k+1}$. Fix such an n. Parts (1), (4), and (5) of $C(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}, n)$ follow immediately from $C(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}, n)$ and Lemma 67.

Suppose $n_{k+1} \perp_{\mathcal{N}} n$. By $C(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}, n)$ and Remark 56(1), we have

$$a_{n_{k+1}}(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}) \ngeq_{\widehat{\mathcal{N}}_k} a_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}).$$

Then, by our definition of f_{k+1} and by Lemma 67,

$$n_{k+1} \ngeq_{\widehat{\mathcal{N}}_{k+1}} a_n(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}).$$

Therefore, part (2) of $C(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}, n)$ holds in the case $i = n_{k+1}$. If $i \neq n_{k+1}$, part (2) of $C(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}, n)$ holds by part (2) of $C(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}, n)$ and Lemma 67. A similar argument, using Remark 56(2) in place of Remark 56(1), shows that part (3) of $C(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}, n)$ holds.

This completes the proof of Lemma 66 for case 1, where n_{k+1} is degenerate for $\widehat{\mathcal{N}}_k$.

3.7 Case 2 (of the proof of Lemma 66)

We assume that n_{k+1} is not degenerate for $\widehat{\mathcal{N}_k}$, and we must exhibit the embedding f_{k+1} described by Lemma 66.

We will construct a Π^0_1 class $\mathcal{Q}_{k+1}$ and then define $f_{k+1} : \mathcal{N}_{k+1} \hookrightarrow \mathcal{P}_s$ by letting f_{k+1} agree with f_k on $\mathcal{N}_k$ and setting $f_{k+1}(n_{k+1}) = deg_s(\mathcal{Q}_{k+1})$.

We need some notation.

3.7.1 Notation for use throughout the priority argument

If $j \in \widehat{\mathcal{N}_k}$ then let $\mathcal{Q}_j$ be a Π^0_1 class such that $\widehat{f}_k(j) = deg_s(\mathcal{Q}_j)$.

Let

$$a = a_{n_{k+1}}(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}})$$

and

$$b = b_{n_{k+1}}(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}).$$

Then, by definition, $\mathcal{Q}_a$ is a Π^0_1 class such that $\widehat{f}_k(a_{n_{k+1}}(\widehat{\mathcal{N}_k}, \mathcal{N}_k, \leq_{\mathcal{N}})) = deg_s(\mathcal{Q}_a)$.

Note that $deg_s(\mathcal{Q}_a) = deg_s(\bigwedge\{\mathcal{Q}_j : j \in A_{n_{k+1}}(\mathcal{N}_k, \leq_{\mathcal{N}})\})$.

Also by definition, $\mathcal{Q}_b$ is a Π^0_1 class such that $\widehat{f}_k(b_{n_{k+1}}(\widehat{\mathcal{N}_k}, \mathcal{N}_k, \leq_{\mathcal{N}})) = deg_s(\mathcal{Q}_b)$.

Note that $deg_s(\mathcal{Q}_b) = deg_s(\bigvee\{\mathcal{Q}_j : j \in B_{n_{k+1}}(\mathcal{N}_k, \leq_{\mathcal{N}})\})$.

3.7.2 Towards a first form for requirements

Lemma 68. *Under the assumptions of case 2 of the proof of Lemma 66, and using the notation just defined:*

1. $\mathcal{Q}_a >_s \mathcal{Q}_b$.
2. $\mathcal{Q}_i \not\geq_s \mathcal{Q}_a$ *for all* $i \in I_{n_{k+1}}(\mathcal{N}_k, \leq_{\mathcal{N}})$.
3. $\mathcal{Q}_i \not\leq_s \mathcal{Q}_b$ *for all* $i \in I_{n_{k+1}}(\mathcal{N}_k, \leq_{\mathcal{N}})$.

Proof. We see that (1) follows from Proposition 47(3) and the fact that n_{k+1} is not degenerate in $\mathcal{N}_k$. We see that (2) follows from part (2) of the definition of $C(\widehat{\mathcal{N}_k}, \mathcal{N}_k, \leq_{\mathcal{N}_k}, n_{k+1})$. We see that (3) follows from part (3) of the definition of $C(\widehat{\mathcal{N}_k}, \mathcal{N}_k, \leq_{\mathcal{N}_k}, n_{k+1})$. □

We now propose to construct $\mathcal{Q}_{k+1}$ to meet the following requirements.

(I) $\mathcal{Q}_a >_s \mathcal{Q}_{k+1} >_s \mathcal{Q}_b$.

(II) $\mathcal{Q}_i \ngeq_s \mathcal{Q}_{k+1}$ for all $i \in I_{n_{k+1}}(\mathcal{N}_k, \leq_{\mathcal{N}})$.

(III) $\mathcal{Q}_i \nleq_s \mathcal{Q}_{k+1}$ for all $i \in I_{n_{k+1}}(\mathcal{N}_k, \leq_{\mathcal{N}})$.

For all $x, y \in \widehat{\mathcal{N}_k}$,

(IV) If $\mathcal{Q}_b \vee \mathcal{Q}_x \ngeq_s \mathcal{Q}_y$, then $\mathcal{Q}_{k+1} \vee \mathcal{Q}_x \ngeq_s \mathcal{Q}_y$.

(V) If $\mathcal{Q}_a \wedge \mathcal{Q}_x \nleq_s \mathcal{Q}_y$, then $\mathcal{Q}_{k+1} \wedge \mathcal{Q}_x \nleq_s \mathcal{Q}_y$.

Remark 69. Requirement (IV) implies that if $\mathcal{Q}_b \ngeq_s \mathcal{Q}_y$ for some $y \in \widehat{\mathcal{N}}_k$, then $\mathcal{Q}_{k+1} \ngeq_s \mathcal{Q}_y$. Similarly, (V) implies that if $\mathcal{Q}_a \nleq_s \mathcal{Q}_y$ for some $y \in \widehat{\mathcal{N}}_k$, then $\mathcal{Q}_{k+1} \nleq_s \mathcal{Q}_y$.

3.7.3 Verification that the requirements complete the proof.

Before giving the actual construction, we explain how a $\mathcal{Q}_{k+1}$ meeting requirements (I)-(V) completes the proof of case 2 of Lemma 66. Namely, if f_{k+1} is the proposed extension given by $f_{k+1}(n_{k+1}) = \mathcal{Q}_{k+1}$, then we show that f_{k+1} is indeed an embedding, that each $n \in \mathcal{N} - \mathcal{N}_{k+1}$ still possesses the property C, now relative to f_{k+1}, and that there are no degenerate pairs in the $\widehat{L}$-structure induced by f_{k+1}.

That f_{k+1} is an L-embedding follows from (I)-(III) and the fact that f_k was an L-embedding.

Recall that, by definition, $\widehat{\mathcal{N}}_{k+1}$ is the $\widehat{L}$-structure minimally extending $\mathcal{N}_{k+1}$ and isomorphic to the closure of the image of $\mathcal{N}_{k+1}$ in $\mathcal{P}_s$ under f_{k+1}.

Lemma 70. *In case 2 of the proof of Lemma 66, with f_{k+1} as described above, if $n \in \mathcal{N}$ is degenerate in $\widehat{\mathcal{N}}_{k+1}$, then n was degenerate in $\widehat{\mathcal{N}}_k$.*

Proof. We show the contrapositive. Suppose n is not degenerate in $\widehat{\mathcal{N}}_k$.

If $n = n_{k+1}$ it is clear that $a_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}) = a_n(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}})$ and that $b_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}) = b_n(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}})$, so that n_{k+1} is not degenerate in $\widehat{\mathcal{N}}_{k+1}$. Hence, we may assume that $n \neq n_{k+1}$.

For notational convenience, let us set

$$a_n = a_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}})$$

$$b_n = b_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}).$$

By the assumption that n is not degenerate in $\widehat{\mathcal{N}}_k$, we have that
$a_n >_{\widehat{\mathcal{N}}_{k+1}} b_n$. If we had

$$a_n(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}) = b_n(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}),$$

then either

$n_{k+1} <_{\mathcal{N}} n$ and $n_{k+1} \bigvee b_n = a_n$

or

$n_{k+1} >_{\mathcal{N}} n$ and $n_{k+1} \bigwedge a_n = b_n$

(where the join and meet are taken in $\widehat{\mathcal{N}}_{k+1}$).

In the former case, this means that $\mathcal{Q}_{k+1} \bigvee \mathcal{Q}_{b_n} \geq_s \mathcal{Q}_{a_n}$, and then by requirement (IV) that $\mathcal{Q}_b \bigvee \mathcal{Q}_{b_n} \geq_s \mathcal{Q}_{a_n}$. However, we also had $n_{k+1} <_{\mathcal{N}} n$, and hence by Proposition 47 (9), in reality $\mathcal{Q}_b \bigvee \mathcal{Q}_{b_n} = \mathcal{Q}_{b_n}$. This implies $\mathcal{Q}_{b_n} \geq_s \mathcal{Q}_{a_n}$, contradicting that $a_n >_{\widehat{\mathcal{N}}_{k+1}} b_n$. A similar contradiction is achieved in the latter case by using requirement (V) and Proposition 47 (8).

□

Since there are no new degenerate elements, it follows that $\widehat{\mathcal{N}}_{k+1}$ has no new degenerate pairs, and hence no degenerate pairs at all, since $\widehat{\mathcal{N}}_k$ had none, by assumption of Lemma 66.

It remains to show that for each $n \in \mathcal{N} - \mathcal{N}_{k+1}$, $C(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}, n)$ holds. If n is degenerate in $\widehat{\mathcal{N}}_{k+1}$, then by our recent Lemma 70, n was already degenerate in $\widehat{\mathcal{N}}_k$. Then by part (1) of $C(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}, n)$, $a_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}) =$ $b_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}) \notin \mathcal{N}_k$. Thus, to show part (1) of $C(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}, n)$, it suffices to show $n_{k+1} \neq_{\widehat{\mathcal{N}}_{k+1}} a_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}) = b_n(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}})$. This follows from Remark 69 and that n_{k+1} was not degenerate for $\widehat{\mathcal{N}}_k$.

If $i \neq n_{k+1}$, then parts (2) and (3) follow from parts (2) and (3) of $C(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}, n)$, along with requirements (IV) and (V). In the case $i = n_{k+1}$, parts (2) and (3) follow from parts (4) and (5) of $C(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}, n)$, along with Remark 69.

Parts (4) and (5) of $C(\widehat{\mathcal{N}}_{k+1}, \mathcal{N}_{k+1}, \leq_{\mathcal{N}}, n)$ follow from parts (4) and (5) of $C(\widehat{\mathcal{N}}_k, \mathcal{N}_k, \leq_{\mathcal{N}}, n)$ and requirements (IV) and (V).

3.8 Shaping the requirements

Here we finish the proof of case 2 of Lemma 66, by constructing the Π^0_1 class $\mathcal{Q}_{k+1}$ meeting requirements (I)-(V), mentioned in Section 3.7.3.

Actually, we construct a Π^0_1 class $\mathcal{Q}$, and then set

$$\mathcal{Q}_{k+1} = (\mathcal{Q} \vee \mathcal{Q}_b) \wedge \mathcal{Q}_a,$$

where $\mathcal{Q}_b$ and $\mathcal{Q}_a$ are as defined in Section 3.7.3.

We will have the following requirements, for each $e \in \omega$.

$$I.N_e \equiv \Phi_e : (\mathcal{Q} \vee \mathcal{Q}_b) \wedge \mathcal{Q}_a \not\to \mathcal{Q}_a.$$

$$I.P_e \equiv \Phi_e : \mathcal{Q}_b \not\to (\mathcal{Q} \vee \mathcal{Q}_b) \wedge \mathcal{Q}_a.$$

$$II.P_{e,i} \equiv \Phi_e : \mathcal{Q}_i \not\to (\mathcal{Q} \vee \mathcal{Q}_b) \wedge \mathcal{Q}_a.$$

where $i \in I_{n_{k+1}}(\mathcal{N}_k, \leq_{\mathcal{N}})$.

$$III.N_{e,i} \equiv \Phi_e : (\mathcal{Q} \vee \mathcal{Q}_b) \wedge \mathcal{Q}_a \not\to \mathcal{Q}_i,$$

where $i \in I_{n_{k+1}}(\mathcal{N}_k, \leq_{\mathcal{N}})$.

$$IV.N_{e,x,y} \equiv \Phi_e : [(\mathcal{Q} \vee \mathcal{Q}_b) \wedge \mathcal{Q}_a] \vee \mathcal{Q}_x \not\to \mathcal{Q}_y,$$

where $x, y \in \widehat{\mathcal{N}}_k$ are such that $\mathcal{Q}_b \vee \mathcal{Q}_x \not\geq_s \mathcal{Q}_y$.

$$V.P_{e,x,y} \equiv \Phi_e : \mathcal{Q}_y \not\to [(\mathcal{Q} \vee \mathcal{Q}_b) \wedge \mathcal{Q}_a] \wedge \mathcal{Q}_x,$$

where $x, y \in \widehat{\mathcal{N}}_k$ are such that $\mathcal{Q}_a \wedge \mathcal{Q}_x \not\leq_s \mathcal{Q}_y$.

Note that, taken together, the requirements of the form $I.N_e$ and $I.P_e$ suffice to satisfy requirement I. For, from the definition that $\mathcal{Q}_{k+1} = (\mathcal{Q} \vee \mathcal{Q}_b) \wedge \mathcal{Q}_a$, it immediately follows that $\mathcal{Q}_a \geq_s \mathcal{Q}_{k+1} \geq_s \mathcal{Q}_b$; then $I.N_e$ and $I.P_e$ take care of the necessary inequalities to give $\mathcal{Q}_a >_s \mathcal{Q}_{k+1} >_s \mathcal{Q}_b$.

It is immediately clear that the satisfaction of all requirements of the form $II.P_{e,i}, III.N_{e,i}, IV.N_{e,x,y}$, and $V.P_{e,x,y}$ guarantees the satisfaction of the original requirements II, III, IV, and V.

Our strategies for satisfying these more specific requirements will be like those in Chapter 2. We give next the two representative strategies and then the full construction.

3.9 Basic strategies

For the basic notation and definitions to be used in the priority argument, refer back to Section 2.3. We add two pieces of notation.

Definition 71. Let $\{\sigma_n\}_{n\in\omega}$ be an effective enumeration of $2^{<\omega}$.

We also need to give names to trees for $\mathcal{Q}_a$ and $\mathcal{Q}_b$.

Definition 72. Let T_a and T_b be computable trees so that $[T_a] = \mathcal{Q}_a$ and $[T_b] = \mathcal{Q}_b$, where $\mathcal{Q}_a$ and $\mathcal{Q}_b$ are as in Section 3.7.1.

Globally we construct disjoint c.e. sets C, D, where $C = \cup_s C_s$, $D = \cup_s D_s$, and define $\mathcal{Q}$ and a tree T as follows:

$$\mathcal{Q} = S(C, D) = \{X \in 2^\omega : (n \in C \Rightarrow X(n) = 1)\&(n \in D \Rightarrow X(n) = 0)\}.$$

$$T = \{\sigma \in 2^{<\omega} : \forall n < |\sigma|[(\sigma(n) = 1 \Rightarrow n \notin D_{|\sigma|})\&(\sigma(n) = 0 \Rightarrow n \notin C_{|\sigma|})]\}.$$

Note $[T] = \mathcal{Q}$.

3.9.1 Strategy for $I.N_e$

To satisfy $\Phi_e : (\mathcal{Q} \vee \mathcal{Q}_b) \wedge \mathcal{Q}_a \not\to \mathcal{Q}_a$ it suffices to satisfy $\Phi_e : \mathcal{Q} \vee \mathcal{Q}_b \not\to \mathcal{Q}_a$. The construction controls $\mathcal{Q}$ but not $\mathcal{Q}_b$ or $\mathcal{Q}_a$. We will use a preservation strategy so that if $\Phi_e : \mathcal{Q} \vee \mathcal{Q}_b \to \mathcal{Q}_a$, then in fact $\mathcal{Q}_b \geq_s \mathcal{Q}_a$, a contradiction to Lemma 68(1). As it seems more true that $\Phi_e : \mathcal{Q} \vee \mathcal{Q}_b \to \mathcal{Q}_a$, according to some length of agreement function, we will work to ensure there is a computable element of $\mathcal{Q} = S(C, D)$, by making C and D computable. Then we would have $\mathcal{Q} \vee \mathcal{Q}_b \equiv_s \mathcal{Q}_b$, and hence $\mathcal{Q}_b \geq_s \mathcal{Q}_a$.

The following length of agreement function for this negative requirement is a re-formulation of one defined by Cenzer and Hinman [5]. (I would like to thank Keng Meng Ng for helping me to see that the stagewise tree approximations used by Cenzer and Hinman and by me in Chapter 2 are not technically necessary.)

Definition 73. Suppose $[T] = S(C, D), [T_a] = \mathcal{Q}_a$, and $[T_b] = \mathcal{Q}_b$, as in Definition 72, and where C, D are disjoint c.e. sets. Then, $\ell_N(s) = max\{y : \forall \sigma \in (T \vee T_b)^s \ [\tau = \Phi^\sigma_{e,s} \upharpoonright y \in T_a \ \& \ \exists \rho \in T^s_a, \rho \supseteq \tau]\}$.

The intuition is that if $\ell_N(s) = y$ then everything long enough in $T \vee T_b$ is mapped to something of length at least y in T_a, and, moreover, none of these images in T_a are currently known to be dead ends in T_a.

Remark 74. $\ell_N(s) = \min\{y : \exists \sigma \in (T \vee T_b)^s \ [\tau = \Phi^\sigma_{e,s} \upharpoonright y + 1 \notin T_a$ or $\forall \rho \in T^s_a, \rho \not\supseteq \tau]\}$. (If $\Phi^\sigma_{e,s} \upharpoonright y + 1$ is undefined, we say it is not in T_a.) This equivalent formulation gives another way to think about $\ell_N(s)$.

Definition 75. $s + 1$ is an *expansionary stage for* $I.N_e$ if $\ell_N(s + 1) > \ell_N(r)$ for all $r \leq s$.

If $s + 1$ is an expansionary stage for $I.N_e$, and $I.N_e$ is the highest priority requirement whose strategy currently requires attention, then we act to restrain T by restraining C and D up to $s + 1$. That is, we make a request that the construction ensure $C \upharpoonright s + 1 = C_s \upharpoonright s + 1$ and $B \upharpoonright s + 1 = B_s \upharpoonright s + 1$. Strategies for lower priority requirements must always respect this request.

If some higher priority strategy at stage $t > s+1$ makes an enumeration so that $C_t \upharpoonright s+1 \neq C_s \upharpoonright s+1$ or $D_t \upharpoonright s+1 \neq D_s \upharpoonright s+1$, we say that $I.N_e$ is *injured* at stage t.

This ends the description of the strategy for satisfying $I.N_e$.

Lemma 76. *Fix nonempty Π^0_1 classes $\mathcal{Q}_b, \mathcal{Q}_a$, an enumeration of disjoint c.e. sets C, D, and set $\mathcal{Q} = S(C, D)$. If $\Phi_e : \mathcal{Q} \vee \mathcal{Q}_b \to \mathcal{Q}_a$ then $lim_{s\to\infty} \ell_N(s) = \infty$, and $I.N_e$ has infinitely many expansionary stages.*

Proof. This is a slightly modified version of Lemma 21(1). Suppose that $\Phi_e : \mathcal{Q} \vee \mathcal{Q}_b \to \mathcal{Q}_a$. We prove that for every $n \in \omega$ there is a stage t such that for all $t' > t$, $\ell_N(t') > n$. Fix n. For each $\sigma \in 2^{<\omega}$, let $\tau_\sigma = \Psi^\sigma_{e,|\sigma|} \upharpoonright n$ if this is defined. If it is not defined, then τ_σ is undefined, and therefore not a substring of any other string. Define

$$Bad = \{\sigma \in T \vee T_b : \neg\exists\rho \in T_a^{|\sigma|}, \rho \supseteq \tau_\sigma\}.$$

Define

$$Good = \{\sigma \in T \vee T_b : \exists\rho \in T_a^{|\sigma|}, \rho \supseteq \tau_\sigma\}.$$

Immediately $Bad \cup Good = (T \vee T_b)$ and $Bad \cap Good = \emptyset$. Also, note that Bad is closed downwards and is therefore a tree.

If Bad were infinite, then by compactness for the standard topology on 2^ω, there would be $X \in [Bad]$. Since $Bad \subseteq T \vee T_b$ and $[T \vee T_b] = \mathcal{Q} \vee \mathcal{Q}_b$, we would have $X \in \mathcal{Q} \vee \mathcal{Q}_b$. By hypothesis $\Phi_e : \mathcal{Q} \vee \mathcal{Q}_b \to \mathcal{Q}_a$, and so there would be $\sigma \subset X$ so that $\Phi^\sigma_{e,t} \upharpoonright n \in \tilde{T}_a$ for some stage t. (Recall $\tilde{T}_a$ is the tree of extendible nodes in T_a.) Letting $s = \max\{|\sigma|, t\}$ we have $\Phi^{X\upharpoonright s}_{e,s} \upharpoonright n \in \tilde{T}_a$, whence $X \upharpoonright s \in Good$, a contradiction.

Since Bad is finite, there is a stage t' so that if $t > t'$ and $\sigma \in (T \vee T_b)^t$, then $\sigma \in Good$. This means that for all $t > t'$, $\ell_N(t') > n$, as desired.

□

Lemma 77. *Suppose $\mathcal{Q}$ is constructed in accord with the strategy given in this subsection for $I.N_e$ and strategies for requirements of higher priority than $I.N_e$ act only finitely often. Then $\Phi_e : \mathcal{Q} \vee \mathcal{Q}_b \not\to \mathcal{Q}_a$.*

Proof. This is a slightly modified version of the proof in Section 2.5. Assume otherwise, that $\Phi_e : \mathcal{Q} \vee \mathcal{Q}_b \to \mathcal{Q}_a$. Then it suffices to show that $\mathcal{Q}$ contains a computable element, as noted at the beginning of this subsection. To show that $\mathcal{Q} = S(C, D)$ contains a computable element, it suffices to show that C and D are computable. Fix a stage s' so that $I.N_e$ is no longer injured after stage s'. (Such a stage exists because $I.N_e$ is only injured by higher priority requirements.) Let $n > s'$. We say how to compute $C(n)$ and $D(n)$. By Lemma 76, there is an expansionary stage $s > n > s'$ so that $\ell_N(s) > n$. The strategy for $I.N_e$ will act at stage s because no strategy for a higher priority requirement acts after stage s'; the strategy for $I.N_e$ will restrain C and D up to s. Since $I.N_e$ is never injured after stage s', $C \upharpoonright s = C_s \upharpoonright s$ and $D \upharpoonright s = D_s \upharpoonright s$. Since $n < s$, $C(n) = C_s(n)$ and $D(n) = D_s(n)$. Note that s' is fixed: it does not depend on n.

□

Lemma 78. *If $I.N_e$ is satisfied, then $I.N_e$ acts only finitely often. Thus, action for $I.N_e$ will not hinder strategies for lower priority requirements more than finitely often.*

Proof. It suffices to show that $I.N_e$ has only finitely many expansionary stages, if $I.N_e$ is satisfied. We will actually show the contrapositive, namely that if

$\limsup_{s\to\infty} \ell_N(s) = \infty$, then $\Phi : \mathcal{Q} \vee \mathcal{Q}_b \to \mathcal{Q}_a$. We simplify the proof we gave for Lemma 21(2).

Suppose $\limsup_{s\to\infty} \ell_N(s) = \infty$. Given $X \in \mathcal{Q} \vee \mathcal{Q}_b$, we want to show that $Y = \Phi_e^X$ exists and $Y \in \mathcal{Q}_a$. It suffices to show that for each n, there is a stage s_n so that $\Phi_{e,s_n}^{X\restriction s_n} \restriction n$ exists and is in T_a. Precisely this is guaranteed by the fact that for each n, there is a stage s_n such that $\ell_N(s_n) \geq n$.

□

3.9.2 Strategy for $I.P_e$

To satisfy $\Phi_e : \mathcal{Q}_b \not\to (\mathcal{Q} \vee \mathcal{Q}_b) \wedge \mathcal{Q}_a$, it suffices to show $\Phi_e : \mathcal{Q}_b \not\to \mathcal{Q} \wedge \mathcal{Q}_a$. Our construction controls $\mathcal{Q}$, but $\mathcal{Q}_b$ and $\mathcal{Q}_a$ are given. We use a strategy based on the Sacks coding strategy, as adapted by Cenzer and Hinman. At expansionary stages for an appropriate length of agreement function, we try to code $\mathcal{Q}_a$ into $\mathcal{Q}$. The result will be that if $\Phi_e : \mathcal{Q}_b \to \mathcal{Q} \wedge \mathcal{Q}_a$, then we will also have $\mathcal{Q} \geq_s \mathcal{Q}_a$, and hence $\mathcal{Q}_b \geq_s \mathcal{Q}_a$, contradicting Lemma 68(1).

We have a length of agreement function in analogy to the one defined in the strategy for $I.N_e$.

Definition 79. Suppose $[T] = S(C, D), [T_a] = \mathcal{Q}_a$, and $[T_b] = \mathcal{Q}_b$, as in Definition 72. Then, let $\ell_P(s) = \max\{y : \forall\sigma \in T_b^s\ [\tau = \Phi_{e,s}^\sigma \restriction y \in (T \wedge T_a)\ \&\ \exists\rho \in (T \wedge T_a)^s, \rho \supseteq \tau]\}$.

Remark 80. $\ell_P(s) = \min\{y : \exists\sigma \in T_b^s\ [\tau = \Phi_{e,s}^\sigma \restriction y+1 \notin (T \wedge T_a)$ or $\forall\rho \in (T \wedge T_a)^s, \rho \not\supseteq \tau]\}$. (If $\Phi_{e,s}^\sigma \restriction y+1$ is undefined, we say it is not in $(T \wedge T_a)$.) This equivalent formulation gives another view of $\ell_P(s)$.

Definition 81. $s+1$ is an *expansionary stage for $I.P_e$* if $\ell_P(s+1) > \ell_P(r)$ for all $r \leq s$.

If $I.P_e$ is expansionary at stage $s+1$ with $\ell_P(s+1) = k$ and is the highest priority requirement that requires attention at stage $s+1$, we set up a means to code into T something about which nodes are extendible in T_a. We use markers. For all $j \le k$ for which $m(\sigma_j)$ has not been defined, we define $m(\sigma_j)$ to be greater than any number yet mentioned in the construction. (Recall σ_j is the j-th finite string.) Then, for each such j, we wait for a stage $t+1 > s+1$ such that no higher priority requirement needs attention at stage $t+1$ and either

(i) there is no $\tau \supseteq \sigma_j{}^\frown 0$ such that $\tau \in T_a^t$ or

(ii) there is no $\tau \supseteq \sigma_j{}^\frown 1$ such that $\tau \in T_a^t$.

If (i) happens at stage $t+1$ and (ii) has not yet happened, or also first happens at stage $t+1$, then we set $m(\sigma_j) \in C_{t+1}$.

If (ii) happens at stage $t+1$ and (i) has not yet happened by stage $t+1$, then we set $m(\sigma_j) \in D_{t+1}$.

The strategy for $I.P_e$ is injured along with the marker $m(\sigma_j)$ at stage v if some higher priority requirement makes a restraint up to $v > s$, where $m(\sigma_j)$ was last defined at stage s, and $m(\sigma_j) \notin C_{v-1} \cup D_{v-1}$. In this case, we do not take any action for (i) or (ii) happening for some $m(\sigma_j)$ at stage v; and before ending stage v, for all injured $m(\sigma_j)$, redefine $m(\sigma_j)$ to be greater than any number yet mentioned in the construction. Then proceed with the construction.

This ends the strategy for satisfying $I.P_e$.

The verification begins as it did for the negative requirement.

Lemma 82. *Fix nonempty Π^0_1 classes $\mathcal{Q}_b, \mathcal{Q}_a$, an enumeration of disjoint c.e. sets C, D, and set $\mathcal{Q} = S(C, D)$. If $\Phi_e : \mathcal{Q}_b \to \mathcal{Q} \wedge \mathcal{Q}_a$, then $\lim_s \ell_P(s) = \infty$, and $I.P_e$ has infinitely many expansionary stages.*

Proof. Analogous to the proof of Lemma 76. ◻

Lemma 83. *Suppose $\mathcal{Q}$ is constructed in accord with the strategy given in this subsection for $I.P_e$ and strategies for requirements of higher priority than $I.P_e$ act only finitely often. Then $\Phi_e : \mathcal{Q}_b \not\to \mathcal{Q} \wedge \mathcal{Q}_a$.*

Proof. This proof is essentially the same as the verification in Section 2.6. Suppose otherwise, that $\Phi_e : \mathcal{Q}_b \to \mathcal{Q} \wedge \mathcal{Q}_a$. By the assumption of this lemma, there is a stage s' after which no strategies for requirements of higher priority than $I.P_e$ act. Let $X \in \mathcal{Q}_b$. We describe, in a uniform way, how to compute $Y \in \mathcal{Q}_a$ from X, contradicting that $\mathcal{Q}_b \not\geq_s \mathcal{Q}_a$.

To begin, $\Phi_e^X \in \mathcal{Q} \wedge \mathcal{Q}_a$. If $\Phi_e^X(0) = 1$, then $Y \in \mathcal{Q}_a$, where $1^\frown Y = \Phi_e^X$.

If $\Phi_e^X(0) = 0$, then $Z \in \mathcal{Q}$, where $0^\frown Z = \Phi_e^X$. We use Z to compute $Y \in \mathcal{Q}_a$. Set $\tau_0 = \emptyset$, the empty string. Inductively assume we have $\tau_n \in \tilde{T}_a^n$, the set of extendible strings in T_a of length n. We uniformly describe how to compute that $\tau_n{}^\frown 0 \in \tilde{T}_a$ or that $\tau_n{}^\frown 1 \in \tilde{T}_a$. First the value of $m(\tau_n)$ at stage s', if any, is determined. If $m(\tau_n)$ is undefined at stage s', there is a stage $s > s'$ at which it is defined, because $\lim_s \ell_P(s) = \infty$ by Lemma 82. Note that once $m(\tau_n)$ is defined at or after stage s', it is ever after defined with the same value, by choice of s'.

Case 1: Suppose $Z(m(\tau_n)) = 1$. Then $m(\tau_n) \notin D$, because $Z \in \mathcal{Q} = S(C, D)$. Thus if (ii) from our strategy for $I.P_e$ happened for τ_n, it happened after (i), or at the same stage as (i). Since τ_n is extendible in T_a, it cannot be that both (i) and (ii) happened. So (ii) did not happen. Hence $\tau_n{}^\frown 1 \in \tilde{T}_a$. Set $\tau_{n+1} = \tau_n{}^\frown 1$.

Case 2: Suppose $Z(m(\tau_n)) = 0$. Then $m(\tau_n) \notin C$. Thus, if (i) from our strategy for $I.P_e$ happened for τ_n, it happened after (ii). Since τ_n is extendible in T_a, it cannot be that both (i) and (ii) happened. Hence (i) did not happen. So

$\tau_n{}^\frown 0 \in \tilde{T}_a$. Set $\tau_{n+1} = \tau_n{}^\frown 0$.

Define $Y = \cup_n \tau_n$. Since $\tau_n \in \tilde{T}_a$ for each n, it follows that $Y \in \mathcal{Q}_a$. So, we have shown $\mathcal{Q}_b \geq_s \mathcal{Q}_a$, the desired contradiction. Hence $\Phi_e : \mathcal{Q}_b \not\to \mathcal{Q} \wedge \mathcal{Q}_a$.

□

Lemma 84. *If $I.P_e$ is satisfied, then the strategy for $I.P_e$ acts only finitely often. Hence enumerations into C and D for the sake of $I.P_e$ will injure strategies for lower priority requirements only finitely often.*

Proof. Analogous to the proof of Lemma 78. □

The other positive and negative strategies work according to the same principles, though they are more involved.

3.10 Full priority construction

The idea for the full construction is simply to generalize the two strategies just described.

Definition 85. Suppose $\mathcal{U}$ and $\mathcal{V}$ are nonempty Π^0_1 classes and U and V are computable trees such that $\mathcal{U} = [U]$ and $\mathcal{V} = [V]$. Let R be a requirement of the form $\Phi_e : \mathcal{U} \not\to \mathcal{V}$. Then, *the length of agreement function associated with R and the trees U and V* is as follows.

$\ell(s) = \max\{y : \forall \sigma \in U^s[\tau = \Phi^\sigma_{e,s} \restriction y \in V \ \&\ \exists \rho \in V^s, \rho \supseteq \tau]\}$.

Remark 86. Where $\ell(s)$ is as in Definition 85, we have

$\ell(s) = \min\{y : \exists \sigma \in U^s\ [\tau = \Phi^\sigma_{e,s} \restriction y + 1 \notin V \text{ or } \forall \rho \in V^s, \rho \not\supseteq \tau]\}$. (If $\Phi^\sigma_{e,s} \restriction y + 1$ is undefined we say it is not in V.) This formulation is another view of $\ell(s)$.

Definition 87. $s+1$ is an expansionary stage for requirement R if $\ell(s+1) > \ell(r)$ for all $r \leq s$, where $\ell(s)$ is the length of agreement function for R.

Lemma 88. *If a requirement R of the form $\Phi : \mathcal{U} \not\to \mathcal{V}$ is not satisfied, then R has infinitely many expansionary stages.*

Proof. Same as the proof of Lemma 76.

□

Lemma 89. *If a requirement R of the form $\Phi : \mathcal{U} \not\to \mathcal{V}$ is satisfied, then R has only finitely many expansionary stages.*

Proof. The proof of this is found in the essence of the proof of Lemma 78. □

Definition 90. Suppose we are constructing a Π^0_1 class $\mathcal{Q}$.

1. A *negative requirement* is a requirement R of the form $\Phi_e : \mathcal{Q} \vee \mathcal{S} \not\to \mathcal{V}$, where $\mathcal{S}$ and $\mathcal{V}$ are fixed Π^0_1 classes such that $\mathcal{S} \not\geq_s \mathcal{V}$.

2. A *positive requirement* is a requirement R of the form $\Phi_e : \mathcal{U} \not\to \mathcal{Q} \wedge \mathcal{S}$, where $\mathcal{U}$ and $\mathcal{S}$ are fixed Π^0_1 classes such that $\mathcal{S} \not\leq_s \mathcal{U}$.

Now, suppose we begin with the setup as in the previous section. We construct disjoint c.e. sets C and D and set $\mathcal{Q} = S(C, D)$. The construction of C and D is in stages; $C = \cup_s C_s$ and $D = \cup_s D_s$. We again have $\mathcal{Q} = [T]$, where
$T = \{\sigma \in 2^{<\omega} : \forall n < |\sigma|[(\sigma(n) = 1 \Rightarrow n \notin D_{|\sigma|}) \;\&\; (\sigma(n) = 0 \Rightarrow n \notin C_{|\sigma|})]\}$.

We suppose we have certain positive and negative requirements in our construction. We put the requirements in some computable ω-ordering.

Strategy for a negative requirement. Suppose R is the negative requirement $\Phi_e : \mathcal{Q} \vee \mathcal{S} \not\to \mathcal{V}$. Let S be a computable tree such that $[S] = \mathcal{S}$ and let V be a computable tree such that $[V] = \mathcal{V}$. We let $\ell(s)$ be the length of agreement function for R as defined by Definition 85, where we take $U = T \vee S$.

If $s+1$ is an expansionary stage for R and it is the highest priority requirement whose strategy currently requires attention, then we act to restrain T by restraining C and D up to $s+1$. That is, we make a request that the construction ensure $C \upharpoonright (s+1) = C_s \upharpoonright (s+1)$ and $D \upharpoonright (s+1) = D_s \upharpoonright (s+1)$. Strategies for lower priority requirements must always respect this request. If some higher priority strategy at stage $t > s+1$ makes an enumeration so that $C_t \upharpoonright (s+1) \neq C_s \upharpoonright (s+1)$ or $D_t \upharpoonright (s+1) \neq D_s \upharpoonright (s+1)$, we say $I.N_e$ is injured at stage t.

This ends the description of the strategy for satisfying a negative requirement.

Verification of the strategy for a negative requirement.

Lemma 91. *If $\mathcal{Q}$ is constructed in accord with the strategy just given for the negative requirement R, and strategies for requirements of higher priority than R act only finitely often, then R is satisfied, and the strategy for R acts only finitely often.*

Proof. Assume contrary to R, that $\Phi_e : \mathcal{Q} \vee \mathcal{S} \to \mathcal{V}$. It suffices to show $\mathcal{Q}$ contains a computable element, because from the definition of the negative requirement R, we have $\mathcal{S} \not\geq_s \mathcal{V}$. To show $\mathcal{Q} = S(C, D)$ contains a computable element, is suffices to show C and D are computable. Fix a stage s' so that no requirement of higher priority than R acts after stage s'. Let $n > s'$. We say how to compute $C(n)$ and $D(n)$. By Lemma 88 there is an expansionary stage $s > n > s'$ so that $\ell(s) > n$. The strategy for R will act at stage s because no strategy for a higher priority requirement acts after stage s'; the strategy for R will restrain C and D up to s at stage s. Since R is never injured after stage s', $C \upharpoonright s = C_s \upharpoonright s$ and $D \upharpoonright s = D_s \upharpoonright s$. Since $n < s$, $C(n) = C_s(n)$ and $D(n) = D_s(n)$. Note that s' is fixed: it does not depend on n.

Since R is satisfied, by Lemma 89 R has only finitely many expansionary stages. Hence, R acts only finitely often. $\square$

Strategy for a positive requirement. Suppose R is the positive requirement $\Phi_e : \mathcal{U} \not\to \mathcal{Q} \wedge \mathcal{S}$. Let U be a tree so that $[U] = \mathcal{U}$ and let S be a tree such that $[S] = \mathcal{S}$. We let $\ell(s)$ be the length of agreement function for R as defined by Definition 85, where we take $V = T \wedge S$.

If R is expansionary at stage $s+1$ with $\ell(s+1) = k$ and is the highest priority requirement that requires attention at stage $s+1$, we set up a means to code into T something about which nodes are extendible in S. We use markers. For all $j \leq k$ for which $m(\sigma_j)$ has not been defined, we define $m(\sigma_j)$ to be greater than any number yet mentioned in the construction. (Recall σ_j is the j-th finite string.) Then, for each such j, we wait for a stage $t+1 > s+1$ such that no higher priority requirement needs attention at stage $t+1$ and either

(i) there is no $\tau \supseteq \sigma_j{}^\frown 0$ such that $\tau \in T_a^t$ or

(ii) there is no $\tau \supseteq \sigma_j{}^\frown 1$ such that $\tau \in T_a^t$.

If (i) happens at stage $t+1$ and (ii) has not yet happened, or also first happens at stage $t+1$, then we set $m(\sigma_j) \in C_{t+1}$.

If (ii) happens at stage $t+1$ and (i) has not yet happened by stage $t+1$, then we set $m(\sigma_j) \in D_{t+1}$.

The strategy for R is injured along with the marker $m(\sigma_j)$ at stage v if some higher priority requirement makes a restraint up to $v > s$, where $m(\sigma_j)$ was last defined at stage s and $m(\sigma_j) \notin C_{v-1} \cup D_{v-1}$. In this case, we do not take any action for (i) or (ii) happening for $m(\sigma_j)$ at stage v; and before ending stage v, for all injured $m(\sigma_j)$, we redefine $m(\sigma_j)$ to be greater than any number yet mentioned in the construction. Then we proceed with the construction.

Verification of the strategy for a positive requirement.

Lemma 92. *If $\mathcal{Q}$ is constructed in accord with the strategy just given for the positive requirement R, and strategies for requirements of higher priority than R act only finitely often, then R is satisfied, and the strategy for R acts only finitely often.*

Proof. Assume, contrary to R, that $\Phi_e : \mathcal{U} \to \mathcal{Q} \wedge \mathcal{S}$. By Lemma 88, R has infinitely many expansionary stages. Let s' be a stage after which no strategy for a requirement of higher priority than R acts. Let $X \in \mathcal{U}$. We describe, in a uniform way, how to compute $Y \in \mathcal{S}$ from X. This contradicts the assumption of the requirement R that $\mathcal{U} \not\geq_s \mathcal{S}$.

By assumption, we have $\Phi_e^X \in \mathcal{Q} \wedge \mathcal{S}$. If $\Phi_e^X(0) = 1$, then $Y \in \mathcal{S}$, where $1^\frown Y = \Phi_e^X$.

If $\Phi_e^X(0) = 0$, then $Z \in \mathcal{Q}$, where $0^\frown Z = \Phi_e^X$. We use Z to compute $Y \in \mathcal{S}$. Set $\tau_0 = \emptyset$, the empty string. Inductively assume we have $\tau_n \in \tilde{S}^n$. We uniformly describe how to compute that $\tau_n{}^\frown 0 \in \tilde{S}$ or that $\tau_n{}^\frown 1 \in \tilde{S}$. First, we determine the value of $m(\tau_n)$ at stage s', if any. If $m(\tau_n)$ is undefined at stage s', there is a stage $s > s'$ at which it is defined, because there are infinitely many expansionary stages, by Lemma 88. Note that once $m(\tau_n)$ is defined at or after stage s', it is ever after defined with the same value, by our choice of s'.

Case 1: Suppose $Z(m(\tau_n)) = 1$. Then $m(\tau_n) \notin D$, because $Z \in \mathcal{Q} = S(C, D)$. Thus if (ii) from our strategy for R happened for τ_n, it happened after (i), or at the same stage as (i). Since τ_n is extendible in S, it cannot be that both (i) and (ii) happened. So (ii) did not happen. Hence $\tau_n{}^\frown 1 \in \tilde{S}$. Set $\tau_{n+1} = \tau_n{}^\frown 1$.

Case 2: Suppose $Z(m(\tau_n)) = 0$. Then $m(\tau_n) \notin C$. Thus if (i) from our strategy for R happened for τ_n, it happened after (ii). Since τ_n is extendible in

S, it cannot be that both (i) and (ii) happened. Hence (i) did not happen. So $\tau_n{}^\frown 0 \in \tilde{S}$. Set $\tau_{n+1} = \tau_n{}^\frown 0$.

Define $Y = \cup_n \tau_n$. Since $\tau_n \in \tilde{S}$ for each n, it follows that $Y \in \mathcal{S}$. So, we have shown $\mathcal{U} \geq_s \mathcal{S}$, as was required. Hence R is satisfied.

By Lemma 89 there are only finitely many expansionary stages for R. Since the strategy defines finitely many markers at each expansionary stage, and acts on a marker at most once, it acts only finitely often.

$\square$

Proposition 93. *In a construction of a Π^0_1 class $\mathcal{Q}$ consisting of a computable list of strategies for positive and negative requirements carried out as just described, each of the corresponding positive and negative requirements is satisfied.*

Proof. By induction on the ordering of priority for the requirements, and Lemmas 91 and 92. $\square$

All that remains is to show that the requirements we need to satisfy in our construction for $\mathcal{Q}$ can be stated as positive and negative requirements in the sense of Definition 90.

Lemma 94. *To satisfy the requirements $I.N_e$, $I.P_e$, $II.P_{e,i}$, $III.N_{e,i}$, and $IV.N_{e,x,y}$ given at the beginning of Section 3.8, it suffices to satisfy the following, slightly simplified, requirements.*

For each $e \in \omega$.

$$I.N'_e \equiv \Phi_e : \mathcal{Q} \vee \mathcal{Q}_b \not\to \mathcal{Q}_a.$$

$$I.P'_e \equiv \Phi_e : \mathcal{Q}_b \not\to \mathcal{Q} \wedge \mathcal{Q}_a.$$

$$II.P'_{e,i} \equiv \Phi_e : \mathcal{Q}_i \not\to \mathcal{Q} \wedge \mathcal{Q}_a.$$

where $i \in I_{n_{k+1}}(\mathcal{N}_k, \leq_{\mathcal{N}})$.

$$III.N'_{e,i} \equiv \Phi_e : \mathcal{Q} \vee \mathcal{Q}_b \not\to \mathcal{Q}_i,$$

where $i \in I_{n_{k+1}}(\mathcal{N}_k, \leq_{\mathcal{N}})$.

$$IV.N'_{e,x,y} \equiv \Phi_e : \mathcal{Q} \vee \mathcal{Q}_b \vee \mathcal{Q}_x \not\to \mathcal{Q}_y,$$

where $x, y \in \widehat{\mathcal{N}}_k$ *are such that* $\mathcal{Q}_b \vee \mathcal{Q}_x \not\geq_s \mathcal{Q}_y$.

$$V.P'_{e,x,y} \equiv \Phi_e : \mathcal{Q}_y \not\to \mathcal{Q} \wedge \mathcal{Q}_a \wedge \mathcal{Q}_x,$$

where $x, y \in \widehat{\mathcal{N}}_k$ *are such that* $\mathcal{Q}_a \wedge \mathcal{Q}_x \not\leq_s \mathcal{Q}_y$.

Proof. Notice that when trying to show $\mathcal{U} \not\geq_s \mathcal{V}$, it suffices to prove the inequality with $\mathcal{U}'$ substituted for $\mathcal{U}$ if $\mathcal{U}' \geq_s \mathcal{U}$, or with $\mathcal{V}'$ substituted for $\mathcal{V}$ if $\mathcal{V}' \leq_s \mathcal{V}$.

□

Proposition 95. *There is a* Π^0_1 *class* $\mathcal{Q}$ *that meets the requirements described at the beginning of Section 3.8.*

Proof. Apply Lemma 94. We shall show how the requirements given by Lemma 94 are in the form of positive and negative requirements in the sense of Definition 90. To show this, for each requirement we describe what to take for $\mathcal{S}$ and $\mathcal{U}$ or $\mathcal{V}$. For each of the first four requirements, we also make use of one of the three inequalities from Lemma 68: $\mathcal{Q}_b \not\geq_s \mathcal{Q}_a, \mathcal{Q}_i \not\geq_s \mathcal{Q}_a$, and $\mathcal{Q}_b \not\geq_s \mathcal{Q}_i$, for all $i \in I_{n_{k+1}}(\mathcal{N}_k, \leq_{\mathcal{N}})$.

For $I.N'_e$ take $\mathcal{S} = \mathcal{Q}_b$ and $\mathcal{V} = \mathcal{Q}_a$. Note $\mathcal{Q}_b \ngeq_s \mathcal{Q}_a$.

For $I.P'_e$ take $\mathcal{U} = \mathcal{Q}_b$ and $\mathcal{S} = \mathcal{Q}_a$. Note $\mathcal{Q}_b \ngeq_s \mathcal{Q}_a$.

For $II.P'_{e,i}$ take $\mathcal{U} = \mathcal{Q}_i$ and $\mathcal{S} = \mathcal{Q}_a$. Note $\mathcal{Q}_i \ngeq_s \mathcal{Q}_a$.

For $III.N'_{e,i}$ take $\mathcal{S} = \mathcal{Q}_b$ and $\mathcal{V} = \mathcal{Q}_i$. Note $\mathcal{Q}_b \ngeq_s \mathcal{Q}_i$.

For $IV.N'_{e,x,y}$, where $x, y \in \widehat{\mathcal{N}}_k$ are such that $\mathcal{Q}_b \vee \mathcal{Q}_x \ngeq_s \mathcal{Q}_y$, take $\mathcal{S} = \mathcal{Q}_b \vee \mathcal{Q}_x$ and $\mathcal{V} = \mathcal{Q}_y$. The condition on x, y is exactly what we need for this to be a negative requirement.

For $V.P'_{e,x,y}$, where $x, y \in \widehat{\mathcal{N}}_k$ are such that $\mathcal{Q}_a \wedge \mathcal{Q}_x \nleq_s \mathcal{Q}_y$, take $\mathcal{U} = \mathcal{Q}_y$ and $\mathcal{S} = \mathcal{Q}_a \wedge \mathcal{Q}_x$. The condition on x, y is exactly what we need for this to be a positive requirement.

An application of Proposition 93 completes the proof.

$\square$

Proposition 95 completes the proof of Lemma 66, which completes the proof of the central result of this chapter, Theorem 58.

CHAPTER 4

OTHER VIEWS OF THE $\forall\exists$-THEORY OF $\mathcal{P}_s$.

4.1 A model-theoretic understanding

Another way to way to state the solution to the extension of embeddings problem for $\mathcal{P}_s$ in the language $L = \{0, 1, \leq\}$ is given in the following theorem. It will be helpful to recall our conventions stated in Section 3.1.1. We only need to consider L-structures that are actually partial orders with greatest and least elements; $\widehat{L}$ structures are in the language $\{0, 1, \leq, \wedge, \vee\}$ and are actually distributive lattices in which $\vee$ and $\wedge$ are defined by $\leq$.

Theorem 96. *Let $\mathcal{M} \subseteq \mathcal{N}$ be L-structures. Every embedding of $\mathcal{M}$ into $\mathcal{P}_s$ extends to an embedding of $\mathcal{N}$ into $\mathcal{P}_s$*

iff

For every $\widehat{L}$-structure $\widehat{\mathcal{M}}$ that minimally extends $\mathcal{M}$ and in which $\mathbf{0}$ is non-branching, there is an $\widehat{L}$-structure $\widehat{\mathcal{N}}$ that extends $\widehat{\mathcal{M}}$ and into which $\mathcal{N}$ embeds.

For $\mathcal{P}_s$ is as accommodating as could possibly be hoped. To begin with, any L-structure does embed into $\mathcal{P}_s$, and in every conceivable $\widehat{L}$-way (so long as $\mathbf{0}$ is non-branching); then, an extension is impossible, only if it is algebraically impossible.

Definition 97. A *positive* $\exists$*-sentence* is one which is in prenex normal form and contains only existential quantifiers. Note that a positive $\exists$-sentence cannot begin with a negation symbol before the string of existential quantifiers. It may, of course, use the negation symbol after the block of existential quantifiers. An $\exists$*-sentence* is either a positive $\exists$-sentence, or the negation of a positive $\exists$-sentence.

Definition 98. Similarly a *positive* $\forall\exists$*-sentence* is one which is in prenex normal form and has a block of universal quantifiers followed by a block of existential quantifiers.

Example 99. If $\varphi(x, y)$ is a quantifier-free formula, then $\forall x \exists y \varphi(x, y)$ is a positive $\forall\exists$-sentence. On the other hand, $\neg\forall x \exists y \varphi(x, y)$ is not.

Recall $L = \{0, 1, \leq\}$ and $\widehat{L} = \{0, 1, \leq, \wedge, \vee\}$. We define a collection of $\widehat{L}$ sentences: $T = \{\varphi : \mathcal{P}_s \models \varphi \ \&\ \varphi \text{ is an } \exists\text{-sentence}\}$.

Proposition 100. *Suppose ψ is a positive $\forall\exists$-sentence in the language L. Then $\mathcal{P}_s \models \psi \iff T$ does not refute ψ.*

Proof. ($\Rightarrow$) Suppose $\mathcal{P}_s \models \psi$. Since $\mathcal{P}_s \models T$, T does not refute ψ.

($\Leftarrow$) We show the contrapositive. Suppose $\mathcal{P}_s \not\models \psi$. We must show T refutes ψ. Using completeness, we prove that if $\widehat{\mathcal{P}}$ is an $\widehat{L}$ structure and $\widehat{\mathcal{P}} \models T$, then $\widehat{\mathcal{P}} \models \neg\psi$.

By the proof of Proposition 44, there are finitely many instances of the multi-extension of embeddings problem in the language L, $(\mathcal{M}_0, \mathcal{N}_{00}, \ldots, \mathcal{N}_{0k_0}), \ldots, (\mathcal{M}_t, \mathcal{Q}_{t0}, \ldots, \mathcal{Q}_{tk_t})$, so that an L-structure $\mathcal{P}$ satisfies ψ just in case each of these instances of the multi-extension of embeddings problem has an affirmative answer for $\mathcal{P}$.

Fix an $\widehat{L}$-structure $\widehat{\mathcal{P}} \models T$; of course, it is also an L-structure.

By supposition, $\mathcal{P}_s \not\models \psi$. Thus, $\mathcal{P}_s$ has a negative answer for one of the above-mentioned instances of the multi-extension of embeddings problem, say $(\mathcal{M}_0, \mathcal{N}_{00}, \ldots, \mathcal{N}_{0k_0})$. Thus, there is an embedding $f : \mathcal{M}_0 \hookrightarrow \widehat{\mathcal{M}_0} \hookrightarrow \mathcal{P}_s$ that does not extend to an embedding into $\mathcal{P}_s$ of any $\mathcal{N}_{00}, \ldots, \mathcal{N}_{0k_0}$. (Where $\widehat{\mathcal{M}_0}$ is an $\widehat{L}$-structure minimally extending $\mathcal{M}_0$ and isomorphic to the closure of the image of $\mathcal{M}_0$ in $\mathcal{P}_s$, as described in Definition 37 and Proposition 39.)

Now, we apply the proof of Theorem 60 to obtain the following condition, which we label $(*)$. For every $0 \leq i \leq k_0$ either

1. There are $n_1, n_2 \in \mathcal{N}_{0i}$ so that (n_1, n_2) is a degenerate pair for $\widehat{\mathcal{M}_0}$ or
2. There is $n \in \mathcal{N}_{0i} - \mathcal{M}_0$ so that $C(\widehat{\mathcal{M}_0}, \mathcal{M}_0, \leq_{\mathcal{N}_{0i}}, n)$ does not hold, where C is as in Definition 55.

Let $\mathcal{M}_0 = \{m_0, \ldots, m_r\}$. Let $\theta(x_0, \ldots, x_r)$ be an $\widehat{L}$-formula so that $\theta(f(m_0), \ldots, f(m_r))$ is the "complete atomic $\widehat{L}$ diagram of $f(m_0), \ldots, f(m_r)$ in $\mathcal{P}_s$." The quotation marks are necessary, because, in reality, the existence of the function symbols $\wedge$ and $\vee$ in $\widehat{L}$ means the complete atomic diagram is infinite. However, the distributive laws for $\mathcal{P}_s$ ensure the existence of a normal form for each element named by a term. Therefore, we may take θ to be a conjunction of atomic formulas of the form $\bigwedge_n \bigvee_{i \in J_n} x_i \leq \bigwedge_{n'} \bigvee_{i \in J_{n'}} x_i$ where each $J_n, J_{n'} \subseteq \{0, \ldots, r\}$ and $\bigwedge_n \bigvee_{i \in J_n} f(m_i) \leq \bigwedge_{n'} \bigvee_{i \in J_{n'}} f(m_i)$ holds in $\mathcal{P}_s$, or the negation of such a formula, such that the negation of the corresponding statement holds in $\mathcal{P}_s$.

Let φ be the $\exists$-sentence $\exists x_0 \ldots \exists x_r(\theta(x_0, \ldots, x_r))$. $\mathcal{P}_s \models \varphi$, so $\varphi \in T$, and so $\widehat{\mathcal{P}} \models \varphi$.

Note that $\{$"$x_a \leq x_b$": $m_a \leq m_b\} \cup \{$"$x_a \not\leq x_b$": $m_a \not\leq m_b\}$ is a subset of the atomic sentences that comprise θ and expresses that $x_0, \ldots, x_r$ are able to be the image of an L-embedding of $\mathcal{M}_0$. So let $g : \mathcal{M}_0 \hookrightarrow \widehat{\mathcal{M}}_0^g \hookrightarrow \widehat{\mathcal{P}}$. (Again, $\widehat{\mathcal{M}}_0^g$ is as in Definition 37 and Proposition 39.) Since θ gives a complete atomic $\widehat{L}$ diagram, we must be able to construct g so that $\widehat{\mathcal{M}}_0^g$ is isomorphic to $\widehat{\mathcal{M}}_0$. Hence, we may write $g : \mathcal{M}_0 \hookrightarrow \widehat{\mathcal{M}}_0 \hookrightarrow \widehat{\mathcal{P}}$.

Now, since $(*)$ depends only on $\mathcal{M}_0$ and $\widehat{\mathcal{M}}_0$, $(*)$ still holds relative to the embedding g. Hence, we may conclude from Theorem 60 that $g : \mathcal{M}_0 \hookrightarrow \widehat{\mathcal{P}}$ does not extend to any embedding of any $\mathcal{N}_{00}, \ldots, \mathcal{N}_{0k_0}$ into $\widehat{\mathcal{P}}$. Then, $\widehat{\mathcal{P}}$ has a negative answer for the instance $(\mathcal{M}_0, \mathcal{N}_{00}, \ldots, \mathcal{N}_{0k})$ of the extension of embeddings problem. Hence, $\widehat{\mathcal{P}} \not\models \psi$, as desired.

$\square$

CHAPTER 5

SOME RESULTS IN $\mathcal{P}_w$.

We do not know the Turing degree of the elementary theory of $\mathcal{P}_w$. In this chapter, we say something about index sets related to it, and a little that is known about the question of density. This first result suggests that the degree of the theory may actually achieve the natural upper bound.

To help with this section, a review of the concept of a homeomorphism in the context of descriptive set theory can be found in [12]. A review of Kleene's $\mathcal{O}$ and of the computation of the complexity of sentences with function quantifiers may be found in [2].

5.1 Index sets related to $\mathcal{P}_w$.

The material in this Section is the result of joint work with Stephen G. Simpson [7].

Definition 101. Let $\Phi_e(\sigma) = \Phi_e^\sigma \upharpoonright n$, where n is the greatest number such that $\Phi_{e,|\sigma|}^\sigma(m)$ is defined for all $m < n$.

Definition 102. A *treemap* is a function $TM : \omega^{<\omega} \to \omega^{<\omega}$ such that $F(\sigma)^\frown i \subseteq F(\sigma^\frown i)$ for all $\sigma \in \omega^{<\omega}$ and all $i \in \omega$.

Definition 103. If $S \subseteq \omega^{<\omega}$ but is not necessarily a tree, let $[S] = [\{\sigma : \exists \tau \in S, \sigma \subseteq \tau\}]$.

Remark 104. Let TM be a treemap, and let $T \subseteq \omega^{<\omega}$ be a tree. Then, $h \in [TM(T)] \iff \exists f \in [T]$ so that $\bigcup_n TM(f \restriction n) = h$.

Definition 105. For $\tau, \sigma \in \omega^{<\omega}$, we say τ is *a substring of* σ if there are strings $\sigma_0, \ldots, \sigma_{|\tau|}$, each allowed to be the emptystring, so that $\sigma = \tau(0)^\frown\sigma_0{}^\frown\tau(1)^\frown\sigma_1{}^\frown\tau(2)^\frown\sigma_2{}^\frown \ldots {}^\frown\tau(|\tau|)^\frown\sigma_{|\tau|}$.

Remark 106. Suppose TM is a treemap. If τ is minimal so that $\sigma \subseteq TM(\tau)$, then τ is a substring of σ.

Proposition 107. *Given Π^0_1 sets $\mathcal{P}, \mathcal{Q} \subseteq \omega^\omega$ such that $\mathcal{Q}$ does not contain a computable function, we can effectively find a Π^0_1 set $H(\mathcal{P}, \mathcal{Q}) \subseteq \omega^\omega$ that is homeomorphic to $\mathcal{P}$ and such that there are not $g \in \mathcal{Q}$ and $h \in H(\mathcal{P}, \mathcal{Q})$ with $g \leq_T h$.*

Proof. We begin by describing $H_{\mathcal{Q}}(\mathcal{P}) \subseteq \omega^\omega$ homeomorphic to $\mathcal{P}$ and missing the cone of Turing degrees above $\mathcal{Q}$. To begin with $H_{\mathcal{Q}}(\mathcal{P})$ will be only Π^0_2, not Π^0_1, but we will be able to convert it to the desired Π^0_1 class $H(\mathcal{P}, \mathcal{Q})$ by a standard trick.

Let $T_{\mathcal{P}} \subseteq \omega^{<\omega}$ be a computable tree such that $[T_{\mathcal{P}}] = \mathcal{P}$. Let $T_{\mathcal{Q}} \subseteq \omega^{<\omega}$ be a computable tree such that $[T_{\mathcal{Q}}] = \mathcal{Q}$. We shall actually construct a tree map $TM : \omega^{<\omega} \to \omega^{<\omega}$ and then take $H_{\mathcal{Q}}(\mathcal{P}) = [TM(T_{\mathcal{P}})]$. Note that by Remark 104, we will have $h \in H_{\mathcal{Q}}(\mathcal{P}) \iff \exists f \in \mathcal{P}$ so that $\bigcup_n TM(f \restriction n) = h$.

We define TM by induction. Let $TM(\emptyset) = \emptyset$. To define $TM(\sigma^\frown i)$, let $e = |\sigma|$ and $\tau_0 = TM(\sigma)^\frown i$. Given τ_n, let τ_{n+1} be the least τ such that $\tau_n \subset \tau$ and $\Phi_e(\tau_n) \subset \Phi_e(\tau) \in T_{\mathcal{Q}}$. If no such τ exists, let τ_{n+1} be undefined. This must happen for some n; otherwise, we would have $\Phi_e(\tau_n) \subset \Phi_e(\tau_{n+1}) \in T_{\mathcal{Q}}$ for all n, and $\bigcup_n \Phi_e(\tau_n)$ would be a computable path in $\mathcal{Q}$, contrary to assumption. So, we may set $TM(\sigma^\frown i) = \tau_n$ for the least n such that τ_{n+1} is undefined.

This construction is similar to the one in the proof of Theorem 2.1 in [4]. This, in turn, is similar to the argument in the proof of Theorem 4.1 in [11].

Lemma 108. *There are not $f \in \omega^\omega$ and $g \in \mathcal{Q}$ so that $\bigcup_n TM(f \restriction n) \geq_T g$.*

Proof. Suppose $f' = \bigcup_n TM(f \restriction n) \geq_T g$. Let e be such that $\Phi_e^{f'} = g$. Consider $\sigma^\frown i = f \restriction (e+1)$, and the definition of $TM(\sigma^\frown i) \subset f'$. There is no $\tau \supset TM(\sigma^\frown i)$ so that $\Phi_e(\tau) \supset \Phi_e(TM(\sigma^\frown i))$ and $\Phi_e(\tau) \in T_{\mathcal{Q}}$. Hence, $\Phi_e^{f'} \notin \mathcal{Q}$, contrary to the assumption. □

Lemma 109. *$H_{\mathcal{Q}}(\mathcal{P}) = [TM(T_{\mathcal{P}})]$ is a Π^0_2 subset of ω^ω, and a Π^0_2 index may be effectively computed from a Π^0_1 index for $\mathcal{P}$.*

Proof. From Remark 106, $H_{\mathcal{Q}}(\mathcal{P}) = [TM(T_{\mathcal{P}})] = \{f : \forall n \exists \tau (\tau \in T_{\mathcal{P}} \ \&\ \tau$ is a substring of $f \restriction n \ \&\ f \restriction n \subseteq TM(\tau))\}$. Here $\exists \tau$ is in effect a bounded quantifier, because, given n, there are only finitely many substrings of $f \restriction n$. From its definition, it can be seen that TM is computable from $0'$. Hence $H_{\mathcal{Q}}(\mathcal{P})$ is $\Pi^{0,0'}_1$. We also note that this index is effectively calculated from a Π^0_1 index for $\mathcal{P}$. Finally, we recall that a $\Pi^{0,0'}_1$ set is a Π^0_2 set and the Π^0_2 index can be effectively calculated from the $\Pi^{0,0'}_1$ index.

□

The following Lemma effectively converts this Π^0_2 class into a satisfactory Π^0_1 class. The technique is “Skolemization.”

Lemma 110. *Let $\mathcal{U}$ be a Π^0_2 subset of ω^ω. From a Π^0_2 index we can effectively obtain a Π^0_1 index for $\mathcal{V} \subseteq \omega^\omega$ that is homeomorphic to $\mathcal{U}$, and with the same Turing degree spectrum as $\mathcal{U}$.*

Proof. Say $\mathcal{U} = \{f : \forall x \exists y R(f, x, y)\}$ where R is a computable relation. Consider $\mathcal{V} = \{f \oplus k : \forall x R(f, x, k(x))$ & for each x, $k(x)$ is the least y such that $R(f, x, y)\}$. Note that for each $f \in \mathcal{U}$ and its corresponding k, $f \equiv_T f \oplus k \in \mathcal{V}$. Also note that $\mathcal{V}$ is homeomorphic to $\mathcal{U}$ (see [12]) and is Π^0_1.

To see $\mathcal{V}$ is Π^0_1: let Φ_e be a computable functional such that $\Phi^f_e(x, y) = 1$ if $R(f, x, y)$ and $\Phi^f_e(x, y) = 0$ if $\neg R(f, x, y)$. We define a computable tree $T_{\mathcal{V}}$ and claim $\mathcal{V} = [T_{\mathcal{V}}]$. For $\sigma \in \omega^{<\omega}$, define $f_\sigma, k_\sigma \in \omega^{<\omega}$ such that $\sigma = f_\sigma \oplus k_\sigma$. If $\exists x \leq |\sigma|$ such that $\Phi^{f_\sigma}_{e,|\sigma|}(x, k_\sigma(x)) = 0$, we set $\sigma \notin T_{\mathcal{V}}$. If $\exists x \leq |\sigma|$ and $\exists y < k_\sigma(x)$ such $\Phi^{f_\sigma}_{e,|\sigma|}(x, y) = 1$, we also set $\sigma \notin T_{\mathcal{V}}$. Otherwise, we set $\sigma \in T_{\mathcal{V}}$.

□

We may apply Lemma 110 to $\mathcal{U} = H_{\mathcal{Q}}(\mathcal{P})$ to obtain $H(P, Q) = \mathcal{V}$. By this lemma and Lemma 109, we gain effectivity and the homeomorphism with $\mathcal{P}$ as sought by Proposition 107. Lemma 108 and Lemma 110 ensure that no element of $H(\mathcal{P}, \mathcal{Q})$ computes an element of $\mathcal{Q}$. This completes the proof of Proposition 107. □

Lemma 111. *Let $\mathcal{U} \subseteq \omega^\omega$ be Σ^0_3. Let $\boldsymbol{u}$ be the weak degree of $\mathcal{U}$. Then $inf(\boldsymbol{u}, \boldsymbol{1})$ belongs to $\mathcal{P}_w$. Moreover, a Π^0_1 index for a representative of $inf(\boldsymbol{u}, \boldsymbol{1})$ can be effectively computed from a Σ^0_3 index for $\mathcal{U}$.*

Proof. This lemma is due to Simpson [21]. □

Theorem 112. *Let $\{\mathcal{Q}_i\}_{i \in \omega}$ be an effective enumeration of the nonempty Π^0_1 subsets of 2^ω. For each i let $\boldsymbol{q}_i$ be the weak degree of $\mathcal{Q}_i$. If j is such that $\boldsymbol{p}_j > \boldsymbol{0}$, then the index sets $\{i : \boldsymbol{p}_i = \boldsymbol{p}_j\}$ and $\{i : \boldsymbol{p}_i \geq \boldsymbol{p}_j\}$ are Π^1_1 complete sets of integers.*

Proof. Let $\{\mathcal{S}_e\}_{e \in \omega}$ be an effective enumeration of all Π^0_1 subsets of ω^ω. By a result of Kleene and Spector (see [2, Theorem 5.14]), the index set $\{e : \mathcal{S}_e = \emptyset\}$

is a Π^1_1 complete set of integers. We shall reduce this Π^1_1 complete set to each of the index sets in question. Fix an index j such that $\mathbf{q}_j > \mathbf{0}$, which implies $\mathcal{Q}_j$ has no computable member. Given an index e, by Proposition 107 we can effectively find an index $h(e,j)$ such that $\mathcal{S}_{h(e,j)} = H(\mathcal{S}_e, \mathcal{Q}_j)$. Also, Lemma 111 tells us that we can effectively find an index $f(e,j)$ such that $\mathcal{Q}_{f(e,j)}$ is weakly equivalent to $\mathcal{S}_e \cup \mathcal{Q}_j$. Combining these two results: given an index e, we can effectively find an index $i = f(h(e,j), j)$ such that $\mathcal{Q}_i \equiv_w H(\mathcal{S}_e, \mathcal{Q}_j) \cup \mathcal{Q}_j$.

We can see the reduction now. If $\mathcal{S}_e = \emptyset$, then $H(\mathcal{S}_e, \mathcal{Q}_j) = \emptyset$, hence $\mathcal{Q}_i$ is weakly equivalent to $\mathcal{Q}_j$. On the other hand, if $\mathcal{S}_e \neq \emptyset$, then $H(\mathcal{S}_e, \mathcal{Q}_j) \neq \emptyset$ and for all $h \in H(\mathcal{S}_e, \mathcal{Q}_j)$, there is no $g \in \mathcal{Q}_j$ such that $g \leq_T h$, hence $\mathcal{Q}_j$ is not weakly reducible to $\mathcal{Q}_i$. Thus we see that the Π^1_1 complete set $\{e : \mathcal{S}_e = \emptyset\}$ is reducible to both $\{i : \mathbf{q}_i = \mathbf{q}_j\}$ and $\{i : \mathbf{q}_i \geq \mathbf{q}_j\}$ via the reduction $e \mapsto f(h(e,j), j)$.

$\square$

Definition 113. Let $\mathcal{O}$ be Kleene's $\mathcal{O}$, a Π^1_1 complete set of integers.

Proposition 114. *An upper bound for the Turing degree of the elementary theory of* $(\mathcal{P}_w, \boldsymbol{0}, \boldsymbol{1}, \leq_w)$ *is the degree of* $\mathcal{O}^{(\omega)}$*, the* ω*-th jump of* $\mathcal{O}$.

Proof. This proof does not depend on Theorem 112. We note that we think of an integer n as representing $\mathcal{Q}_n$, the n-th nonempty Π^0_1 class in our effective listing of them. We further suppose that $\mathcal{Q}_n = [T_n]$, where $\{T_n\}_{n\in\omega}$ is an effective listing of the infinite computable binary-branching trees.

If $f \in 2^\omega$, then $f \in \mathcal{Q}_m \iff \forall n (f \upharpoonright n \in T_m)$. This gives a Π^0_1 predicate on a function and a number.

$\mathcal{Q}_m \leq_w \mathcal{Q}_n \iff \forall f (f \in \mathcal{Q}_n \Rightarrow \exists e (\Phi^f_e \in \mathcal{Q}_m))$. This gives a Π^1_1 predicate on pairs of numbers.

$\mathcal{Q}_m \equiv_w \mathcal{Q}_n \iff \mathcal{Q}_m \leq_w \mathcal{Q}_n$ & $\mathcal{Q}_n \leq_w \mathcal{Q}_m$. This gives a Π^1_1 predicate on pairs of numbers.

$\mathcal{Q}_m \equiv_w \mathbf{0} \iff \forall n(\mathcal{Q}_n \geq_w \mathcal{Q}_m)$. This gives a Π^1_1 predicate on numbers.

$\mathcal{Q}_m \equiv_w \mathbf{1} \iff \forall n(\mathcal{Q}_n \leq_w \mathcal{Q}_m)$. This gives a Π^1_1 predicate on numbers.

Thus all atomic sentences about elements of $\mathcal{P}_w$ are decidable by a Π^1_1 complete oracle. It then immediately follows that $\deg_T(\mathcal{O}^{(\omega)})$ is an upper bound for the Turing degree of the elementary theory of $(\mathcal{P}_w, \mathbf{0}, \mathbf{1}, \leq_w)$.

□

From Theorem 112 and Proposition 114 one can conjecture that the Turing degree of the elementary theory of $\mathcal{P}_w$ in the language $\{0, 1, \leq\}$ is $\mathcal{O}^{(\omega)}$, the ω-th jump of Kleene's $\mathcal{O}$. This is reasonable especially in light of similar results, for example in [17].

5.2 A note on the downward density of $\mathcal{P}_w$.

We have discussed the proof that $\mathcal{P}_s$ is dense; and that its techniques seem insufficient for showing the density of $\mathcal{P}_w$. Yet, there are a few partial results in the area. We mention one. We have seen Binns' [3] result that every non-zero weak degree splits. This shows that $\mathcal{P}_w$ is downward dense, at least: there is a degree between any degree and $\mathbf{0}$. It is interesting to note that a proof of the downward density of $\mathcal{P}_w$ is available through classical theorems about c.e. Turing degrees and Π^0_1 classes.

Proposition 115. *$\mathcal{P}_w$ is downward dense.*

Proof. The proof is based on the following two results from a paper by Jockusch and Soare [10].

Theorem 116. *If $\mathcal{P}$ is a Π^0_1 class with no computable member, then there exists a nonzero c.e. Turing degree $\boldsymbol{a}$ such that $\mathcal{P}$ has no member of degree $\leq \boldsymbol{a}$ [10, Theorem 2].*

Theorem 117. *If $\boldsymbol{a}$ is a Turing degree and $0 < \boldsymbol{a} \leq 0'$, then $\boldsymbol{a}$ is the degree of a member of some Π^0_1 class with no computable member [10, Corollary 1.1].*

Given a Π^0_1 class $\mathcal{Q}$ with no computable member, by Theorem 116 let $\mathbf{a}$ be a nonzero c.e. Turing degree such that $\mathcal{Q}$ has no member of degree $\leq \mathbf{a}$. By Theorem 117 there is a Π^0_1 class $\mathcal{U}$ with an element of degree $\mathbf{a}$ and no computable member. Consider $\mathcal{Q} \cup \mathcal{U}$. Note that $0 <_w \mathcal{Q} \cup \mathcal{U}$ because neither $\mathcal{Q}$ nor $\mathcal{U}$ contains a computable member. Also, $\mathcal{Q} \cup \mathcal{U} \leq_w \mathcal{Q}$ because $\mathcal{Q} \cup \mathcal{U} \supseteq \mathcal{Q}$. Now, there is $f \in \mathcal{Q} \cup \mathcal{U}$ of degree $\mathbf{a}$. By choice of $\mathbf{a}$, anything computed by f is not in $\mathcal{Q}$. Hence $\mathcal{Q} \cup \mathcal{U} \not\geq_w \mathcal{Q}$. Thus $0 <_w \mathcal{Q} \cup \mathcal{U} <_w \mathcal{Q}$, as desired.

$\square$

BIBLIOGRAPHY

[1] Christopher P. Alfeld. Classifying the branching degrees in the Medvedev lattice of Π^0_1 classes. *Notre Dame Journal of Formal Logic*, 49(3):227–243, 2008.

[2] C.J. Ash and Julia F. Knight. *Computable Structures and the Hyperarithmetical Hierarchy*. Studies in Logic and the Foundations of Mathematics. Elsevier, 2000.

[3] Stephen E. Binns. A splitting theorem for the Medvedev and Muchnik lattices. *Mathematical Logic Quarterly*, 49:327–335, 2003.

[4] Stephen E. Binns and Stephen G. Simpson. Embededings into the Medvedev and Muchnik lattices of Π^0_1 classes. *Archive for Mathematical Logic*, 43:399–414, 2004.

[5] Douglas Cenzer and Peter G. Hinman. Density of the Medvedev lattice of Π^0_1 classes. *Archive for Mathematical Logic*, 42:583–600, 2003.

[6] Joshua A. Cole. Embedding $FD(\omega)$ densely in $\mathcal{P}_s$. *Archive for Mathematical Logic*, 46(7-8):649–664, 2007.

[7] Joshua A. Cole and Stephen G. Simpson. Mass problems and hyperarithmeticity. *Journal of Mathematical Logic*, 7(2):125–153, 2008.

[8] J. Dekker, editor. *Proceedings of the Symposium on Pure Mathematics*, volume 5. American Mathematical Society, Providence, Rhode Island, 1962.

[9] George Grätzer. *General Lattice Theory*. Birkhäuser-Verlag, second edition, 1978.

[10] Carl G. Jockusch Jr. and Robert I. Soare. Degrees of members of Π^0_1 classes. *Pacific Journal of Mathematics*, 40(3):605–616, 1972.

[11] Carl G. Jockusch Jr. and Robert I. Soare. Π^0_1 classes and degrees of theories. *Transactions of the American Mathematical Society*, 173:33–56, 1972.

[12] Alexander S. Kechris. *Descriptive Set Theory*. Graduate Texts in Mathematics. Springer, 1995.

[13] Manuel Lerman. *Degrees of Unsolvability*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, Heidelberg, 1980.

[14] David Marker. *Model Theory: An Introduction*. Springer, New York, 2002.

[15] Antonio Montalbán. Extensions of embeddings in the Δ^0_2 Turing degrees. Slides from the "Computability, Complexity, and Randomness" conference in Nanjing, China. Accessed 8 Jan 2009 from http://ims.nju.edu.cn/conference/randomness/slides.html, May 2008.

[16] Dana S. Scott. Algebras of sets binumerable in complete extensions of arithmetic. Appears in [8], 1962.

[17] Stephen G. Simpson. First order theory of the degrees of recursive unsolvability. *Annals of Mathematics*, 105:121–139, 1977.

[18] Stephen G. Simpson. Π^0_1 sets and models of WKL_0. Appears in [20], 2000.

[19] Stephen G. Simpson. Mass problems and randomness. *Bulletin of Symbolic Logic*, 11:1–27, 2005.

[20] Stephen G. Simpson, editor. *Reverse Mathematics 2001*, volume 21 of *Lecture Notes in Logic*. Association for Symbolic Logic, 2005.

[21] Stephen G. Simpson. An extension of the recursively enumerable Turing degrees. *Journal of the London Mathematical Society*, 75:287–297, 2007.

[22] Stephen G. Simpson. Mass problems and almost everywhere domination. *Mathematical Logic Quarterly*, 53:483–492, 2007.

[23] Stephen G. Simpson and Theodore A. Slaman. Medvedev degrees of Π^0_1 subsets of 2^ω. Preprint, 2001.

[24] Theodore A. Slaman and Robert I. Soare. Algebraic aspects of the computably enumerable degrees. *Proceedings of the American Mathematical Society*, 92(2):617–621, 1995.

[25] Robert I. Soare. *Recursively Enumerable Sets and Degrees*. Springer-Verlag, 1987.

This document was prepared & typeset with pdfLaTeX, and formatted with NDdiss2$_{\varepsilon}$ *classfile (v3.0[2005/07/27]) provided by Sameer Vijay.*