EXTENSION THEOREMS, ORBITS, AND AUTOMORPHISMS OF THE COMPUTABLY ENUMERABLE SETS

PETER A. CHOLAK AND LEO A. HARRINGTON

ABSTRACT. We prove an algebraic extension theorem for the computably enumerable sets, $\mathcal{E}$. Using this extension theorem and other work we then show if A and $\hat{A}$ are automorphic via Ψ then they are automorphic via Λ where $\Lambda \upharpoonright \mathcal{L}^*(A) = \Psi$ and $\Lambda \upharpoonright \mathcal{E}^*(A)$ is Δ^0_3. We give an algebraic description of when an arbitrary set $\hat{A}$ is in the orbit of a computably enumerable set A. We construct the first example of a definable orbit which is not a Δ^0_3 orbit. We conclude with some results which restrict the ways one can increase the complexity of orbits. For example, we show that if A is simple and $\hat{A}$ is in the same orbit as A then they are in the same Δ^0_6-orbit and furthermore we provide a classification of when two simple sets are in the same orbit.

1. INTRODUCTION

We will work in the structure of the computably enumerable sets. The language is just inclusion, $\subseteq$. This structure is called $\mathcal{E}$. There have been a large number of papers, see [7, 8, 19] for some recent surveys, studying $\mathcal{E}$ and the interaction within $\mathcal{E}$ among the following four mathematical concepts:

- Automorphisms: Is there a classification of the orbits of $\mathcal{E}$. Which sets are automorphic, i.e., in the same orbit?
- Definability: What computably enumerable sets can be defined (in the language of just $\{\subset\}$)? Is there a formula which distinguishes one set from another within $\mathcal{E}$?
- Dynamic Properties: How fast (or slow) can a set be enumerated compared to another set? or with respect to the standard enumeration of all computably enumerable sets?
- Complexity: How do sets in an orbit interact with each other via Turing reducibility? How do the sets in an orbit fit into jump classes,

Date: August 31, 2005.

2000 *Mathematics Subject Classification.* Primary 03D25.

Research partially supported NSF Grants DMS-96-34565, 99-88716, 02-45167 (Cholak), DMS-96-22290 and DMS-99-71137 (Harrington). We would like to thank Bob Soare and Mike Stob for their interest and helpful comments.

in particular, the low_n and high_n classes? This interaction is part of our connection to the computably enumerable degrees.

In this paper we focus on automorphisms and orbits although some aspects of the remaining concepts will arise.

Our understanding of automorphisms of $\mathcal{E}$ is unique to $\mathcal{E}$. In most structures with nontrivial automorphisms we can construct automorphisms via the normal "back and forth" argument. But this is not the case with $\mathcal{E}$. To construct automorphisms we use the properties of being *well-visited* and *well-resided*. Well-visited is Π^0_2 and not being well-resided is Σ^0_3 (we use the negation). Since the complexity of these properties is at most Σ^0_3, the construction of the desired automorphism can be placed on a tree. (We will not discuss the details on this placement nor of the construction of an automorphism of $\mathcal{E}$ but direct the reader to Harrington and Soare [11] or Cholak [3].) This method is called the Δ^0_3 automorphism method. If an automorphism Φ is constructed on a tree then Φ has a presentation computable in the true path (which is Δ^0_3). Hence all automorphisms constructed in this way are Δ^0_3-automorphisms. (In some cases we can make the automorphism effective.)

One step above using the Δ^0_3 automorphism method is to use an *extension* theorem. Basically, an extension theorem extends an isomorphism between two substructures of $\mathcal{E}$ to an automorphism of $\mathcal{E}$. The isomorphism between two substructures of $\mathcal{E}$ can be given in a number of ways and the same can be said about the resulting automorphism.

Generally, extension theorems are introduced to prove new automorphism results but they also allow us to reflect back and understand old automorphism results. Our philosophy is to argue modularly as much as possible. The hope is that an extension theorem provides an "understandable" module in the construction of an automorphism of $\mathcal{E}$.

The first major automorphism result, Soare's result [17] that the maximal sets form an orbit, used Soare's Extension Theorem. In Cholak [3], several more extension theorems were introduced and used to show that every noncomputable computably enumerable set is automorphic to a high set. In Cholak [2], the Modified Extension Theorem was introduced which allowed many of the automorphism constructions to be recast as using an extension theorem. For example, in Cholak [2], the results about orbits of hhsimple sets in Maass [15] and the result that the hemimaximal sets form an orbit found in Downey and Stob [10] were recast in this fashion. The Modified Extension Theorem has a weaker hypothesis than Soare's Extension Theorem. Soare has recently proven the "New Extension Theorem" and in addition to proving several new automorphism results with Harrington

he has recast almost all known automorphism results using this and similar theorems (see Soare [19] and Soare [16]).

All of these extension theorems share several common features. First they *always* produce Δ^0_3 automorphisms. All but Soare's Extension Theorem used the Δ^0_3 automorphism method as described in Cholak [3] and Harrington and Soare [11]. Soare's Extension Theorem was done effectively. The isomorphism which these extension theorems extend and the resulting automorphism are given *dynamically*.

The big issue before applying any extension theorem is to "match" up "entry states" which is done dynamically. The work done in Section 3.1 illustrates what we mean by dynamic, entry states, and matching.

One of the goals of this paper is to prove two new extension theorems (Theorems 3.1 and 4.9). These two theorems differ from the previous extension theorems. Theorem 4.9 allows the possibility that the resulting automorphism is not Δ^0_3. Both of them are stated "algebraically" (or "statically"). We have come up with an algebraic description of entry states and matching using *extendible Boolean algebras* and *supports*. Theorem 4.9 follows algebraically from Theorem 3.1. However we are not free from the use of dynamic methods. For example, the proof of Theorem 3.1 is dynamic and uses Soare's Extension Theorem along with other dynamic theorems.

(One word of caution: We use the word algebraic to mean facts or results about the structures we are considering. The structures we consider are Boolean algebras and lattices which are ordered structures where all the definable relations and functions can be defined just using the order, not necessarily the structures, a model theorist or algebraist might wish to study. So a model theorist or algebraist might wish to read "order-theoretic" in place of "algebraic".)

Theorem 4.9 shows that whether an isomorphism between $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$ can be extended to an automorphism depends on the existence of a "nice" isomorphism among "some of the entry states", where "some of the entry states" corresponds to extendible Boolean algebras and "nice" means some properties of the presentation of the algebras and the isomorphism.

As with any extension theorem, our extension theorems allow us to both reflect on old automorphism results and prove new automorphism results. In Section 5, we reprove some of the automorphism results mentioned above using Theorems 4.9 and 5.3. One current shortcoming of our extension theorem is with results where one is given a computably enumerable set A and constructs an automorphic $\hat{A}$ with certain properties (such as highness, for example); this is what Soare calls a "type 2" automorphism result (see Soare [19, Section 7]). But this might change.

By our extension theorems, the main result from Cholak and Harrington [6] (which depends heavily on Cholak and Harrington [5]) and a result about automorphisms and extendible Boolean algebras which resembles an automorphism construction, we can show that if A and $\hat{A}$ are automorphic via Ψ then the isomorphism between $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$ induced via Ψ can be extended into an automorphism Λ where $\Lambda \upharpoonright \mathcal{E}^*(A)$ is Δ^0_3. In other words we can convert Ψ into an automorphism Λ with some nice properties.

The Conversion Theorem (Theorem 6.3). *If A and $\hat{A}$ are automorphic via Ψ then they are automorphic via Λ where $\Lambda \upharpoonright \mathcal{L}^*(A) = \Psi$ and $\Lambda \upharpoonright \mathcal{E}^*(A)$ is Δ^0_3.*

Hence the complexity of an automorphism comes from the induced isomorphism between $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$. The impact of this theorem is that if we want to show A and $\hat{A}$ are automorphic we are not handicapped by using an extension theorem or the Δ^0_3 automorphism method. If we show A and $\hat{A}$ are automorphic via Λ, where Λ is built using an extension theorem or the Δ^0_3 automorphism method, then $\Lambda \upharpoonright \mathcal{E}^*(A)$ is always Δ^0_3. Our result says if there is an automorphism taking A to $\hat{A}$ then there is an automorphism taking A to $\hat{A}$ which is Δ^0_3 on the inside of A and $\hat{A}$.

As a result we get an algebraic description, in terms of the $\mathcal{L}^*(A)$, $\mathcal{L}^*(\hat{A})$, and extendible algebras, of when an arbitrary set $\hat{A}$ is in the orbit of a computably enumerable set A (see Theorem 6.4). Not surprisingly the algebraic description is Σ^1_1; it begins "does there exist an isomorphism between $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$".

In Section 7, we use our extension theorems to show that there is an elementary definable Δ^0_5 orbit $\mathcal{O}$, which is not an orbit under Δ^0_3 automorphisms. All the previously known orbits are orbits under Δ^0_3 automorphisms.

What is surprising is that this complexity comes from how $A \in \mathcal{O}$ interacts with sets which are disjoint from A. It was long thought this complexity would come from how A interacts with sets W such that $W \cap A \neq^* \emptyset$ and $W - A$ is infinite. For more details see Section 7.3 and Theorem 7.17. In Theorem 8.7, we improve Theorem 7.17 to all A; we show given an arbitrary computably enumerable set A the complexity of the orbit of A is determined by the sets disjoint from A.

There will be a sequel to this paper. In the forthcoming paper we show that there are orbits which are orbits under $\Delta^0_{\alpha+1}$ automorphisms but not Δ^0_α automorphisms, for all computable α. Cholak, Downey, and Harrington

have shown that the conjecture of Slaman-Woodin that $\{(A, \hat{A}) : A$ is automorphic to $\hat{A}\}$ is Σ^1_1-complete is correct. We hope to use our extension theorems to provide an understandable and manageable proof of the Slaman-Woodin conjecture. In fact, we want to show that there is an A such that whether $\hat{A}$ is in the orbit of A is Σ^1_1-complete. Theorems 7.17 and 8.7 will have great impact on how we approach these forthcoming results; they force us to use techniques similar to those used in Sections 7.1.1 and 7.2.5. Our extension theorems seem the best tool for these tasks since we must build non-Δ^0_3 automorphisms in all cases.

Our results certainly justify our philosophy to argue modularly as much as possible with the use of Soare's Extension Theorem as a module. It would be very difficult, if not impossible, to argue that building automorphisms of $\mathcal{E}$ all at once would be more enlightening.

In Section 2, we introduce and discuss the algebraic notations needed for our extension theorems. The remaining sections have been discussed above.

2. Splits of A

2.1. **Notation and definitions.** Our notation and definitions are standard and follow Cholak and Harrington [8] which follows Soare [18].

We will be dealing with isomorphisms between various substructures of $\mathcal{E}$ and automorphisms of $\mathcal{E}$. In all cases we will think of the isomorphism (automorphism) as a map from ω to another copy of ω, $\hat{\omega}$. All subsets of $\hat{\omega}$ will wear hats. We refer to $\hat{\omega}$ as the *hatted* side and sometimes we refer to ω as the *unhatted* side. When we define something on the unhatted side there is, of course, the hatted dual. We will use this duality frequently without mention.

2.2. **The structure $\mathcal{S}_{\mathcal{R}}(A)$.** Fix a computably enumerable set A.

Definition 2.1. Let $\mathcal{S}(A) = \{B : \exists C(B \sqcup C = A)\}$. $\mathcal{S}(A)$ is the splits of A and $\mathcal{S}(A)$ forms a Boolean algebra. $\mathcal{F}(A)$ is the finite subsets of A and is an ideal of $\mathcal{S}(A)$. Let $\mathcal{S}^*(A)$ be the quotient structure $\mathcal{S}(A)$ modulo $\mathcal{F}(A)$. Let $\mathcal{R}(A) = \{R : R \subseteq A$ and R is computable$\}$. $\mathcal{R}(A)$ is the computable subsets of A and is an ideal of $\mathcal{S}(A)$. Let $\mathcal{S}_{\mathcal{R}}(A)$ be the quotient structure $\mathcal{S}(A)$ modulo $\mathcal{R}(A)$.

Let W be in $\mathcal{S}(A)$. Then let $\breve{W} = A - W$ (a computably enumerable set) and $W^{\mathcal{R}}$ be the equivalence class of W in $\mathcal{S}_{\mathcal{R}}(A)$. From Cholak and Harrington [6, Lemma 2.2], we know that if A is noncomputable, then $\mathcal{S}_{\mathcal{R}}(A)$ is the atomless Boolean algebra and hence every Boolean algebra can be embedded in $\mathcal{S}_{\mathcal{R}}(A)$.

2.3. Σ^0_3 **Boolean algebras.** Recall from Soare [18] the following definition.

Definition 2.2. A countable Boolean algebra $\mathcal{B} = (\{X_i\}_{i\in\omega}, \leq, \cup, \cap, \bar{})$ is a Σ^0_3 *Boolean algebra* if the listing $\{X_i\}_{i\in\omega}$ is uniformly computable and there are computable functions f and g and a Σ^0_3 relation R such that $X_i \cup X_j = X_{f(i,j)}$, $X_i \cap X_j = X_{g(i,j)}$, and $X_i \leq X_j$ iff $R(i,j)$. (An element of $\mathcal{B}$ must appear at least once in $\{X_i\}_{i\in\omega}$ but there is no bound on the number of times an element may appear in $\{X_i\}_{i\in\omega}$.)

We should be familiar with Σ^0_3 Boolean algebras. There is a beautiful theorem of Lachlan (see Soare [18, X.7.2]) that says if $\mathcal{B}$ is any Σ^0_3 Boolean algebra then there is an hhsimple set H such that $\mathcal{L}^*(H)$ is isomorphic to $\mathcal{B}$. Let $\widetilde{\mathcal{L}}(H)$ be the quotient substructure of $\mathcal{S}_{\mathcal{R}}(H)$ given by $\{R \cap H : R \text{ is computable}\}$ modulo $\mathcal{R}(H)$. Clearly, as given, $\widetilde{L}(H)$ is definable in $\mathcal{E}$ with a parameter for H. In Cholak and Harrington [6, Lemma 11.2], it is shown that $\mathcal{L}^*(H)$ and $\tilde{\mathcal{L}}(H)$ are isomorphic. Hence there is a substructure of $\mathcal{S}_{\mathcal{R}}(A)$ which ranges over all Σ^0_3 Boolean algebras as A ranges over all computably enumerable sets.

All of the Boolean algebras we consider will be substructures of $\mathcal{S}_{\mathcal{R}}(A)$, $\mathcal{L}^*(A)$, or $\mathcal{E}$. So we will always consider the list $\{X_i\}_{i\in\omega}$ as a list of computably enumerable sets. The operations will be union, intersection, and complementation on computably enumerable sets; and hence the functions f and g are clearly computable. The relation R will reflect either $X \subseteq Y$, $X \subseteq_{\mathcal{R}} Y$, or $X \subseteq^* Y$.

Lemma 2.3. *Given two splits X and Y, whether $X \subseteq_{\mathcal{R}} Y$ is Σ^0_3.*

Proof. Given the index for X, it is possible to find in a Δ^0_3 way an index for $\breve{X}$. Similarly for Y. Hence we can find an index for $X \triangle Y$ in a Δ^0_3 fashion. Now $X \subseteq_{\mathcal{R}} Y$ iff $X \triangle Y$ is computable iff there is an l such that $W_l \sqcup (X \triangle Y) = \omega$. Since "$W_l \sqcup (X \triangle Y) = \omega$" is Π^0_2, the last clause in the above sentence is Σ^0_3. □

Theorem 2.4. *Let $\{X_i : i \in \omega\}$ be a uniformly computable list of computably enumerable sets (not necessarily splits of A) and a Σ^0_3 set B such that $\{X_i : i \in B\}$ generates a subalgebra $\mathcal{B}$ of $\mathcal{S}_{\mathcal{R}}(A)$. Then there is a list $\{Y_i : i \in \omega\}$ where all the Y_is are splits of A, which witnesses that $\mathcal{B}$ is a Σ^0_3 Boolean algebra. Furthermore there is a Δ^0_3 function g from B to ω such that $X_i = Y_{g(i)}$.*

Proof. Basically we are going to pad the Σ^0_3 list, $\{X_i : i \in B\}$, with lots of finite sets to make it a computable list of computably enumerable sets all of which are splits of A. This padding will be done on a tree, $2^{<\omega}$. It will be a standard Π^0_2 tree argument.

Assume $i \in B$ iff $\exists k \varphi(i,k)$, where $\varphi(i,k)$ is Π^0_2. Assume that $\varphi(i,k)$ is $(\forall x)(\exists y)[\Theta(i,k,x,y)]$, where Θ is Δ^0_0. We define the true path by induction as follows: Let $\alpha \subset f$ such that $|\alpha| = \langle i,k\rangle$. If $\varphi(i,k)$ then $\alpha\hat{}0 \subset f$; otherwise $\alpha\hat{}1 \subset f$.

The approximation to the true path is also defined by induction. Let $\alpha \subseteq f_s$ such that $|\alpha| = \langle i,k\rangle$ and $|\alpha| \leq s$. We need a length of agreement function: $l_\alpha(s)$ is the greatest z such that for all $x \leq z$ there is a y with $\Theta(i,k,x,y)$. Let $t < s$ be the last stage that $\alpha \subseteq f_s$ (if such a stage does not exist let $t = 0$). If $l_\alpha(t) < l_\alpha(s)$ (an α-expansionary stage) then $\alpha\hat{}0 \subseteq f_s$; otherwise $\alpha\hat{}1 \subseteq f_s$. It is not too hard to show that $f = \liminf_s f_s$.

At $\beta = \alpha\hat{}0$ we will construct a set Y_j. If $\beta \subseteq f_s$ for the first time ever or the first time after being initialized, choose the least j such that Y_j is not being constructed and start constructing Y_j. If $\beta \subseteq f_s$ and β is building Y_j, let $Y_{j,s} = X_{i,s}$, where $|\alpha| = \langle i,k\rangle$. If β is to the right of f_s we will initialize β at stage s (and end the construction of the current Y_j).

If $\beta = \alpha\hat{}0 \subset f$ then, by the nature of the tree construction, at some stage β will be assigned a permanent Y_j and never be initialized after that stage. Then $Y_j = X_i$, where $|\alpha| = \langle i,k\rangle$. If Y_j is not permanently assigned to such a β then Y_j is finite. □

Corollary 2.5. *$\mathcal{I}_{\mathcal{R}}(A)$ is a Σ^0_3 Boolean algebra.*

Proof. Given a computably enumerable set W_e, it is Σ^0_3 to decide if W_e is a split of A (is there a j such that $W_e \sqcup W_j = A$). □

Definition 2.6. Following Theorem 2.4, given $\mathcal{B}$ a Σ^0_3 Boolean algebra of $\mathcal{I}_{\mathcal{R}}(A)$ ($\mathcal{L}^*(A)$ or $\mathcal{E}$), if there is a uniformly computable list $\mathcal{X} = \{X_i\}_{i\in\omega}$ of computably enumerable sets and a Σ^0_3 set B such that $\{X_i : i \in B\}$ generates $\mathcal{B}$, we say $\mathcal{X}$ and B is a *representation* for $\mathcal{B}$. (B might be all of ω.)

2.4. **Listings of splits of A.** We are concerned with the certain well-represented subalgebras of $\mathcal{I}_{\mathcal{R}}(A)$. Even if we know X is a split of A we still need $\mathbf{0}''$ to find a Y such that $X \sqcup Y = A$. We want to limit ourselves to considering just splits S where we can find $A - S$ effectively.

Definition 2.7. A uniformly computable listing, $\mathcal{I} = \{S_i : i \in \omega\}$, of splits of A is an *effective listing* of splits of A iff there is another uniformly computable listing $\{\breve{S}_i : i \in \omega\}$ of splits of A such that $S_i \sqcup \breve{S}_i = A$.

Lemma 2.8. *Let $S_e = W_e \searrow A$; this is an entry set. Then the entry sets, $\mathcal{I} = \{S_e : e \in \omega\}$, is an effective listing of splits.*

Proof. $(W_e \searrow A) \sqcup (A \backslash W_e) = A$. □

With an entry set the corresponding split is determined at the moment x enters A; either x enters A in W_e or not. The entry sets are the canonical example of an effective listing of splits. This list depends on the enumeration of A.

Lemma 2.9. *Let $\mathcal{S} = \{S_i : i \in \omega\}$ be an effective listing of splits of A. Then there is an enumeration of A, an effective listing of splits of A, $\tilde{\mathcal{S}} = \{\tilde{S}_i : i \in \omega\}$, and an effective listing of splits of A, $\breve{\tilde{\mathcal{S}}} = \{\breve{\tilde{S}}_i : i \in \omega\}$, such that, for all i, w.r.t. the new enumeration of A, $\tilde{S}_i =^* S_i$, $A \searrow \tilde{S}_i = \emptyset$ (so $\tilde{S}_i = \tilde{S}_i \searrow A$), $A \searrow \breve{\tilde{S}}_i = \emptyset$, $\tilde{S}_i \sqcup \breve{\tilde{S}}_i \sqcup (A \cap \{0, 1, \dots i\}) = A$, and if $x \in \tilde{S}_{i,s} \sqcup \breve{\tilde{S}}_{i,s}$ then $x \in S_{j,s} \sqcup \breve{S}_{j,s}$, for all $j \leq i$.*

Proof. Let x enter A (under the old enumeration). Wait for x to enter S_i or $\breve{S}_i$ for $i < x$; adding x to $\tilde{S}_i$ or $\breve{\tilde{S}}_i$, respectively. Then allow x to enter A (under the new enumeration).

Clearly $\tilde{\mathcal{S}} = \{\tilde{S}_i : i \in \omega\}$ and $\breve{\tilde{\mathcal{S}}} = \{\breve{\tilde{S}}_i : i \in \omega\}$ are uniformly computable listings of splits of A. The uniformly computable listing of splits of A, $\{\breve{\tilde{S}}_i \cup (A \cap \{0, 1, \dots, i\}) : i \in \omega\}$ witnesses that $\tilde{\mathcal{S}}$ is an effective listing of splits. Similarly $\{\tilde{S}_i \cup (A \cap \{0, 1, \dots, i\}) : i \in \omega\}$ witnesses that $\breve{\tilde{\mathcal{S}}}$ is an effective listing of splits. □

Remark 2.10. It is necessary that $\mathcal{S}$ be an effective listing of splits of A for the above lemma to hold. The key point of this lemma is that when x enters A it has been determined whether x is in $\tilde{S}_i$ or not. So $\tilde{S}_i \sqcup (A \backslash \tilde{S}_i) = A$.

This lemma will be essential. It is used in Lemma 2.15 which in turn plays a key role in Section 3.3. Also see the proof of Lemma 3.8.

Hence as we vary the enumeration of A we get almost all effective listing of splits of A as entry sets. However we do not get all (noneffective) listing of splits this way.

Lemma 2.11. *No effective listing of splits of infinite computably enumerable set A contains all splits of A.*

Proof. We will provide two proofs of this lemma.

Let $\mathcal{S} = \{S_e : e \in \omega\}$ be an effective list of splits of A. Let $\{a_i : i \in \omega\}$ be a computable listing of the elements of A without repeats. Let $S = \{a_i : a_i \notin S_i\} = \{a_i : a_i \in \breve{S}_i\}$. If $S = S_j$ then $a_j \in S$ iff $a_j \in S_j$ iff $a_j \notin S_j$. So $S \neq S_j$, for all j.

By Lemma 2.9, we can assume $S_i = S_i \searrow A$ and $\breve{S}_i = \breve{S}_i \searrow A$, for all i. By easily modifying the Friedberg Splitting Theorem (see Soare [18, X.2.1]), we can build a split S and $\breve{S}$ such that if $S_i \searrow A$ ($\breve{S}_i \searrow A$) is infinite

then $S_i \searrow S$ ($\check{S}_i \searrow S$) is infinite and similarly for $\check{S}$. The split S is not in $\mathcal{S}$. □

2.5. **Extendible subalgebras.** We would like to consider subalgebras of $\mathcal{S}_{\mathcal{R}}(A)$ which have a representation that is an effective listing of splits of A.

Definition 2.12. A Σ^0_3 subalgebra $\mathcal{B}$ of $\mathcal{S}_{\mathcal{R}}(A)$ is *extendible* iff there is representation $\mathcal{S}$ and B of $\mathcal{B}$ such that $\mathcal{S}$ is an effective listing of splits of A and B is a Δ^0_3 set.

We will assume that if $\mathcal{B}$ is extendible then the given representation is always an effective listing of splits of A. From this point further $\mathcal{S} = \{S_i : i \in \omega\}$ will always refer to an effective listing of splits of A and $\mathcal{X} = \{X_i : i \in \omega\}$ to a uniformly computable list of computably enumerable sets.

Lemma 2.13. *The trivial subalgebra of $\mathcal{S}_{\mathcal{R}}(A)$ is extendible.*

Proof. Let $S_{2e} = \emptyset$, $\check{S}_{2e} = A$, $S_{2e+1} = A$, $\check{S}_{2e+1} = \emptyset$, and $B = \omega$. □

Lemma 2.14. *The subalgebra $\mathcal{E}_A$ generated by the entry sets is extendible (this is what we call an entry set Boolean algebra for A).*

Proof. Use the listing from Lemma 2.8 and $B = \omega$. □

Lemma 2.15. *Let $\mathcal{B} \subseteq \mathcal{S}_{\mathcal{R}}(A)$ be extendible via $\mathcal{S}$ and B. There is an enumeration of A and an effective listing of splits, $\tilde{\mathcal{S}} = \{\tilde{S}_i : i \in \omega\}$, such that $\tilde{\mathcal{S}}$ and B witness that $\mathcal{B}$ is extendible and, for all i, $A \searrow \tilde{S}_i = \emptyset$ (and so $\tilde{S}_i \sqcup (A \setminus \tilde{S}_i) = A$).*

Proof. Apply Lemma 2.9 to $\mathcal{S}$ to get the desired enumeration of A and the effective listing of splits of A, $\tilde{\mathcal{S}}$. $\{\tilde{S}_i : i \in B\}$ generates $\mathcal{B}$. □

Hence every extendible Boolean algebra is an extendible subalgebra of an entry set Boolean algebra. Clearly every extendible Boolean algebra is a Σ^0_3 Boolean algebra.

Lemma 2.16. *If $\mathcal{B}$ and $\mathcal{B}'$ are extendible then $\mathcal{B} \oplus \mathcal{B}'$ are extendible.*

Proof. Let $\{S_i\}_{i \in \omega}$ and B witness that $\mathcal{B}$ is extendible and similarly for $\mathcal{B}'$. Let $T_{2i} = S_i$ and $T_{2i+1} = S'_i$. Then $\{T\}_{i \leq \omega}$ and $\{2i : i \in B\} \cup \{2i+1 : i \in B'\}$ witness that $\mathcal{B} \oplus \mathcal{B}'$ is extendible. □

Theorem 2.17. *There is an extendible algebra $\mathcal{B}$ of $\mathcal{S}_{\mathcal{R}}(A)$ such that*

(1) *for all $i \in B$, S_i is computable,*
(2) *for all $R \in \mathcal{R}(A)$, there is $i \in B$ such that $R = S_i$, and*
(3) *B is infinite.*

Proof. For this proof fix an enumeration of A (with $A_1 = \emptyset$). The idea is that if R is a computable split of A then there are i_0, i_1, i_2 such that $R = W_{i_0}$, $A \searrow W_{i_0} = \emptyset$ (w.r.t. this fixed enumeration), $\breve{W}_{i_0} = W_{i_1}$, $A \searrow W_{i_1} = \emptyset$, $W_{i_0,s+1} \sqcup W_{i_1,s+1} = A_{s+2}$, $\overline{W}_{i_0} = W_{i_2}$, and $W_{i_1,s+1} \subseteq W_{i_2,s+1}$, for all s, (before x enters A determine which of R or $\overline{R} = W_{i_2}$ x is in and add x to W_{i_0} or W_{i_1} and W_{i_2} accordingly). In this case, we can let $S_i = W_{i_0}$ and $\breve{S}_i = W_{i_1}$, where $i = \langle i_0, i_1, i_2\rangle$. But to make $\mathcal{S}$ a uniformly computable list of computably enumerable sets we must be more careful.

Let $i = \langle i_0, i_1, i_2\rangle$. Assume that $S_{i,s}$ and $\breve{S}_{i,s}$ have been defined and i has not been declared *unusable*. If $(A \searrow W_{i_0})_{s+1} = \emptyset$, $(A \searrow W_{i_1})_{s+1} = \emptyset$, $W_{i_0,s+1} \sqcup W_{i_1,s+1} = A_{s+2}$, $W_{i_0,s+1} \cap W_{i_2,s+1} = \emptyset$, and $W_{i_1,s+1} \subseteq W_{i_2,s+1}$, then let $S_{i,s+1} = W_{i_0,s+1}$ and $\breve{S}_{i,s+1} = W_{i_1,s+1}$. Otherwise declare i *unusable* and, for all $s' > s$, let $S_{i,s'} = S_{i,s}$ and $\breve{S}_{i,s'} = A_{s'+1} - S_{i,s}$. $\{S_i\}_{i\in\omega}$ is an effective listing of splits of A.

Let $i \in B$ iff $W_{i_0} \sqcup W_{i_1} = A$, $A \searrow W_{i_0} = \emptyset$, $A \searrow W_{i_1} = \emptyset$, $W_{i_0,s+1} \sqcup W_{i_1,s+1} = A_{s+2}$, $\overline{W}_{i_0} = W_{i_2}$, and $W_{i_1,s+1} \subseteq W_{i_2,s+1}$, for all s. B is Δ^0_3.

$\{S_i\}_{i\in\omega}$ and B represent our extendible algebra $\mathcal{B}$. If $i \in B$ then $S_i = W_{i_0}$, $\breve{S}_i = W_{i_1}$, and $S_i \sqcup W_{i_2} = \omega$ and hence S_i is computable. Given a computable subset R of A, by the first paragraph of this proof, there is an corresponding $i \in B$ with $R = W_{i_0}$. Since there are infinitely many such R, B is infinite. □

2.6. **Isomorphisms.**

Definition 2.18. We consider Θ a partial map between splits of A and splits of $\hat{A}$ an *isomorphism* between a substructure $\mathcal{B}$ of $\mathcal{S}_{\mathcal{R}}(A)$ and a substructure $\hat{\mathcal{B}}$ of $\mathcal{S}_{\mathcal{R}}(\hat{A})$ if Θ preserves $\subseteq_{\mathcal{R}}$, for each equivalence class $S_{\mathcal{R}}$ of $\mathcal{B}$ if $S \in S_{\mathcal{R}}$, $\Theta(S)$ exists, and for each equivalence class $\hat{S}_{\mathcal{R}}$ of $\hat{\mathcal{B}}$ if $\hat{S} \in \hat{S}_{\mathcal{R}}$, $\Theta^{-1}(\hat{S})$ exists. There is a function h such that $\Theta(W_i) = \hat{W}_{h(i)}$ and $\Theta^{-1}(\hat{W}_i) = W_{h^{-1}(i)}$. If h is Δ^0_3 then so is Θ.

Definition 2.19. We say two extendible Boolean algebras $\mathcal{B}$ and $\hat{\mathcal{B}}$ are *extendibly isomorphic* via Θ iff

- there is an effective listing of splits $\{S_i\}_{i\in\omega}$ and a B which witness that $\mathcal{B}$ is an extendible algebra,
- there are $\{\hat{S}_i\}_{i\in\omega}$ and $\hat{B}$ which witness $\hat{\mathcal{B}}$ is an extendible algebra,
- for all $i \in B$, there is a $j \in \hat{B}$ such that $\Theta(S_i) = \hat{S}_j$,
- for all $j \in \hat{B}$ there is an $i \in B$ such that $\Theta^{-1}(\hat{S}_j) = S_i$, and

- this partial map induces an isomorphism Θ' between $\mathcal{B}$ and $\hat{\mathcal{B}}$ as in Definition 2.18.

In this case, we say that Θ is an *extendible isomorphism*. There is a function h such that $\Theta(S_i) = \hat{S}_{h(i)}$ and $\Theta^{-1}(\hat{S}_i) = S_{h^{-1}(i)}$. If h is Δ^0_3 then so is Θ. We write $\Theta(S_i) = \hat{S}_{\Theta(i)}$ and $\Theta^{-1}(\hat{S}_j) = S_{\Theta^{-1}(j)}$. If S is not an S_i, for all i, but $S_{\mathcal{R}} \in \mathcal{B}$ we let $\Theta(S) = \Theta'(S)$ and similarly for $\hat{S}$. Hence we will also consider Θ to be an isomorphism (as in Definition 2.18) between $\mathcal{B}$ and $\hat{\mathcal{B}}$.

Lemma 2.20. *Let $\mathcal{B}$ be a Σ^0_3 substructure of $\mathcal{S}_{\mathcal{R}}(A)$ and $\hat{\mathcal{B}}$ be a Σ^0_3 substructure of $\mathcal{S}_{\mathcal{R}}(\hat{A})$. Assume that Θ is a map between $\{X_i : i \in B\}$ and $\{\hat{X}_i : i \in \hat{B}\}$. Furthermore assume that for $i, j \in B$, $X_i - X_j$ is computable iff $\Theta(X_i) - \Theta(X_j)$ is computable and, dually, for all $i, j \in \hat{B}$, $\hat{X}_i - \hat{X}_j$ is computable iff $\Theta^{-1}(\hat{X}_i) - \Theta^{-1}(\hat{X}_j)$ is computable. Then Θ induces an isomorphism Θ' between $\mathcal{B}$ and $\hat{\mathcal{B}}$.*

Proof. Θ and Θ^{-1} preserve $\subseteq_{\mathcal{R}}$. $X_j \subseteq_{\mathcal{R}} X_i$ iff $X_j - X_i$ is computable iff $\Theta(X_j) - \Theta(X_i)$ is computable iff $\Theta(X_j) \subseteq_{\mathcal{R}} \Theta(X_i)$. And similarly for Θ^{-1}. Given $S_{\mathcal{R}} \in \mathcal{B}$ find i such that $X_i \in S_{\mathcal{R}}$ and, for all $S \in S_{\mathcal{R}}$, let $\Theta'(S) = \Theta(X_i)$. Θ' is well defined and preserves $\subseteq_{\mathcal{R}}$ since Θ does. Define Θ^{-1} dually. □

If Θ is an extendible isomorphism and we apply Lemma 2.15 to the effective listing of splits then Θ remains an extendible isomorphism between these two extendible algebras with regard to the new listing of splits.

Lemma 2.21. *The trivial subalgebras of $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\hat{A})$ are effectively extendibly isomorphic as extendible subalgebras of $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\hat{A})$.*

Proof. Let $\{S_i\}_{i<\omega}$ be the listing of splits given in Lemma 2.13 for the trivial subalgebra of $\mathcal{S}_{\mathcal{R}}(A)$. Let $\{\hat{S}_i\}_{i<\omega}$ be the listing of splits given in Lemma 2.13 for the trivial subalgebra of $\mathcal{S}_{\mathcal{R}}(\hat{A})$. Let $\Theta(S_i) = \hat{S}_i$ and $\Theta^{-1}(\hat{S}_i) = S_i$. □

Lemma 2.22. *Assume that $\mathcal{B}$ and $\hat{\mathcal{B}}$ are extendible subalgebras which are extendibly isomorphic via Θ. Assume that $\mathcal{B}'$ and $\hat{\mathcal{B}}'$ are extendible subalgebras which are extendibly isomorphic via Θ'. Then, by Lemma 2.16, $\mathcal{B} \oplus \mathcal{B}'$ and $\hat{\mathcal{B}} \oplus \hat{\mathcal{B}}'$ are extendible subalgebras which are extendibly isomorphic via Δ, where $\Delta(T_{2e}) = \Theta(S_e)$, $\Delta(T_{2e+1}) = \Theta'(S'_e)$, $\Delta^{-1}(\hat{T}_{2e}) = \Theta^{-1}(\hat{S}_e)$, and $\Delta^{-1}(\hat{T}_{2e+1}) = (\Theta')^{-1}(\hat{S}'_e)$.*

3. Extensions to isomorphisms

Recall that $\mathcal{E}^*(A)$ is the structure $(\{W_e \cap A : e \in \omega\}, \subseteq)$ modulo the finite sets. An isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$ is a one-to-one,

onto (both of these items are in terms of $*$-equivalence classes) function, Ξ, from $\{W_e \cap A : e \in \omega\}$ to $\{\hat{W}_e \cap \hat{A} : e \in \omega\}$ such that $W_e \cap A \subseteq^* W_i \cap A$ iff $\Xi(W_e \cap \hat{A}) \subseteq^* \Xi(W_i \cap \hat{A})$. Note the Ξ is applied to $W_e \cap A$, not W_e.

The goal of this section is to prove and discuss the import of the following extension theorem.

Theorem 3.1. *Let $\mathcal{B} \subseteq \mathcal{I}_{\mathcal{R}}(A)$ and $\hat{\mathcal{B}} \subseteq \mathcal{I}_{\mathcal{R}}(\hat{A})$ be two extendible Boolean algebras which are Δ^0_3 extendibly isomorphic via Θ. Then there is a Φ such that Φ is a Δ^0_3 isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$, for all $i \in B$, $\Phi(S_i) =_{\mathcal{R}} \Theta(S_i)$, and for all $i \in \hat{B}$, $\Phi^{-1}(\hat{S}_i) =_{\mathcal{R}} \Theta^{-1}(\hat{S}_i)$.*

What is important about this theorem is that we can *extend* the extendible isomorphism between $\mathcal{B}$ and $\hat{\mathcal{B}}$ to an isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$.

The first clause of the conclusion should not be very surprising. After all, if A and $\hat{A}$ are infinite then there is an effective isomorphism Ψ between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$. Let f be an effective map from A to $\hat{A}$ and $\Psi(W) = f(W)$. Moreover, if A and $\hat{A}$ are computable then Ψ clearly computably agrees with Θ on all S_i and hence the second clause of the conclusion holds with Ψ.

The main use of Theorem 3.1 is in the proof of Theorem 4.9 and Theorem 5.4. These are the only examples of the use of Theorem 3.1 in this paper. However, we will provide several examples of the use of Theorem 4.9 and Theorem 5.4.

There are several possible ways to prove this theorem. For example, one could use some of Soare's recent work on extension theorems. We had used such a proof in an earlier version of this paper. In this version we will base our proof on published theorems. However, we will have to use them in novel ways and, in a few cases, note that these proofs prove more than what is actually stated.

We will base our proof on a theorem, the Translation Theorem, from Cholak [2]. The proof will have a few parts. First we will restate the Translation Theorem in a slightly strengthened form and show why this version follows from the proof in Cholak [2]. Then we construct a $\mathbf{0}''$ enumeration witnessing that Θ is an extendible isomorphism and meeting the hypothesis of the Translation Theorem. Then we apply the modified Translation Theorem followed by Soare's original Extension Theorem to this enumeration to get the desired isomorphism.

The proof of Theorem 3.1 is one of the few places where we have to go into the difficult details of actually building an isomorphism by a dynamic construction and the use of states.

3.1. **The Modified Translation Theorem.** These next definitions are a repeat of the first six definitions in Section 1 of Cholak [2] using slightly different notation.

Definition 3.2. (1) $\{X_n\}_{n<\omega}$ is a *uniformly computable collection* of c.e. sets if there is a computable function h such that for all n, $X_n = W_{h(n)}$.

(2) $\{X_n\}_{n<\omega}$ is a *uniformly $\mathbf{0}''$-computable collection* of c.e. sets if there is a function $h \leq_T \mathbf{0}''$ such that for all n, $X_n = W_{h(n)}$.

(3) $\{X_{n,s}\}_{n<\omega,s<\omega}$ is a *uniformly $\mathbf{0}''$-computable enumeration* of c.e. sets if there is a function $h \leq_T \mathbf{0}''$ such that for all n and s, $X_{n,s} = W_{h(n),s}$.

Definition 3.3. For any e, if we are given uniformly computable enumerations of $\{X_{n,s}\}_{n\leq e,s<\omega}$ and $\{Y_{n,s}\}_{n\leq e,s<\omega}$ of c.e. sets $\{X_n\}_{n\leq e}$ and $\{Y_n\}_{n\leq e}$, define the *full e-state of x at stage s*, $\nu(e,x,s)$, with respect to (w.r.t.) $\{X_{n,s}\}_{n\leq e,s<\omega}$ and $\{Y_{n,s}\}_{n\leq e,s<\omega}$ to be the triple

$$\nu(e,x,s) = \langle e, \sigma(e,x,s), \tau(e,x,s)\rangle$$

where

$$\sigma(e,x,s) = \{i \leq e : x \in X_{i,s}\}$$

and

$$\tau(e,x,s) = \{i \leq e : x \in Y_{i,s}\}.$$

Definition 3.4. For any collection of c.e. sets $\{X_n\}_{n\leq e}$ and $\{Y_n\}_{n\leq e}$, define the *final e-state of x*, $\nu(e,x)$, w.r.t $\{X_n\}_{n\leq e}$ and $\{Y_n\}_{n\leq e}$ to be the triple

$$\nu(e,x) = \langle e, \sigma(e,x), \tau(e,x)\rangle$$

where

$$\sigma(e,x) = \{i \leq e : x \in X_i\}$$

and

$$\tau(e,x) = \{i \leq e : x \in Y_i\}.$$

Definition 3.5. Assume that $\{A_s\}_{s<\omega}$ is a uniformly computable enumeration of A, an infinite c.e. set. For any e, assume we are given uniformly computable enumerations of $\{X_{n,s}\}_{n\leq e,s<\omega}$ and $\{Y_{n,s}\}_{n\leq e,s<\omega}$ of c.e. sets $\{X_n\}_{n\leq e}$ and $\{Y_n\}_{n\leq e}$. For each full e-state ν, define the c.e. set

$$D_\nu^A = \{x : \exists t \text{ such that } x \in A_{s+1} - A_s \text{ and } \nu = \nu(e,x,s) \text{ w.r.t. } \{X_{n,s}\}_{n\leq e,s<\omega} \text{ and } \{Y_{n,s}\}_{n\leq e,s<\omega}\}.$$

If $x \in D_\nu^A$, we say that ν is the *entry state* of x w.r.t. $\{X_{n,s}\}_{n\leq e,s<\omega}$ and $\{Y_{n,s}\}_{n\leq e,s<\omega}$ into A. We say that D_ν^A is measured w.r.t. $\{X_{n,s}\}_{n\leq e,s<\omega}$ and $\{Y_{n,s}\}_{n\leq e,s<\omega}$.

The following definition is new and is used for notation ease.

Definition 3.6. We write $X \doteq_{\mathcal{R}} Y$ iff $X \subseteq Y$ and $X =_{\mathcal{R}} Y$.

Theorem 3.7 (The Modified Translation Theorem). *Assume that* $\{A^\dagger_s\}_{s\in\omega}$, $\{\hat{A}^\dagger_s\}_{s\in\omega}$, $\{U^\dagger_{n,s}\}_{n<\omega,s<\omega}$, $\{\hat{V}^\dagger_{n,s}\}_{n<\omega,s<\omega}$, $\{\hat{U}^\dagger_{n,s}\}_{n<\omega,s<\omega}$, *and* $\{V^\dagger_{n,s}\}_{n<\omega,s<\omega}$ *are uniformly* $\mathbf{0}''$*-computable enumerations of the infinite c.e. sets* $A^\dagger$ *and* $\hat{A}^\dagger$ *and the uniformly* $\mathbf{0}''$*-computable collection of c.e. sets* $\{U^\dagger_n\}_{n<\omega}$, $\{\hat{V}^\dagger_n\}_{n<\omega}$, $\{\hat{U}^\dagger_n\}_{n<\omega}$, *and* $\{V^\dagger_n\}_{n<\omega}$ *satisfying the following conditions:*

$$(\forall n)[\hat{A}^\dagger \searrow \hat{U}^\dagger_n = A^\dagger \searrow \hat{V}^\dagger_n = \emptyset], \tag{3.1}$$

$$(\forall \nu)[D^{A^\dagger}_\nu \text{ is infinite iff } D^{\hat{A}^\dagger}_\nu \text{ is infinite}], \tag{3.2}$$

where, for all e-states, $D^{A^\dagger}_\nu$ *is measured w.r.t* $\{U^\dagger_{n,s}\}_{n\leq e,s<\omega}$ *and* $\{\hat{V}^\dagger_{n,s}\}_{n\leq e,s<\omega}$, *and* $D^{\hat{A}^\dagger}_\nu$ *is measured w.r.t* $\{\hat{U}^\dagger_{n,s}\}_{n\leq e,s<\omega}$ *and* $\{V^\dagger_{n,s}\}_{n\leq e,s<\omega}$.

Then there is a collection of uniformly computable c.e. sets $\{U_n\}_{n<\omega}$, $\{\hat{V}^+_n\}_{n<\omega}$, $\{\hat{U}^+_n\}_{n<\omega}$, *and* $\{V_n\}_{n<\omega}$ *and uniformly computable enumerations* $\{A_s\}_{s\in\omega}$, $\{\hat{A}_s\}_{s\in\omega}$, $\{U_{n,s}\}_{n<\omega,s<\omega}$, $\{\hat{V}^+_{n,s}\}_{n<\omega,s<\omega}$, $\{\hat{U}^+_{n,s}\}_{n<\omega,s<\omega}$, *and* $\{V_{n,s}\}_{n<\omega,s<\omega}$ *of these sets such that*

$$A_{s+1} = A^\dagger_s \text{ and } \hat{A}_{s+1} = \hat{A}^\dagger_s, \tag{3.3}$$

$$(\forall n)[\hat{A} \searrow \hat{U}^+_n = A \searrow \hat{V}^+_n = \emptyset], \tag{3.4}$$

$$(\forall n)(\exists e_n)[U^\dagger_n =^* U_{e_n},\ \hat{V}^+_{e_n} \doteq_{\mathcal{R}} \hat{V}^\dagger_n,\ \hat{U}^+_{e_n} \doteq_{\mathcal{R}} \hat{U}^\dagger_n, \text{ and } V^\dagger_n =^* V_{e_n}], \tag{3.5}$$

$$\begin{aligned}(\forall e)[&\textit{either } [U_e\backslash A =^* \hat{V}^+_e\backslash A =^* \hat{U}^+_e\backslash\hat{A} =^* V\backslash\hat{A} =^* \emptyset]\\ &(\textit{hence, by Equation (3.4)}, \hat{U}^+_e = \hat{V}^+_e =^* \emptyset) \textit{ or}\\ &[\textit{there is an } n \textit{ such that } e = e_n \textit{ (from Equation (3.5))}]],\end{aligned} \tag{3.6}$$

$$(\forall \nu)[D^{\hat{A}}_\nu \text{ is infinite implies } (\exists \nu' \geq \nu) D^A_{\nu'} \text{ is infinite}], \tag{3.7}$$

$$(\forall \nu)[D^A_\nu \text{ is infinite implies } (\exists \nu' \leq \nu)[D^{\hat{A}}_{\nu'} \text{ is infinite}]], \tag{3.8}$$

where, for all e-states, D^A_ν *is measured w.r.t* $\{U_{n,s}\}_{n\leq e,s<\omega}$ *and* $\{\hat{V}^+_{n,s}\}_{n\leq e,s<\omega}$, *and* $D^{\hat{A}}_\nu$ *is measured w.r.t* $\{\hat{U}^+_{n,s}\}_{n\leq e,s<\omega}$ *and* $\{V_{n,s}\}_{n\leq e,s<\omega}$.

3.2. **Proving the Modified Translation Theorem.** We will show that the Modified Translation Theorem follows from the version of the Translation Theorem published in Cholak [2]. Equations labeled "3.x" refer to the Modified Translation Theorem and equations labeled "1.x" refer to the Translation Theorem.

First note that rather than $A^\dagger$, A, $\hat{A}^\dagger$, $\hat{A}$, $\hat{U}^+$, and $\hat{V}^+$ the published version of the Translation Theorem used $T^\dagger$, T, $\hat{T}^\dagger$, $\hat{T}$, $\hat{U}$, and $\hat{V}$. So Equation 3.1 is the same as Equation 1.7. Equation 3.2 implies Equations 1.8 and 1.9. Hence this version is weaker than the published version. We could weaken the hypothesis of this version but for our current uses there is no need.

In the conclusions, Equation 3.3 is the same as Equation 1.10, Equation 3.4 is the same as Equation 1.11, Equation 3.7 is the same as Equation 1.14, and Equation 3.7 is the same as Equation 1.15.

That leaves Equations 3.5 and 3.6. Equations 1.12 and 1.13 are shown true on page 95 of Cholak [2] (lines -13 to -11). (Note in Equation 1.12, the first and only "$\cup$" should be a "$\cap$".) We will start from the middle of page 95 and show that Equations (3.5) and (3.6) hold.

Recall g is an onto, one-to-one, computable function from ω to Tr. In [2], $U_e = U_{g(e)}$ and similarly for $\hat{V}^+$, $\hat{U}^+$, and V, while $U^\dagger_{g(e)} = U^\dagger_{|g(e)|}$ and similarly for $\hat{V}^\dagger$, $\hat{U}^\dagger$, and $V^\dagger$. If $g(e) \not\subset f$ then the first clause of Equation (3.6) holds. If $\beta = g(e) \subset f$ and $n = |g(e)|$ then it is enough to show $e = e_n$. (That is, it is enough to show Equation (3.5) holds for n and e.) So rather than showing $\hat{V}^\dagger_n \cap \overline{A} =^* \hat{V}^+_e \cap \overline{A}$ we must show $\hat{V}^+_e \doteq_{\mathcal{R}} \hat{V}^\dagger_n$ and similarly for $\hat{U}^+$ and $\hat{A}$ and we will be done.

By Lemma 2.12 of Cholak [2], the fact that for all x, $\alpha(x,0) = \lambda$ (see Stage 0 of the construction on page 96 of [2]), and if x enters A at stage s then $\alpha(x, s+1) \uparrow$ (see Step 1 on page 97), then, for almost all x, there is a least stage s_β such that either $\alpha(x, s_\beta) \uparrow$ or $\beta \subseteq \alpha(x, s_\beta)$. Let $R = \{x | x \in A_{s_\beta}\}$. R is a computable subset of A. Assume $x \in \overline{R}$ enters $\hat{V}^\dagger_n = \hat{V}^\dagger_{g(e)} = \hat{V}^\dagger_\beta$ at stage s. Let $s' = \max\{s, s_\beta\}$. By Equation (3.1) and the definition of R, $x \notin A_{s'}$ and hence $\beta \subseteq \alpha(x, s')$. Then, by the last clause of $\mathcal{Q}_\alpha$ (on page 95), $x \in \hat{V}^+_{\beta,s'} = \hat{V}^+_{g(e),s'} = \hat{V}^+_{e,s'}$. By $\mathcal{Q}_\alpha$, $\hat{V}^+_e \subseteq \hat{V}^\dagger_n$. Hence $\hat{V}^+_e \doteq_{\mathcal{R}} \hat{V}^\dagger_n$. The proof that $\hat{U}^+_e \doteq_{\mathcal{R}} \hat{U}^\dagger_n$ is similar. □

3.3. **Meeting the hypothesis of the Modified Translation Theorem.** By the hypothesis of Theorem 3.1 and Definition 2.19, we can assume that there are an effective listing of splits of A, $\{S_i\}_{i\leq\omega}$, and a Δ^0_3 set B such that $\{S_i\}_{i\in B}$ generates $\mathcal{B}$ and $\{\hat{S}_i\}_{i\leq\omega}$ is a similar listing of splits of $\hat{A}$ for $\hat{\mathcal{B}}$,

$\hat{B}$, and $\hat{A}$ such that $\Theta(S_i) = \hat{S}_{\Theta(i)}$ and $\Theta^{-1}(\hat{S}_i) = \hat{S}_{\Theta^{-1}(i)}$ is an extendible isomorphism between $\mathcal{B}$ and $\hat{\mathcal{B}}$.

By Lemmas 2.21 and 2.22, we can assume that the split $\emptyset$ and A appears as some S_i and $\breve{S}_i$ for some $i \in B$. Since $\{S_i\}_{i\leq\omega}$ is effective we can assume for all i, $S_{2i+1} = \breve{S}_{2i}$ and that $2i \in B$ iff $2i+1 \in B$. Similarly for $\{\hat{S}_i\}_{i\leq\omega}$ and $\hat{B}$. Without loss, we can assume that $S_{\Theta^{-1}(2e+1)} = \breve{S}_{\Theta^{-1}(2e)}$ and $\hat{S}_{\Theta(2e+1)} = \breve{\hat{S}}_{\Theta(2e)}$. Since $\{S_i\}_{i\leq\omega}$ and $\{\hat{S}_i\}_{i\leq\omega}$ are effective listings of splits, Θ remains Δ^0_3. By Lemma 2.15, we will also assume that for all i, $A \searrow S_i = \emptyset$, for some fixed enumeration of $\{A\}_{s\leq\omega}$. Dually for $\{\hat{S}_i\}_{i\leq\omega}$ and $\hat{A}$.

Furthermore, since at this point we no longer need an effective enumeration of splits, if $2i \notin B$, let $S_{2i} = \emptyset$, $\hat{S}_{\Theta(2i)} = \emptyset$, $S_{2i+1} = A$ (with the enumeration $\{A_{s+1}\}_{s\in\omega}$ so $A^\dagger \searrow S_{2i+1} = \emptyset$) and $\hat{S}_{\Theta(2i+1)} = \hat{A}$ (with the enumeration $\{\hat{A}_{s+1}\}_{s\in\omega}$ so $\hat{A}^\dagger \searrow \hat{S}_{\Theta(2i+1)} = \emptyset$) and dually for $\{\hat{S}_i\}_{i\leq\omega}$ and $\hat{B}$.

We want to, using an oracle for $\mathbf{0}''$, inductively construct an enumeration of the c.e. sets $\{U^\dagger_n\}_{n<\omega}$, $\{\hat{V}^\dagger_n\}_{n<\omega}$, $\{\hat{U}^\dagger_n\}_{n<\omega}$, and $\{V^\dagger_n\}_{n<\nu}$ which meets the two hypotheses of Theorem 3.7. Let $\mathcal{N}_e$ be the set of $(2e+1)$-states ν such that D^A_ν is infinite and $D^{\hat{A}}_\nu$ is infinite, where D^A_ν is measured w.r.t. $\{S_{n,s}\}_{n\leq 2e+1}$ and $\{S_{\Theta^{-1}(n),s}\}_{i\leq 2e+1}$ and $D^{\hat{A}}_\nu$ is measured w.r.t. $\{\hat{S}_{\Theta(n),s}\}_{n\leq 2e+1}$ and $\{\hat{S}_{n,s}\}_{i\leq 2e+1}$, for all $s<\omega$. Determining $\mathcal{N}_e$ is the only place $\mathbf{0}''$ is used.

Let $x \in A_{s+1} - A_s$. Let $\nu = \nu(2e+1, x, s)$ (as measured above). If $\nu \in \mathcal{N}_e$ then let $x \in U^\dagger_{2e,s}$ iff $x \in S_{2e,s}$, $x \in U^\dagger_{2e+1,s}$ iff $x \in S_{2e+1,s}$, $x \in \hat{V}^\dagger_{2e,s}$ iff $x \in S_{\Theta^{-1}(2e),s}$, and $x \in \hat{V}^\dagger_{2e+1,s}$ iff $x \in S_{\Theta^{-1}(2e+1),s}$. We act dually if $\hat{x} \in \hat{A}_{s+1} - \hat{A}_s$. For all s, let $A^\dagger_s = A_s$ and $\hat{A}^\dagger_s = \hat{A}_s$.

Since only finitely much information, mainly $\mathcal{N}_e$, is used in the above construction of the sets $U^\dagger_{2e}$, $U^\dagger_{2e+1}$, $\hat{V}^\dagger_{2e}$, $\hat{V}^\dagger_{2e+1}$, $\hat{U}^\dagger_{2e}$, $\hat{U}^\dagger_{2e+1}$, $V^\dagger_{2e}$, and $V^\dagger_{2e+1}$, these sets are computably enumerable. Hence $\{U^\dagger_{n,s}\}_{n,s<\omega}$, $\{\hat{V}^\dagger_{n,s}\}_{n,s<\omega}$, $\{\hat{U}^\dagger_{n,s}\}_{n,s<\omega}$, and $V^\dagger_{n,s}\}_{n,s<\omega}$, is a $\mathbf{0}''$-enumeration of $\{U^\dagger_n\}_{n<\omega}$, $\{\hat{V}^\dagger_n\}_{n<\omega}$, $\{\hat{U}^\dagger_n\}_{n<\omega}$, and $\{V^\dagger_n\}_{n<\omega}$ satisfying Condition (3.1). By induction on e, we can easily show that for all $(2e+1)$-states ν, $\nu \in \mathcal{N}_e$ iff $D^{A^\dagger}_\nu$ is infinite iff $D^{\hat{A}^\dagger}_\nu$ is infinite, where $D^{A^\dagger}_\nu$ and $D^{\hat{A}^\dagger}_\nu$ are measured as in Theorem 3.7. Therefore Condition (3.2) is satisfied.

Lemma 3.8. *For all e, $U^\dagger_{2e} \doteq_{\mathcal{R}} S_{2e}$, $U^\dagger_{2e+1} \doteq_{\mathcal{R}} S_{2e+1}$, $\hat{V}^\dagger_{2e} \doteq_{\mathcal{R}} S_{\Theta^{-1}(2e)}$, and $\hat{V}^\dagger_{2e+1} \doteq_{\mathcal{R}} S_{\Theta^{-1}(2e+1)}$. For all e, $\hat{U}^\dagger_{2e} \doteq_{\mathcal{R}} \hat{S}_{\Theta(2e)}$, $\hat{U}^\dagger_{2e+1} \doteq_{\mathcal{R}} \hat{S}_{\Theta(2e+1)}$, $V^\dagger_{2e} \doteq_{\mathcal{R}} \hat{S}_{2e}$, and $V^\dagger_{2e+1} \doteq_{\mathcal{R}} \hat{S}_{2e+1}$.*

Proof. Since Θ is an isomorphism between $\mathcal{B}$ and $\hat{\mathcal{B}}$, for each $(2e+1)$-state ν, $\{x : \nu(2e+1, x) = \nu\}$ is noncomputable iff $\{\hat{x} : \hat{\nu}(2e+1, \hat{x}) = \nu\}$ is noncomputable, where $\nu(2e+1, x)$ is measured w.r.t. $\{S_i\}_{i \leq 2e+1}$ and $\{S_{\Theta^{-1}(i)}\}_{i \leq 2e+1}$ and $\nu(2e+1, \hat{x})$ is measured w.r.t. $\{\hat{S}_{\Theta(i)}\}_{i \leq 2e+1}$ and $\{\hat{S}_i\}_{i \leq 2e+1}$.

By our carefully chosen enumerations of splits of A, $\{S_{i,s}\}_{i,s \leq \omega}$, the set $\{x : \nu = \nu(2e+1, x, s) \wedge x \in A_{s+1} - A_s\}$ is noncomputable iff $\{x : \nu = \nu(2e+1, x)\}$ is noncomputable, where $\nu(2e+1, x, s)$ is measured as above. Dually for $\hat{A}$.

Let $\mathcal{A}_e$ be the set of all $(2e+1)$-states ν. For $\nu \in \mathcal{A}_e$, let $S_{2e,\nu}$ be the set $\{x : \nu = \nu(2e+1, x, s) \wedge x \in A_{s+1} - A_s \wedge x \in S_{2e,s}\}$. If $\nu \notin \mathcal{N}_e$ then $S_{2e,\nu}$ is computable. $S_{2e} = \bigsqcup_{\nu \in \mathcal{A}_e} S_{2e,\nu}$. By the above construction, $U^\dagger_{2e} = \bigsqcup_{\nu \in \mathcal{N}_e} S_{2e,\nu}$. Hence $U^\dagger_{2e} \doteq_{\mathcal{R}} S_{2e}$. We can argue similarly for the remaining sets. □

Since Θ is an isomorphism between substructures of $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\hat{A})$, A is noncomputable iff $\hat{A}$ is noncomputable. As we noted shortly after the statement of Theorem 3.1, Theorem 3.1 holds when A and $\hat{A}$ are computable.

3.4. **Constructing the isomorphism Φ.** In the above section we built a $\mathbf{0}''$-enumeration meeting the hypothesis of Theorem 3.7 and satisfying Lemma 3.8. Now apply Theorem 3.7 to this enumeration. Conditions (3.4), (3.7), and (3.8) of Theorem 3.7 are the three conditions in the hypothesis of Soare's original Extension Theorem (see Soare [18] Theorem XV.4.5). Now apply Soare's original Extension Theorem to the enumeration given to us by Theorem 3.7. This gives us the c.e. sets $\{U_n\}_{n<\omega}$, $\{\hat{V}_n\}_{n<\omega}$, $\{\hat{U}_n\}_{n<\omega}$, and $\{V_n\}_{n<\omega}$. The Extension Theorem only adds elements to $\hat{V}^+_n$ to get $\hat{V}_n$ and similarly for $\hat{U}_n$. $\Phi(U_n) = \hat{U}_n$ and $\Phi^{-1}(V_n) = \hat{V}_n$ is an isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$ (see Soare [18] Section XV.4 for details).

By Lemma 3.8, for all n, $U^\dagger_n \doteq_{\mathcal{R}} S_n$ and $\hat{U}^\dagger_n \doteq_{\mathcal{R}} \hat{S}_{\Theta(n)}$. By (3.5) of Theorem 3.7, $U_{e_n} =^* U^\dagger_n$ and $\hat{U}^+_{e_n} \doteq_{\mathcal{R}} \hat{U}^\dagger_n$. Therefore for all n, $U_{e_n} \doteq_{\mathcal{R}} S_n$, $\hat{U}^+_{e_n} \doteq_{\mathcal{R}} \hat{S}_{\Theta(n)}$. Since Θ is an isomorphism, $\Theta(U_{e_n}) =_{\mathcal{R}} \Theta(S_n)$.

By our careful choice of $\{S_i\}_{i<\omega}$ and our modification of Θ in Section 3.3 we have that for all n, $S_{2n} \sqcup S_{2n+1} = A$ and $\hat{S}_{\Theta(2n)} \sqcup \hat{S}_{\Theta(2n+1)} = \hat{A}$. Hence

for all n, $U_{e_{2n}} \sqcup U_{e_{2n+1}} \sqcup R_n = A$ and $\hat{U}^+_{e_{2n}} \sqcup \hat{U}^+_{e_{2n+1}} \sqcup \hat{R}_n = \hat{A}$, for some computable sets R_n and $\hat{R}_n$.

Since Φ is an isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$ and the sets S_{2n} and $U_{e_{2n+1}}$ are disjoint, $\hat{U}_{e_{2n}} - \hat{U}^+_{e_{2n}} \subseteq^* \hat{R}_n$ and $\Phi(S_{2n}) - \hat{U}_{e_{2n}} \subseteq^* \hat{R}_n$. Therefore $\Phi(U_{e_{2n}}) =^* \hat{U}_{e_{2n}} =_{\mathcal{R}} \hat{S}_{\Theta(2n)}$ and $\Phi(S_{2n}) =_{\mathcal{R}} \Phi(U_{e_{2n}})$. So $\Phi(S_{2n}) =_{\mathcal{R}} \Theta(S_{2n})$. We argue similarly to show $\Phi(S_{2n+1}) =_{\mathcal{R}} \Theta(S_{2n+1})$ and $\Phi^{-1}(\hat{S}_n) =_{\mathcal{R}} \Theta^{-1}(\hat{S}_n)$. □

4. Extensions to automorphisms

Our goal to find an algebraic extension theorem which allows us to find an automorphism Λ of $\mathcal{E}$ taking A to $\hat{A}$ if and when possible. Clearly we will have to add some extra hypotheses to Theorem 3.1 about the outside of A and $\hat{A}$.

Recall that $\mathcal{L}^*(A)$ is the structure $(\{W_e \cup A : e \in \omega\}, \subseteq)$ modulo the finite sets. A substructure $\mathcal{L}$ of $\mathcal{L}^*(A)$ is a subcollection of the sets $(\{W_e \cup A : e \in \omega\}, \subseteq)$ modulo the finite sets. An isomorphism between $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$ is a one-to-one, onto (both of these items are in terms of $*$-equivalence classes) function Ξ from $\{W_e : e \in \omega\}$ to $\{\hat{W}_e : e \in \omega\}$ such that $W_e \cup A \subseteq^* W_i \cup A$ iff $\Xi(W_e \cup \hat{A}) \subseteq^* \Xi(W_i \cup \hat{A})$. Note that Ξ is applied to $W \cup A$.

Assume that $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$ are isomorphic via Ψ and that $\mathcal{B}$ and $\hat{\mathcal{B}}$ are Δ^0_3 isomorphic via Θ. We wish to use the isomorphism Φ from Theorem 3.1 to extend this pair of isomorphisms into an automorphism Λ of $\mathcal{E}$ such that $\Lambda(A) = \hat{A}$.

Notice that $W = (W - A) \sqcup (W \cap A)$. It would be nice to define $\Lambda(W) = (\Psi(W \cup A) - \hat{A}) \sqcup \Phi(W \cap A)$. Clearly this is order preserving. But why is $(\Psi(W \cup A) - \hat{A}) \sqcup \Phi(W \cap A)$ a computably enumerable set? To answer that we must explore more carefully the complex relation between $\mathcal{L}^*(A)$ and $\mathcal{B}$.

Definition 4.1. *S supports X* iff $S \subseteq X$ and $(X - A) \sqcup S$ is a computably enumerable set.

Lemma 4.2. *Whether S supports X is Σ^0_3.*

Proof. S supports X iff there exists an e where $W_e = (X - A) \sqcup S$ and $S \subseteq X$. □

Lemma 4.3. *$W \searrow A$ supports W.*

Proof. $W = (W - A) \sqcup (W \searrow A) \sqcup (A \searrow W)$ and $(W - A) \sqcup (W \searrow A)$ is the computably enumerable set $W \backslash A$. □

Definition 4.4. An extendible subalgebra $\mathcal{B}$ *supports* $\mathcal{L}$ if for all $W \in \mathcal{L}$ there an $i \in B$ such that S_i supports W.

Lemma 4.5. *$\mathcal{E}_A$ supports $\mathcal{L}^*(A)$.*

Lemma 4.6. *If S supports X and T is a split of A such that $T \subseteq S$ and $S =_{\mathcal{R}(A)} T$ then T supports X.*

Proof. $(X-A)\sqcup S$ is a computably enumerable set. If $S-T$ is a computable set R then $(X-A)\sqcup T = (((X-A)\sqcup S)\cap \overline{R})$ is a computably enumerable set. □

Definition 4.7. Assume that

- $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$ are isomorphic via Ψ,
- $\mathcal{B}$ and $\hat{\mathcal{B}}$ are isomorphic via Θ,
- $\mathcal{B}$ supports $\mathcal{L}$, and
- $\hat{\mathcal{B}}$ supports $\hat{\mathcal{L}}$.

Then the isomorphisms Ψ and Θ *preserve* the supports of $\mathcal{L}$ and $\hat{\mathcal{L}}$ if

- for $W^* \in \mathcal{L}$, there is an $i \in B$ such that S_i supports W and $(\Psi(W \cup A) - \hat{A}) \sqcup \Theta(S_i)$ is a computably enumerable set, and
- for all $\hat{W}^* \in \hat{\mathcal{L}}$, there is an $i \in \hat{B}$ such that $\hat{S}_i$ supports $\hat{W}$ and $(\Psi^{-1}(\hat{W} \cup \hat{A}) - A) \sqcup \Theta^{-1}(\hat{S}_i)$ is a computably enumerable set.

For shorthand we just say isomorphisms Ψ and Θ *preserve supports*.

If S_i supports W then $S_i \subseteq W$. But if isomorphisms Ψ and Θ preserve supports, then, while $(\Psi(W \cup A) - \hat{A}) \sqcup \Theta(S_i)$ is a computably enumerable set, we do not require that $\Theta(S_i)$ be contained in $\Psi(W)$. Hence $\Theta(S_i)$ might not be a support of $\Psi(W)$.

Theorem 4.8. *Assume that*

(1) *$\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$ are isomorphic via Ψ,*
(2) *$\mathcal{B}$ and $\hat{\mathcal{B}}$ are extendible algebras which are extendibly Δ^0_3 isomorphic via Θ,*
(3) *$\mathcal{B}$ supports $\mathcal{L}^*(A)$,*
(4) *$\hat{\mathcal{B}}$ supports $\mathcal{L}^*(\hat{A})$,*
(5) *Ψ and Θ preserves supports,*
(6) *Φ is an isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$ such that if $i \in B$ then $\Theta(S_i) =_{\mathcal{R}} \Phi(S_i)$ and if $i \in \hat{B}$ then $\Theta^{-1}(\hat{S}_i) =_{\mathcal{R}} \Phi^{-1}(\hat{S}_i)$.*

Then $\Lambda(W) = (\Psi(W \cup A) - \hat{A}) \sqcup \Phi(W \cap A)$ is an automorphism of $\mathcal{E}$ taking A to $\hat{A}$.

Proof. It is enough to show that $(\Psi(W \cup A) - \hat{A}) \sqcup \Phi(W \cap A)$ is a computably enumerable set. First note that $W \cap A = S_i \sqcup (\breve{S}_i \cap W)$, where

S_i supports W and $i \in B$. Since Φ is an isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$, $\Phi(W \cap A) = \Phi(S_i) \sqcup \Phi(\breve{S}_i \cap W)$. Since Ψ and Θ preserve supports, for some support S_i of W, $(\Psi(W \cup A) - \hat{A}) \sqcup \Theta(S_i)$ is a computably enumerable set. Since $\Theta(S_i) =_{\mathcal{R}} \Phi(S_i)$, $(\Psi(W \cup A) - \hat{A}) \sqcup \Phi(S_i)$ is a computably enumerable set. Hence $(\Psi(W \cup A) - \hat{A}) \sqcup \Phi(S_i) \sqcup \Phi(\breve{S}_i \cap W)$ is a computably enumerable set. Similarly we can show $\Lambda^{-1}(\hat{W})$ is a computably enumerable set. □

Theorem 4.9. *Assume that*

(1) $\mathcal{L}^*(A)$ *and* $\mathcal{L}^*(\hat{A})$ *are isomorphic via* Ψ,
(2) $\mathcal{B}$ *and* $\hat{\mathcal{B}}$ *are extendible algebras which are extendibly* Δ^0_3 *isomorphic via* Θ,
(3) $\mathcal{B}$ *supports* $\mathcal{L}^*(A)$,
(4) $\hat{\mathcal{B}}$ *supports* $\mathcal{L}^*(\hat{A})$,
(5) Ψ *and* Θ *preserve supports.*

Then there is an automorphism Λ *of* $\mathcal{E}$ *such that* $\Lambda(A) = \hat{A}$, $\Lambda \restriction \mathcal{L}^*(A) = \Psi$, *and* $\Lambda \restriction \mathcal{E}^*(A)$ *is* Δ^0_3.

Proof. Apply Theorem 3.1 to get Φ as required by Theorem 4.8.6. Φ is Δ^0_3. Apply Theorem 4.8 to get Λ. □

The way we put together the automorphism in Theorem 4.9 is very similar to the way in which Herrmann showed that the Herrmann sets (along with the hemimaximal sets and other such orbits) form an orbit (see Cholak et al. [4, Sections 5 and 6]). Both methods are algebraic or "static".

In Section 6, we will show that Theorem 4.9 can be improved to be an "if and only if' statement (see Theorem 6.4).

5. Preserving the computable subsets

Definition 5.1. A map Ξ from a substructure of $\mathcal{G} \subseteq \mathcal{E}(A)$ to $\hat{\mathcal{G}} \subseteq \mathcal{E}(\hat{A})$ *preserves the computable subsets* if $R \in \mathcal{R}(A) \cap \mathcal{G}$ iff $\Xi(R) \in \mathcal{R}(\hat{A}) \cap \hat{\mathcal{G}}$.

There is no guarantee that any of the maps we have been considering preserves the computable subsets; this includes Θ. And the same can be said about Soare's original Extension Theorem (see Soare [18, XV.4.5]) (applied by itself). To see this: If $X \in \mathcal{R}(A)$ and Θ is an isomorphism Θ between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$, then there is a Y such that $X \sqcup Y = A$ and $\Theta(X) \sqcup \Theta(Y) = \hat{A}$ but there may not be a Z such that $\Theta(X) \sqcup Z = \hat{\omega}$. Of course, there is such a Z if $\hat{A}$ is computable (and dually if A is computable).

It might be useful to consider the following example: If A and $\hat{A}$ are infinite then there is an effective isomorphism Ψ between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$

(let f be an effective map from A to $\hat{A}$ and let $\Psi(W) = f(W)$). If A is computable but $\hat{A}$ is not then Ψ cannot preserve the computable subsets.

From this point on we will always consider A and $\hat{A}$ to be noncomputable. We will point out that it is known that there is an isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$ which preserves the computable subsets (see Theorem 5.3). The goal of this section is to provide another proof of fact using our methods.

Definition 5.2. $\mathcal{C}(A)$ is the set of W_e such that either $\overline{A} \subseteq W_e$ or $W_e \subseteq^* A$.

Theorem 5.3 (Soare's Automorphism Theorem [17])**.** *Let A and $\hat{A}$ be two noncomputable computably enumerable sets.*

(1) *Then there is a Δ^0_3 isomorphism Λ between $\mathcal{E}(A) \cup \mathcal{C}(A)$ and $\mathcal{E}(\hat{A}) \cup \mathcal{C}(\hat{A})$. Furthermore a Δ^0_3-index for Λ can be found uniformly from indexes for A and $\hat{A}$.*
(2) *In addition, Λ preserves the computable subsets of A.*

Soare [17] explicitly stated Theorem 5.3.1. Soare's result that maximal sets are automorphic follows since A is maximal iff $\mathcal{C}(A) = \mathcal{E}^*$.

Theorem 5.3.2 was observed, in unpublished work, by Herrmann. Assume that R is a computable subset of A. Herrmann's observation was that $\overline{R} \in \mathcal{C}(A)$ and hence $\Lambda(R) \sqcup \Lambda(\overline{R}) =^* \hat{\omega}$ and therefore Λ maps R to a computable subset of $\hat{A}$. This observation of Herrmann was never published and is one of the key facts he used in showing that the Herrmann sets form an orbit; see Cholak et al. [4].

5.1. **Another proof of Theorem 5.3.** We would like to show Theorem 5.3 using the methods of this paper.

First note that an isomorphism Λ between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$ preserving the computable subsets induces an isomorphism Λ' between $\mathcal{E}^*(A) \cup \mathcal{C}(A)$ and $\mathcal{E}^*(\hat{A}) \cup \mathcal{C}(\hat{A})$ taking A to $\hat{A}$. If $\overline{A} \subseteq W$ then $A \cup W = \omega$ and there is a computable set $R \subseteq A$ ($R = A \backslash W$) such that $\overline{R} \subseteq W$ which implies $W = \overline{R} \sqcup (W \cap R)$. So for $W \in \mathcal{C}(A)$, let $\Lambda'(W)$ be $\overline{\Lambda(R)} \sqcup (\Lambda(W \cap R))$.

We would like to prove a theorem along the lines of Theorem 4.8.

Theorem 5.4. *Assume that*

(1) *$\mathcal{B}$ and $\hat{\mathcal{B}}$ are extendible algebras which are Δ^0_3 extendibly isomorphic via Θ;*
(2) *for all $R \in \mathcal{R}(A)$, there is an $i \in B$ such that S_i is computable and $R \subseteq S_i$;*
(3) *for all $\hat{R} \in \mathcal{R}(\hat{A})$, there is an $i \in \hat{B}$ such that $\hat{S}_i$ is computable and $\hat{R} \subseteq \hat{S}_i$;*

(4) *for all* $i \in B$, $\Theta(S_i)$ *is computable iff* S_i *is computable and for all* $i \in \hat{B}$, $\Theta^{-1}(\hat{S}_i)$ *is computable iff* $\hat{S}_i$ *is computable.*

Then there is a Λ *such that* Λ *is a* Δ^0_3 *isomorphism between* $\mathcal{E}^*(A)$ *and* $\mathcal{E}^*(\hat{A})$ *which preserves the computable subsets, for all* $i \in B$, $\Lambda(S_i) =_{\mathcal{R}} \Theta(S_i)$, *and if* $i \in \hat{B}$, *then* $\Lambda^{-1}(\hat{S}_i) =_{\mathcal{R}} \Theta^{-1}(\hat{S}_i)$.

Proof. First apply Theorem 3.1 to get Φ. We will show that Φ is the desired isomorphism Λ. It is enough to show Φ preserves the computable subsets.

Let $R \in \mathcal{R}(A)$. There is an i such that S_i is computable and $R \subseteq S_i$. $\Theta(S_i)$ is computable. By Theorem 3.1, $\Phi(S_i) \equiv_{\mathcal{R}} \Theta(S_i)$. Hence $\Phi(S_i)$ is computable. Therefore, since the set $A - R$ is c.e., the set $\overline{\Phi(R)} =^* \overline{\Phi(S_i)} \sqcup \Phi(S_i \cap (A - R))$ is computably enumerable and $\Phi(R)$ is computable. The other direction is similar. □

It is actually reasonably easy to meet the hypothesis of the above theorem; it is enough that A and $\hat{A}$ both be noncomputable.

Theorem 5.5. *Let* A *and* $\hat{A}$ *be two noncomputable computably enumerable sets. Then there are* $\mathcal{B}$ *and* $\hat{\mathcal{B}}$ *such that*

(1) $\mathcal{B}$ *and* $\hat{\mathcal{B}}$ *are extendible algebras which are* Δ^0_3 *extendibly isomorphic via* Θ*;*
(2) *for all* $R \in \mathcal{R}(A)$, *there is an* $i \in B$ *such that* S_i *is computable and* $R = S_i$*;*
(3) *for all* $\hat{R} \in \mathcal{R}(\hat{A})$, *there is an* $i \in \hat{B}$ *such that* $\hat{S}_i$ *is computable and* $\hat{R} = \hat{S}_i$*;*
(4) *for all* $i \in B$, $\Theta(S_i)$ *is computable iff* S_i *is computable and for all* $i \in \hat{B}$, $\Theta^{-1}(\hat{S}_i)$ *is computable iff* $\hat{S}_i$ *is computable.*

Proof. Apply Theorem 2.17 and its dual to get $\mathcal{B}$ and $\hat{\mathcal{B}}$. Now both B and $\hat{B}$ are infinite and Δ^0_3. We will inductively define θ. If $i + 1 \in B$, let $\theta(i + 1)$ be the least element of $\hat{B}$ which is not yet in the range of θ. Otherwise $\theta(i + 1)$ is undefined. Let $\Theta(S_i) = \hat{S}_{\theta(i)}$. Similarly for Θ^{-1}. Clearly Θ is Δ^0_3.

Since everything in B and $\hat{B}$ are computable splits of A, $\mathcal{B}$ and $\hat{\mathcal{B}}$ are classically isomorphic to the trivial Boolean algebra. Therefore Θ induces an isomorphism between $\mathcal{B}$ and $\hat{\mathcal{B}}$. Hence Θ is clearly the desired extendible isomorphism. □

By combining Theorems 5.4 and 5.5 we get another proof of Theorem 5.3.

5.2. **Some examples of the use of Theorem 5.3.**

5.2.1. *The hemimaximal sets.* We include this example as it has not appeared previously in print in this form and it hints of things to come in later sections. Assume $A_1 \sqcup A_2 = A$ where the A_is are not computable. Dually for $\hat{A}$. Assume that Θ_i is an isomorphism from $\mathcal{E}^*(A_i)$ to $\mathcal{E}^*(\hat{A}_i)$ that preserves the computable subsets (from Theorem 5.3).

As with the maximal sets, it is enough to define an isomorphism Λ between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$ preserving the computable subsets. If $X \subseteq^* A$ then let $\Lambda(X) = \Theta_1(X \cap A_1) \sqcup \Theta_2(X \cap A_2)$. Let $R \in \mathcal{R}(A)$. Then $R \cap A_i$ is computable. So $\Theta_i(R \cap A_i)$ is computable. Hence $\Theta_1(R \cap A_1) \sqcup \Theta_2(R \cap A_2)$ is computable. The complexity of the resulting automorphism is Δ^0_3.

Downey and Stob's proof used the fact that if $W \cup A = \omega$ then $W \searrow A_i$ is infinite: a very dynamic property. Our proof only relies on algebraic facts.

5.2.2. *The atomless Boolean Algebra $\mathcal{S}_{\mathcal{R}}(A)$.* As we know, all atomless Boolean Algebras are isomorphic but with $\mathcal{S}_{\mathcal{R}}(A)$ something stronger is true.

Theorem 5.6 (Nies, see Cholak and Harrington [6])**.** *If A and $\hat{A}$ are non-computable, then $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\hat{A})$ are Δ^0_3 isomorphic.*

Proof. The isomorphism Λ, from Theorem 5.3, is an isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$ preserving the computable sets. Hence Λ induces an isomorphism between $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\hat{A})$. □

5.3. **Extendible Algebras of Computable Sets.** This section was added after the rest of the paper was completed. As we mentioned in the Introduction (third to last paragraph) and last sentence, this paper has a sequel. The goal of this section is to provide a clear, clean interface between the two papers. In particular, we will proof a theorem, Theorem 5.10, which we hope we can use as a black box in the sequel.

Theorem 5.10 is an improved version of Theorem 5.3. In Theorem 5.3 the computable sets are preserved. In Theorem 5.10 the computable sets are preserved plus an external isomorphism determines where some of the computable sets are mapped.

Definition 5.7. An extendible algebra $\mathcal{B}$ of $\mathcal{S}_{\mathcal{R}}(\omega)$ is called an *extendible algebra of computable sets* as the splits of ω are the computable sets.

Lemma 5.8. *If $\mathcal{B} = \{R_i : i \in B\}$ is an extendible algebra of computable sets then $\mathcal{B}_A = \{R_i \cap A : i \in B\}$ is an extendible algebra of $\mathcal{S}_{\mathcal{R}}(A)$.*

Proof. $\{\tilde{R}_i \cap A : i \in \omega\}$ witnesses that $\{R_i \cap A : i \in \omega\}$ is an effective listing of splits of A. □

Lemma 5.9. *Assume that $\mathcal{B}$ and $\hat{\mathcal{B}}$ are extendible subalgebras of computable sets which are extendibly isomorphic via Π. $\Pi_A(R\cap A) = \Pi(R)\cap\hat{A}$ is an extendible isomorphism between $\mathcal{B}_A$ and $\mathcal{B}_{\hat{A}}$.*

Theorem 5.10. *Let $\mathcal{B}$ be a extendible algebra of computable sets and similarly for $\hat{\mathcal{B}}$. Assume the two are extendibly isomorphic via Π. Then there is a Φ such that Φ is a Δ^0_3 isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$, Φ maps computable subsets to computable subsets, and, for all $R \in \mathcal{B}$, $(\Pi(R) - \hat{A}) \sqcup \Phi(R \cap A)$ is computable (and dually).*

Proof. Apply Lemmas 2.16 and 2.22 to $\mathcal{B}_A$, $\hat{\mathcal{B}}_{\hat{A}}$, Π_A, and the extendible algebras and extendible isomorphism from Theorem 5.5 to get $\tilde{\mathcal{B}}$, $\hat{\tilde{\mathcal{B}}}$ and $\tilde{\Theta}$. Now apply Theorem 3.1 to get Φ. By the proof of Theorem 5.4, Φ preserves the computable sets.

Since Π is an isomorphism between extendible algebras of computable sets, $\Pi(R)$ is a computable set. By Theorem 3.1, $\Theta(R \cap A)\triangle\Pi_A(R) = R_0$ is a computable subset of $\hat{A}$. Since $\Theta(R\cap A)$ is a split of $\hat{A}$, $\Theta(R\cap A)\cap R_0 = R_1$ is a computable subset of $\hat{A}$. Similarly, $\Pi_A(R) \cap R_0 = R_2$ is a computable subset of $\hat{A}$. So $\Phi(R \cap A) = (\Pi_A(R) \sqcup R_1) \cap \overline{R}_2$. Hence

$$\begin{aligned}(\Pi(R) \sqcup R_1) \cap \overline{R}_2 =& \big((\Pi(R) - \hat{A}) \sqcup \Pi_A(R) \sqcup R_1\big) \cap \overline{R_2}\\ =&(\Pi(R) - \hat{A}) \sqcup \Phi(R \cap A).\end{aligned}$$

So $(\Pi(R) - \hat{A}) \sqcup \Phi(R \cap A)$ is computable as desired. The dual is proved in a similar fashion. □

6. Automorphisms back to automorphisms

Assume that A and $\hat{A}$ are automorphic via Ψ. Hence $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$ are isomorphic via Ψ. Since A and $\hat{A}$ are automorphic, the structures $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\hat{A})$ are isomorphic structures (since they are definable structures). In fact, from Cholak and Harrington [6], we know much more is true.

Theorem 6.1 (**The Restriction Theorem**; Theorem 1.2 of Cholak and Harrington [6])**.** *If A and $\hat{A}$ are automorphic via Ψ then the structures $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\hat{A})$ are Δ^0_3-isomorphic structures via an isomorphism Γ induced by Ψ.*

In other words there is an isomorphism Γ between $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\hat{A})$ such that for all splits of A, $\Gamma(S) =_{\mathcal{R}} \Psi(S)$; for all splits $\hat{S}$ of $\hat{A}$, $\Gamma^{-1}(\hat{S}) =_{\mathcal{R}} \Psi^{-1}(\hat{S})$; and a Δ^0_3-function f such that for $W_e \in \mathcal{S}(A)$, $W_{f(e)} =_{\mathcal{R}} \Gamma(W_e)$. (For more about this theorem we direct the reader to Cholak and Harrington [6].)

Theorem 6.2. *Assume A and $\hat{A}$ are automorphic via Ψ. Let $\tilde{\mathcal{B}}$ be an extendible algebra (of $\mathcal{S}_{\mathcal{R}}(A)$). Then there are extendible $\hat{\mathcal{B}}$ (of $\mathcal{S}_{\mathcal{R}}(\hat{A})$) and Θ such that*

(1) *$\hat{\mathcal{B}}$ and $\tilde{\mathcal{B}}$ are extendibly Δ^0_3-isomorphic via Θ,*
(2) *if $i \in \tilde{B}$ and S_i supports W then $\Theta(S_i)$ supports $\Psi(W)$.*

The proof of this theorem appears in Section 6.1. We should note that we must argue dynamically in this proof. We can use this result to show the following theorem.

Theorem 6.3 (The Conversion Theorem). *If A and $\hat{A}$ are automorphic via Ψ then they are automorphic via Λ where $\Lambda \upharpoonright \mathcal{L}^*(A) = \Psi$ and $\Lambda \upharpoonright \mathcal{E}^*(A)$ is Δ^0_3.*

Proof. $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$ are isomorphic via Ψ. Recall from Lemma 2.14, $\mathcal{E}_A$ is the extendible algebra generated by the entry sets. Recall from Lemma 4.5, $\mathcal{E}_A$ supports $\mathcal{L}^*(A)$. Apply Theorem 6.2 to $\mathcal{E}_A$ to get $\hat{\mathcal{E}}_A$ and Θ_A and dually to $\mathcal{E}_{\hat{A}}$ to get $\hat{\mathcal{E}}_{\hat{A}}$ and $\Theta_{\hat{A}}$. By Lemmas 2.16 and 2.22, $\mathcal{B} = \mathcal{E}_A \oplus \hat{\mathcal{E}}_{\hat{A}}$ and $\hat{\mathcal{B}} = \hat{\mathcal{E}}_A \oplus \mathcal{E}_{\hat{A}}$ are extendible algebras Δ^0_3-isomorphic via Θ. Since $\mathcal{E}_A$ supports $\mathcal{L}^*(A)$, $\mathcal{B}$ does too. Similarly for $\hat{\mathcal{B}}$ and $\mathcal{L}^*(\hat{A})$. By the last property of Theorem 6.2, isomorphisms Ψ and Θ preserve supports. Now apply Theorem 4.9. □

Also using Theorem 4.9 we can algebraically describe an orbit of A.

Theorem 6.4. *The computably enumerable sets A and $\hat{A}$ are automorphic iff there are Ψ, $\mathcal{B}$, $\hat{\mathcal{B}}$, and Θ such that*

(1) *$\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$ are isomorphic via Ψ,*
(2) *$\mathcal{B}$ and $\hat{\mathcal{B}}$ are extendible algebras which are extendibly Δ^0_3 isomorphic via Θ,*
(3) *$\mathcal{B}$ supports $\mathcal{L}^*(A)$,*
(4) *$\hat{\mathcal{B}}$ supports $\mathcal{L}^*(\hat{A})$,*
(5) *the isomorphisms Ψ and Θ preserve supports.*

6.1. **Proof of Theorem 6.2.** To make life notationally easier we will prove the dual. So let $\tilde{\mathcal{B}}$ be an extendible algebra of $\mathcal{S}_{\mathcal{R}}(\hat{A})$ and we will build $\mathcal{B}$.

By Theorem 6.1, $\tilde{\mathcal{B}}$ and $\Gamma^{-1}(\tilde{\mathcal{B}})$ are Σ^0_3 algebras which are Δ^0_3 isomorphic via Γ^{-1}. But Δ^0_3 images and preimages of extendible algebras need not be extendible. Hence we cannot let $\mathcal{B} = \Gamma^{-1}(\tilde{\mathcal{B}})$. We will construct $\mathcal{B}$ to be extendible and extendibly isomorphic to $\tilde{\mathcal{B}}$ via Θ (and hence isomorphic to $\Gamma^{-1}(\tilde{\mathcal{B}})$). In fact we are going to show something stronger; we will show $\mathcal{E}_A \oplus \mathcal{B}$ is isomorphic to $\Gamma(\mathcal{E}_A) \oplus \tilde{\mathcal{B}}$.

We are going to construct $\mathcal{B}$ and Θ via a standard tree agreement. We will construct a tree, Tr. At each node α of the tree, we will construct the splits of A, S_α and $\check{S}_\alpha$. We are going to build these splits as entry sets. So for all α, if x enters A at stage $s+1$ then x enters either S_α or $\check{S}_\alpha$ at stage s.

The list $\{S_\alpha\}_{\alpha \in Tr}$ is an effective listing of splits. $B = \{\alpha | \alpha \subset f \wedge |\alpha| \in \tilde{B}\}$ is a Δ^0_3 set. So an extendible algebra, $\mathcal{B}$, is created.

If $i \in \tilde{B}$ then let $\Theta(S_\alpha) = \tilde{S}_i$ and $\Theta^{-1}(\tilde{S}_i) = S_\alpha$, where $\alpha \subset f$ and $|\alpha| = i$. If we can show Θ induces an isomorphism between $\mathcal{B}$ and $\tilde{\mathcal{B}}$ then Θ will be a Δ^0_3-extendible isomorphism between $\mathcal{B}$ and $\tilde{\mathcal{B}}$. Hence without loss we can assume that if $i \notin \tilde{B}$ then $\tilde{S}_i = \emptyset$ and $\Gamma^{-1}(\tilde{S}_i) = \emptyset$.

For the rest of this proof we will use *e-splits states* rather than *e*-states.

Definition 6.5. (1) For any e, if we are given a uniform enumeration of splits of A $\{S_{i,s}\}_{i \leq e, s<\omega}$, $\{\check{S}_{i,s}\}_{i \leq e, s<\omega}$, $\{T_{i,s}\}_{i \leq e, s<\omega}$, and $\{\check{T}_{i,s}\}_{i \leq e, s<\omega}$ define the *e-split state of x at stage s*, $\nu^S(e, x, s)$, to be the full $2e$-state of x w.r.t. $\{X_{i,s}\}_{i \leq 2e, s<\omega}$ and $\{Y_{i,s}\}_{i \leq 2e, s<\omega}$, where $X_{2i,s} = S_{i,s}$, $X_{2i+1,s} = \check{S}_{i,s}$, $Y_{2i,s} = T_{i,s}$, and $Y_{2i+1,s} = \check{T}_{i,s}$.

(2) Let $\nu^S(\alpha, x, s) = \nu^S(e, x, s)$ where $|\alpha| = e$ and $\nu^S(e, x, s)$ is measured w.r.t. $\{(W_i \searrow A)_s\}_{i \leq e, s<\omega}$, $\{(A \backslash W_i)_s\}_{i \leq e, s<\omega}$, $\{S_{\beta,s}\}_{\beta \subseteq \alpha, s<\omega}$, and $\{\check{S}_{\beta,s}\}_{\beta \subseteq \alpha, s<\omega}$.

(3) For any collection of splits of A, $\{S_i\}_{i \leq e}$ and $\{T_i\}_{i \leq e}$, define the *final e-split state of x* to be the final full $2e$-state of x w.r.t. $\{X_i\}_{i \leq 2e}$ and $\{Y_i\}_{i \leq 2e}$, where $X_{2i} = S_i$, $X_{2i+1} = \check{S}_i$, $Y_{2i} = T_i$, and $Y_{2i+1} = \check{T}_i$.

(4) Let $\nu^S(e, x)$ be the final e-split state of x measured w.r.t. $\{W_i \searrow A\}_{i \leq e}$ and $\{\Gamma^{-1}(\tilde{S}_i)\}_{i \leq e}$. Let $\hat{\nu}^S(e, \hat{x})$ be the final e-split state of $\hat{x}$ measured w.r.t. $\{\Gamma(W_i \searrow A)\}_{i \leq e}$ and $\{\tilde{S}_i\}_{i \leq e}$.

(5) Let $\nu^S(\alpha, x)$ be the final $|\alpha|$-split state of x measured w.r.t. $\{(W_i \searrow A)\}_{i \leq e}$ and $\{S_\beta\}_{\beta \subseteq \alpha}$. (Careful—this is not the same as $\nu^S(|\alpha|, x)$.)

(6) Every $2e$-state is an *e-split state* and $\nu = \langle 2e, \sigma, \tau \rangle$ is a *reasonable e-split state* if for all $i \leq e$, exactly one of $2i$ or $2i+1$ is in σ, and exactly one of $2i$ or $2i+1$ is in τ.

(7) For every e-split state ν and α such that $|\alpha| = e$, let

$$D^A_{\nu,\alpha} = \{x : \exists s \text{ such that } x \in A_{s+1} - A_s \text{ and } \nu = \nu^S(e, x, s) \text{ w.r.t. } \{(W_i \searrow A)_s\}_{i \leq e, s<\omega}, \{(A \backslash W_i)_s\}_{i \leq e, s<\omega}, \{S_{\beta,s}\}_{\beta \subseteq \alpha, s<\omega}, \text{ and } \{\check{S}_{\beta,s}\}_{\beta \subseteq \alpha, s<\omega}.$$

Let ν be a reasonable e-split state. Then $X_\nu = \{x | \nu^S(e, x) = \nu\}$ is a Boolean combination of splits of A and hence X_ν is also a split of A. Γ is an isomorphism between $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\hat{A})$ (modulo the computable subsets of A). Hence Γ is an isomorphism between $\mathcal{E}_A \oplus \Gamma^{-1}(\tilde{\mathcal{B}})$ and

$\Gamma(\mathcal{E}_A) \oplus \tilde{\mathcal{B}}$ (again modulo the computable subsets of A). Therefore, X_ν is computable iff $\Gamma(X_\nu)$ is computable. So, for all reasonable e-split states ν, $\{x|\nu^S(e,x) = \nu\}$ is computable iff $\{\hat{x}|\hat{\nu}^S(e,\hat{x}) = \nu\}$ is computable.

Since S_α are entry sets, $x \in D^A_{\nu,\alpha}$ iff $\nu^S(\alpha, x) = \nu$. Therefore $\{x : \nu^S(\alpha, x) = \nu$ is computable iff $D^A_{\nu,\alpha}$ is computable.

By Lemma 2.20, to show $\mathcal{B}$ is isomorphic via Θ to $\tilde{\mathcal{B}}$ it is enough to show, for all β, γ, $S_\beta - S_\gamma$ is computable iff $\Theta(S_\beta) - \Theta(S_\gamma) = \tilde{S}_{|\beta|} - \tilde{S}_{|\gamma|}$ is computable. Let α be the longer of β and γ. Then

$$S_\beta - S_\gamma = \bigsqcup\{D^A_{\nu,\alpha} : \nu = \langle 2|\alpha|, \sigma, \tau\rangle, 2|\beta| \in \tau, \text{ and } 2|\gamma| \notin \tau\}.$$

Therefore, it is more than enough to show, for all reasonable e-split states ν and all $\alpha \subset f$, if $|\alpha| = e$ then $D^A_{\nu,\alpha}$ is computable iff $\{x|\nu^S(e,x) = \nu\}$ is computable.

Hence from this point forward we will just work on constructing S_α and $\breve{S}_\alpha$ such that for all reasonable e-split states ν and all $\alpha \subset f$, if $|\alpha| = e$ then

$\mathcal{R}_\nu$ $\quad D^A_{\nu,\alpha}$ is computable iff $\{x|\nu^S(e,x) = \nu\}$ is computable.

(Let $\Theta(W_i \searrow A) = \Gamma(W_i \searrow A)$ and $\Theta^{-1}(\Gamma(W_i \searrow A)) = W_i \searrow A$. Then almost the same argument shows that Θ is an isomorphism between $\mathcal{E}_A$ and $\Gamma(\mathcal{E}_A)$ and, in fact, $\mathcal{E}_A \oplus \mathcal{B}$ is isomorphic via Θ to $\Gamma(\mathcal{E}_A) \oplus \tilde{\mathcal{B}}$.)

If we succeed in meeting $\mathcal{R}_\alpha$ then Θ will be an isomorphism as desired. As we will see it turns out to do this it enough to know for which for all reasonable e-splits states and α, $\{x|\nu^S(e,x) = \nu\}$ is infinite.

Determining whether $\{x|\nu^S(e,x) = \nu\}$ is infinite is Δ^0_3: Are there i_k and j_k, for $k \leq e$, and infinitely many x and stages s such that for all $k \leq e$, $\Gamma^{-1}(\tilde{S}_k) = W_{i_k}$, $\breve{W}_{i_k} = W_{j_k}$, $x \in W_{i_k,s} \sqcup W_{j_k,s}$, and $\nu^S(e,x,s) = \nu$, where $\nu^S(e,x,s)$ is measured w.r.t. $\{(W_k \searrow A)_s\}_{k\leq e, s<\omega}$, $\{(A\backslash W_k)_s\}_{i\leq e, s<\omega}$, $\{W_{i_k,s}\}_{k\leq e, s<\omega}$, and $\{W_{j_k,s}\}_{k\leq e, s<\omega}$. Recall Γ is Δ^0_3 and since we know $S = \Gamma^{-1}(\tilde{S}_k)$ is a split of A we can find $\breve{S}$ using an oracle for $\mathbf{0}''$. This also shows that $\{x|\nu^S(e,x) = \nu\}$ is a computably enumerable set and a split of A.

Hence it is straightforward to construct a tree Tr, with a true path f and an approximation f_s to f such that $f = \liminf_s f_s$, if $\alpha \in Tr$ then α is outfitted with a set of reasonable $|\alpha|$-split states, $\mathcal{M}_\alpha$, and if $\alpha \subset f$ then $\nu \in \mathcal{M}_\alpha$ iff $\{x|\nu^S(e,x) = \nu\}$ is infinite. Furthermore we can assume that if $\beta \subset \alpha$ and $\nu \in \mathcal{M}_\alpha$ then $\nu \upharpoonright 2|\beta| \in \mathcal{M}_\beta$ and that $|f_s| = s$, for all s. In the interest of space and energy we are not going to go into the details. Similar constructions with all the details can be found in Section 7.2.6, Cholak [3], Cholak [2], and Weber [20].

Using the approximation to the true path we will construct a function $\alpha(x,s)$ for all x and s. If $s < x$, let $\alpha(x,s)\uparrow$. Let $\alpha(x,x) = f_x$. For $s \geq x$, if $f_{s+1} <_L \alpha(x,s)$ then let $\alpha(x,s+1) = f_{s+1}$.

If x enters A at stage $s+1$ look for the greatest $\beta \subseteq \alpha(x,s)$ where we can enumerate x into $S_{\gamma,s}$ and $\breve{S}_{\gamma,s}$, for $\gamma \subseteq \beta$, such that $\nu^S(\beta,x,s) = \nu \in \mathcal{M}_\beta$ and, for all $\beta' \subset \beta$, if we can enumerate x into $S_{\gamma,s}$ and $\breve{S}_{\gamma,s}$, for $\gamma \subseteq \beta'$, such that $\nu^S(\beta,x,s) = \nu' \in \mathcal{M}_{\beta'}$ then $D^A_{\nu',\beta',s} \neq \emptyset$. If there are several possible ν, arbitrarily choose the one where $D^A_{\nu,\beta,s}$ is the smallest. Enumerate x such that $\nu^S(\beta,x,s) = \nu$. For all γ, if $\gamma \not\subseteq \beta$ or β does not exist, add x to $\breve{S}_{\gamma,s}$.

For any $\beta \subset f$, let s_β be such that if $f_t <_L \beta$ then $t < s_\beta$, if $\{x|\nu^S(|\beta|,x) = \nu'\}$ is finite and $\nu^S(|\beta|,x) = \nu'$ then $x < s_\beta$, and if $\nu \in \mathcal{M}_\beta$ then $D^A_{\nu,\beta,s} \neq \emptyset$ (by induction on β it is not hard to show that such a stage exists). For each $x \geq s_\beta$ we can effectively find a stage $s_{\beta,x}$ such that for all $s' \geq s_{\beta,x}$, $\beta \subseteq \alpha(x,s')$. Let R_β be the set of x such that either $x < s_\beta$ and $x \in A$ or $x \geq s_\beta$ and $x \in A_{s_{\beta,x}}$. R_β is a computable subset of A.

Lemma 6.6. *If $\alpha \subset f$ and ν is a reasonable $|\alpha|$-split state then $D^A_{\nu,\alpha}$ is computable iff $\{x|\nu^S(|\alpha|,x) = \nu\}$ is computable.*

Proof. Let $|\alpha| = e$ and $\nu = \langle 2e, \sigma, \tau\rangle$.

($\Rightarrow$) Assume $\{x|\nu^S(e,x) = \nu\}$ is not computable. We must show $D^A_{\nu,\alpha}$ is not computable. Assume otherwise. Hence there is an $i > e$ such that $W_i = D^A_{\nu,\alpha}$, and $A \searrow W_i = \emptyset$. There must exist a reasonable i-split state $\nu' = \langle 2i, \sigma', \tau'\rangle$ such that $\sigma' \upharpoonright 2e = \sigma$, $2i+1 \in \sigma'$, $\tau' \upharpoonright 2e = \tau$, and $\{x|\nu^S(i,x) = \nu'\}$ is not computable. (Otherwise $\{x|\nu^S(e,x) = \nu\}$ is computably contained in a computable set, W_i, and hence is computable.) Therefore $\{x|\nu^S(i,x) = \nu'\} - R_\beta$ is infinite. Hence, by the above construction, there is an x such that $x \in D^A_{\nu',\beta}$. This same x is in $D^A_{\nu,\alpha}$ but not in W_i. Contradiction.

($\Leftarrow$) Assume $\{x|\nu^S(e,x) = \nu\}$ is computable. Hence there is an $i > e$ such that $W_i = \{x|\nu^S(e,x) = \nu\}$, and $A \searrow W_i = \emptyset$. Let $\beta \subset f$ and $|\beta| = i$. For $j \geq i$, if $\nu' = \langle 2j, \sigma', \tau'\rangle$, $\sigma' \upharpoonright 2e = \sigma$, $2i+1 \in \sigma'$, and $\tau' \upharpoonright 2e = \tau$, then $\{x|\nu^S(j,x) = \nu'\}$ is not infinite. Hence for all $\gamma \supseteq \beta$, $\nu' \notin \mathcal{M}_\gamma$. Let $x \in \overline{W_i} - R_\beta$ enter A at stage $s+1$. Then $\nu^S(i,x) = \nu' \in \mathcal{M}_\beta$ and $\nu' \upharpoonright 2e \neq \nu$. Hence, by the above construction $\nu^S(\beta,x,s) \neq \nu$. Therefore if x enters A at stage $s+1$ and $\nu^S(\beta,x,s) = \nu$ then $x \in R_\beta$ or $x \in W_i$. Thus $D^A_{\nu,\alpha}$ is computable. □

Therefore Θ is an isomorphism between $\mathcal{B}$ and $\hat{\mathcal{B}}$. Thus (1) holds. The next lemma proves (2).

Lemma 6.7. *If $\alpha \subset f$, $|\alpha| = e$, and $\tilde{S}_e$ supports $\hat{W}$, then S_α supports $X = \Psi^{-1}(\hat{W})$.*

Proof. Since Ψ is an automorphism of $\mathcal{E}^*$ taking A to $\hat{A}$, $\Psi^{-1}(\tilde{S}_e)$ supports X. Since Γ is induced by Ψ, $\Gamma^{-1}(\tilde{S}_e)$ supports X. Let $i > e$ such that $W_i = (X - A) \sqcup \Gamma^{-1}(\tilde{S}_e)$. Hence $W_i \searrow A$ supports X. If $(W_i \searrow A) \subseteq_{\mathcal{R}} Y$ then Y supports X. Hence it is enough to show $(W_i \searrow A) \subseteq_{\mathcal{R}} S_\alpha$.

Let $\beta \subset f$ such that $|\beta| = i$. For $j \geq i$, if $\nu = \langle 2j, \sigma, \tau \rangle$, $2i \in \sigma$, and $\{x | \nu^S(j, x) = \nu\}$ is infinite then $2e \in \tau$. Hence for all $\gamma \supseteq \beta$, if $\nu = \langle |\gamma|, \sigma, \tau \rangle \in \mathcal{M}_\gamma$ and $2i \in \sigma$ then $2e \in \tau$. Let $x \in W_i - R_\beta$ enter A at stage $s + 1$. Then $\nu^S(i, x) = \nu \in \mathcal{M}_\beta$. Hence, by the above construction, for almost all such x, $x \in S_\alpha$. Hence $(W_i \searrow A) \subseteq^* S_\alpha \cup R_\beta$. □

7. A definable orbit which is not a Δ^0_3 orbit

For $\mathcal{E}^*$, all the previously known orbits are actually orbits under Δ^0_3-automorphisms. And a good number of those are also definable in the sense that there is an elementary formula, $\varphi(X)$, in the language of $\mathcal{E}^*$ such that $\varphi(A)$ iff A is in the orbit under question. Examples include maximal sets, creative sets, hemimaximal sets, and quasi-maximal sets.

The following is a definable orbit $\mathcal{O}$, which is not a Δ^0_3 orbit. It is the first example of an orbit which is not an orbit under Δ^0_3-automorphisms. It is an orbit under Δ^0_5-automorphisms.

In the mid 1990s, Cholak and Downey incorrectly claimed to construct a pair of Δ^0_4-automorphic computably enumerable sets which were not Δ^0_3-automorphic. In addition, we show this claim is correct by showing there are two such sets in $\mathcal{O}$.

7.1. **The orbit $\mathcal{O}$.** Assume that A is not computable.

Definition 7.1. F is *A-special* if F is not computable, $F \cap A = \emptyset$, and, for all V, if $V \cap A = \emptyset$ then $V - F$ is computably enumerable.

Lemma 7.2. *Assume F_0 and F_1 are A-special sets and R is computable set disjoint from A.*

1. *Either $F_1 - F_0$ is computable or A-special.*
2. *If $F_0 \cap R = \emptyset$ then $F_0 \sqcup R$ is A-special.*
3. *If $F_0 \cap F_1 = \emptyset$ then $F_0 \sqcup F_1$ is A-special.*
4. *$F_0 \cup F_1$ is A-special.*
5. *If $W \subseteq R$ then W is not A-special.*

Proof. (1) $V - (F_1 - F_0) = (V - F_1) \cup (V \cap F_0)$. So if $F_1 - F_0$ is not computable, it is A-special.

(2) $V - (F_0 \sqcup R) = (V - F_0) - R$. If $F_0 \sqcup R$ is computable then F_0 is computable.

(3) $V - (F_0 \cup F_1) = (V - F_0) - F_1$. If $F_0 \sqcup F_1$ is computable then F_0 is computable.

(4) $F_0 \cup F_1 = F_0 \sqcup (F_1 - F_0)$. Now apply (1) followed by one of (2) or (3).

(5) If for all V, if $V \cap A = \emptyset$ then $V - W$ is computably enumerable then $\overline{W} = (R - W) \cup \overline{R}$. □

Definition 7.3. Let $\varphi(A)$ be the conjunction of the following 3 statements:

(1) $\forall F$ if F is A-special then $\exists G$ such that G is A-special and $F \cap G = \emptyset$;
(2) $\forall W$ if $W \cap A = \emptyset$ then $\exists F$ such that F is A-special and $W \subseteq^* F$;
(3) $\forall W \exists F$ such that $F \cap A = \emptyset$ and either $W \subseteq^* F \sqcup A$ or $W \cup F \cup A =^* \omega$.

Definition 7.4. A list of computably enumerable sets, $\mathcal{F} = \{F_i : i \in \omega\}$, is an *$A$-special list* iff $\mathcal{F}$ is a list of pairwise disjoint noncomputable sets, $F_0 = A$, and for all W there is an i such that $W \subseteq^* \bigsqcup_{l \leq i} F_l$ or $W \cup \bigsqcup_{l \leq i} F_l =^* \omega$. We say that $\mathcal{F}$ is a Γ *A-special list* if $\mathcal{F}$ is an A-special list and there is a function f with property Γ such that $F_i = W_{f(i)}$.

Note that for any i, $\bigsqcup_{l \geq i} F_l$ is not computably enumerable and hence there cannot be an effective A-special list. The automorphic image under Φ of an A-special list is a $\Phi(A)$-special list.

Lemma 7.5. *Assume that an A-special list exists and that $V \cap A = \emptyset$. Then $V \subseteq^* \bigsqcup_{0<l \leq i} F_l$, for some i.*

Proof. If $V \cup \bigsqcup_{l \leq i} F_l =^* \omega$, for some i, then $(V \cup \bigsqcup_{0<l \leq i} F_l) \sqcup A =^* \omega$ and hence A is computable. Contradiction. □

Lemma 7.6. *$\varphi(A)$ iff an $\mathbf{0}^{(4)}$ A-special list exists.*

Proof. ($\Rightarrow$) Let $F_0 = A$. Assume, by induction, for $0 < j < i$, that F_j are defined such that they are pairwise disjoint, A-special, either $W_j \subseteq^* \bigsqcup_{l \leq j} F_l$ or $W_j \cup \bigsqcup_{l \leq j} F_l =^* \omega$, and $\bigsqcup_{0<j<i} F_j$ is A-special. Since $\varphi(A)$ holds, the third clause of Definition 7.3 holds for W_i and hence there is an F such that $F \cap A = \emptyset$ and either $W_i \subseteq^* F \sqcup A$ or $W_i \cup F \cup A =^* \omega$. By the second clause of Definition 7.3 and the fact that A-special sets are disjoint from A, we can assume F is A-special. Hence, by Lemma 7.2, $\bigsqcup_{j<i} F_j \cup F$ is A-special and $F - \bigsqcup_{j<i} F_j$ is either computable or A-special. If $F - \bigsqcup_{j<i} F_j$ is A-special let $F_i = F - \bigsqcup_{j<i} F_j$. Otherwise apply the first clause of Definition 7.3 to $\bigsqcup_{j<i} F_j \cup F$ to get

an A-special G and let $F_i = G \sqcup (F - \bigsqcup_{j<i} F_j)$ which is A-special by Lemma 7.2. Again by Lemma 7.2, $\bigsqcup_{0<j\leq i} F_j$ is A-special.

If X and Y are computably enumerable sets then whether $Y - X$ is computably enumerable is Σ^0_3. So whether F is A-special is Π^0_4. Since $\varphi(A)$ holds, given W, there exists an A-special set F such that either $W \subseteq^* F \cup A$ or $W \cup F \cup A =^* \omega$. Hence we can try all possible F using $\mathbf{0}^{(4)}$ to test if the F being considered has the correct properties. Since such an F exists this algorithm will converge and is computable in $\mathbf{0}^{(4)}$. Going from F to F_i is also computable in $\mathbf{0}^{(4)}$. Hence the A-special list constructed is computable in $\mathbf{0}^{(4)}$.

($\Leftarrow$) By Lemma 7.2, it is enough to show that for all $j \geq 1$, F_j is A-special. To show F_j is A-special it is enough to show that if $V \cap A = \emptyset$ then $V - F_j$ is computably enumerable. Assume $V \cap A = \emptyset$. Then, by Lemma 7.5, $V \subseteq^* \bigsqcup_{0<l\leq i} F_l$, for some i. So $V - F_j =^* V \cap \bigsqcup_{0<l\leq i \wedge l\neq j} F_l$ is a computably enumerable set. □

Theorem 7.7. *Given an* $\mathbf{a}$ *A-special list, $\mathcal{F}$, and an* $\hat{\mathbf{a}}$ *$\hat{A}$-special list, $\hat{\mathcal{F}}$, there is a* $\mathbf{0}'' \oplus \mathbf{a} \oplus \hat{\mathbf{a}}$*-automorphism Θ of $\mathcal{E}^*$ taking A to $\hat{A}$.*

Proof. By Theorem 5.3, there is an isomorphism Θ_i between F_i to $\hat{F}_i$ preserving computable sets. Given W_e define $\Theta(W_e)$ as follows: If $W_e \subseteq^* \bigsqcup_{l\leq i} F_l$ then $\Theta(W_e) = \bigsqcup_{l\leq i} \Theta_l(W_e \cap F_l)$. Otherwise there is a computable set R such that $R \subseteq^* \bigsqcup_{l\leq i} F_l$ and $R \cup W_e =^* \omega$. For all $l \leq i$, $R \cap F_l$ is computable. Therefore, since Θ_l preserves computable sets, $\Theta(R) = \bigsqcup_{l\leq i} \Theta_l(R \cap F_l)$ is computable. Let

$$\Theta(W_e) = \overline{\Theta(R)} \sqcup \bigsqcup_{l\leq i} \Theta_l(W_e \cap R \cap F_l).$$

Θ is an automorphism of $\mathcal{E}^*$ such that $\Theta(A) = \hat{A}$. By Theorem 5.3, an index for Θ_i can be found uniformly from indices for F_i and $\hat{F}_i$. The remaining division into cases can be done using a $\mathbf{0}''$ oracle. □

Theorem 7.8. *The collection of A such that $\varphi(A)$ forms a Δ^0_5 orbit $\mathcal{O}$.*

Proof. This follows from Theorems 7.6 and 7.7. □

Corollary 7.9. *If $\mathcal{F}$ is an A-special list then, for all i, F_i is automorphic to A.*

Proof. The list formed by switching F_i and A is an F_i-special list. □

7.1.1. *$\mathcal{O}$ is not a Δ^0_3 orbit.*

Theorem 7.10. *There are computably enumerable sets A and $\hat{A}$ such that $\varphi(A)$ and $\varphi(\hat{A})$, A and $\hat{A}$ are Δ^0_4-automorphic but not Δ^0_3-automorphic.*

This theorem follows from the next two lemmas.

Lemma 7.11. *There exists A such that a $\mathbf{0}''$ A-special list $\mathcal{F}$ exists.*

Lemma 7.12. *There exists $\hat{A}$ such that a $\mathbf{0}'''$ $\hat{A}$-special list $\hat{\mathcal{F}}$ exists but no $\mathbf{0}''$ $\hat{A}$-special list exists.*

The proofs of these lemmas follow in Section 7.2.

Proof of Theorem 7.10 from Lemmas 7.11 and 7.12. Assume that $\mathcal{F}$ is the A-special list given by Lemma 7.11 and $\hat{\mathcal{F}}$ is the $\hat{A}$-special list given by Lemma 7.12. By Lemma 7.7, A and $\hat{A}$ are in $\mathcal{O}$ and are Δ^0_4-automorphic.

Let f witness that $\mathcal{F}$ is a $\mathbf{0}''$ A-special list. Assume that A and $\hat{A}$ are Δ^0_3 automorphic via $\Phi(W_e) = W_{g(e)}$ then $\{W_{g((f(i))}|i \in \omega\}$ is $\mathbf{0}''$ $\hat{A}$-special list. Therefore A and $\hat{A}$ cannot be in the same Δ^0_3 orbit. □

The following lemma and corollary are needed for the proof of Lemma 7.12.

Lemma 7.13. *If a $\mathbf{0}''$ A-special list $\mathcal{F} = \{F_i : i \in \omega\}$ exists then there is a function d computable in $\mathbf{0}''$ such that if $W_e \cap A = \emptyset$ then $W_{d(e)} \cap (W_e \cup A) = \emptyset$ and $W_{d(e)}$ is A-special.*

Proof. If $W_e \cap A \neq \emptyset$ (whether this occurs is computable in $\mathbf{0}''$) then let $d(e) = 0$. Assume $W_e \cap A = \emptyset$. Let f witness that $\mathcal{F}$ is $\mathbf{0}''$. Then, by Lemma 7.5, $W_e \subseteq^* \bigsqcup_{0<l\leq i} F_l$, for some i. Using f, the least such i can be found computably in $\mathbf{0}''$. Let $d(e) = f(i+1)$. □

Corollary 7.14. *Assume for all e, there are e' and d such that $W_{e'} \cap A = \emptyset$ and if $W_{\varphi(\langle e',d\rangle)}$ is cofinite then either $W_d \cap (W_{e'} \cup A) \neq \emptyset$ or W_d is not A-special. Then A does not have a $\mathbf{0}''$ A-special list.*

Proof. Assume A has a $\mathbf{0}''$ A-special list. Apply Lemma 7.13 to get g. The graph of g is a Δ^0_3 set and hence a Σ^0_3 set. Cof is Σ^0_3-complete. Hence there is an e such that, for all e', $W_{\varphi(\langle e',d(e')\rangle)}$ is cofinite and if $W_{e'} \cap A = \emptyset$ then $W_{d(e')} \cap (W_{e'} \cup A) = \emptyset$ and $W_{d(e')}$ is A-special. Furthermore, since we are reducing the graph of a function to Cof, for all e', if $d \neq d(e')$ then $W_{\varphi(\langle e',d\rangle)}$ is not cofinite. Contradiction. □

7.2. **Proofs of Lemmas 7.11 and 7.12.** First we will focus on Lemma 7.11. Rather than focusing on A we will first focus on constructing the A-special list $\mathcal{F}$. This will be a tree argument and very similar to the Δ^0_3-isomorphism method. At each node $\alpha \in T$ we will build a computably enumerable set, F_α. The goal is to build the F_αs such that if $F_i = F_{|\alpha|}$, for $\alpha \subset f$, where f is the true path, then $\mathcal{F} = \{F_i : i \in \omega\}$ is an A-special list.

7.2.1. *The requirements.* We will construct the F_αs as pairwise disjoint noncomputable sets, for $\alpha \subset f$. F_α must be noncomputable. Hence we must meet the following requirements for all $\alpha \subset f$ and all e:

$\mathcal{R}_{\alpha,e}$: $\qquad \overline{F}_\alpha \neq W_e$.

In addition, we will meet the following requirement for all $\alpha \subset f$:

$\mathcal{N}_\alpha$: $\qquad$ either $W_{|\alpha|} \subseteq^* \bigsqcup_{\beta\subseteq\alpha} F_\beta$ or $W_{|\alpha|} \cup \bigsqcup_{\beta\subseteq\alpha} F_\beta =^* \omega$.

Before we can discuss how we will meet these requirements we need the following remark.

Remark 7.15 (The position function $\alpha(x, s)$). Given the approximation to the true path at stage s, f_s, we will determine the position function $\alpha(x, s)$ by the following rules: x is *α-legal at stage s* if $\alpha(x, s-1) = \alpha^-$ (recall α^- is the node before α in the tree), x is α^--allowed (defined below) and for all stages t, if $x \leq t \leq s$, then $\alpha \leq_L f_t$. If $\alpha \subseteq f_s$ and x is α-legal then let $\alpha(x, s) = \alpha$ (move x downward into α). If $f_s <_L \alpha(x, s-1)$ then let $\alpha(x, s) = \alpha(x, s-1) \cap f_s$.

7.2.2. *Action for $\mathcal{R}_{\alpha,e}$.* Meeting $\mathcal{R}_{\alpha,e}$ is straightforward. But we are going to break it into parts, ensuring that there are possible witnesses and actually taking action to meet $\mathcal{R}_{\alpha,e}$.

Getting witnesses: For each β and each stage s, we will pick a $x_{\beta,s}$. We will hold $x_{\beta,s}$ out of all F_γ, for $\gamma \supset \beta$ but allow $x_{\beta,s}$ to possibly enter F_γ, for $\gamma \subseteq \beta$. If $x_{\beta,s}$ enters some F_γ at stage s (or does not exist yet), then, at the next stage t, such that $\beta \subseteq f_t$ and there is an x with $\alpha(x, s) = \beta$ and $x \notin \bigsqcup_{\beta\in T} F_{\beta,s}$, we will choose the least such x as $x_{\beta,t}$; until that stage t, $x_{\beta,s}$ does not exist. Otherwise $x_{\beta,s}$ remains the same from stage to stage.

Placing witnesses into F_α: Now if $\alpha \subseteq f_s$, $W_{e,s} \cap F_{\alpha,s} = \emptyset$, $|\alpha| \leq e$, and there is an x where $|\alpha(x, s)| \geq |\alpha| + e$, $x \in W_{e,s}$ and $x \notin \bigsqcup_{\beta\in T} F_\beta$, then add x to F_α at stage s.

Assume that for all $\gamma \subset f$, $x_\gamma = \lim_s x_{\gamma,s}$ exists. Then if $\overline{F}_\alpha = W_e$ then it is straightforward to show that at some stage s we will add an x to F_α to meet $\mathcal{R}_{\alpha,e}$.

Notice that only finitely many $\mathcal{R}_{\alpha,e}$ are possibly interested in $x_{\gamma,s}$. So if x_γ fails to exist it is not due to action for $\mathcal{R}_{\alpha,e}$ but action for some $\mathcal{N}_\beta$.

7.2.3. *Action for $\mathcal{N}_\alpha$.* We will meet $\mathcal{N}_\alpha$ as follows: First of all no action is taken at stage s if $x_{\alpha,s}$ does not exist. Furthermore, we never α-allow $x_{\alpha,s}$. Otherwise the desired action at α breaks into cases depending on whether W_α is infinite or not, where

$$W_\alpha = \{x | \exists s(\alpha^- \subseteq \alpha(x, s) \wedge x \text{ is } \alpha^-\text{-allowed } \wedge x \in W_{|\alpha|,s})\}.$$

If α believes W_α is finite we α-*allow* half of the balls which arrive at α (hence these balls can move downward) and put all but one ball, $x_{\alpha,s}$, of the other half into F_α (like $x_{\alpha,s}$, the balls in F_α, are never α-allowed). Assume α believes W_α is infinite. Half of the balls which arrive at α in W_α will be α-allowed immediately. Otherwise if $\alpha(x,s) = \alpha$ and there have been x many balls α-allowed, we will place x into F_α.

7.2.4. *The Verification.* Assume that for all $\alpha \subset f$, infinitely many balls are α-allowed (we will show this later). Then, by induction on $\alpha \subset f$, it is straightforward to show that x_α exists and hence $\mathcal{R}_{\beta,e}$ is met for $\beta \subset f$ and all e. And, again by induction on $\alpha \subset f$, is straightforward to show, using the standard facts about f_s and $\alpha(x,s)$ and the above assumption, for almost all $x \notin \bigsqcup_{\beta\subset\alpha} F_\beta$, there is a stage such that either x enters F_α or x is α-allowed. Hence if W_α is finite then $W_{|\alpha|} \subseteq^* \bigsqcup_{\beta\subset\alpha} F_\beta$ and otherwise $W_{|\alpha|} \cup \bigsqcup_{\beta\subseteq\alpha} F_\beta =^* \omega$. Therefore, under the above assumption, $\mathcal{N}_\alpha$ is met.

Now we will show, by induction on $\alpha \subset f$, that infinitely many balls are α-allowed. Assume this is true for α^-. Almost all of the balls which are α^--allowed will arrive at α at some later stage (i.e., there is a stage t such that $\alpha \subseteq \alpha(x,t)$). Hence at almost all stages, $x_{\alpha,s}$ exists. Therefore if W_α is finite then half of those balls which arrive at α will be α-allowed. If W_α is infinite then infinitely many balls arrive at α in W_α, half of which are α-allowed.

Hence the only thing needed to complete the proof of Lemma 7.11 is to construct the tree T, the true path f, and the approximation to the true path at stage s, f_s. But since we want to use the same tree and related materials for the proof of Lemma 7.12, we will delay this until Section 7.2.6.

7.2.5. *Changes needed for the proof of Lemma 7.12.* Rather than proving Lemma 7.12 we will prove its unhatted dual. We are going to make use of Lemma 7.14. We must meet the requirements:

$\mathcal{Q}_e$: there are e' and d such that $W_{e'} \cap A = \emptyset$ and if $W_{\varphi(\langle e',d\rangle)}$ is cofinite then either $W_d \cap (W_{e'} \cup A) \neq \emptyset$ or W_d is not A-special.

By the Recursion Theorem we can assume there are computable functions g and h such that $W_{g(\alpha)} = F_\alpha$ and $W_{h(\alpha)} = \bigcup_{\lambda\subset\beta\subseteq\alpha} F_\beta$, for all $\alpha \in T$ and $\alpha \neq \lambda$. Recall λ is the empty node and $F_\lambda = A$. For all $\alpha \neq \lambda$, $W_{h(\alpha)} \cap A = \emptyset$.

Assume that α is assigned to meet $\mathcal{Q}_e$. α will use $W_{h(\alpha)}$ as $W_{e'}$. We want to look for the least d and l such that $[l,\infty) \subseteq W_{\varphi_e(\langle h(\alpha),d\rangle)}$. We will use the tree to find k and l and to assign α to $\mathcal{Q}_e$.

We will define the tree such that there are d, l where $[l,\infty) \subseteq W_{\varphi_e(\langle h(\alpha),d\rangle)}$ iff there is a unique β such that $\alpha \subset \beta \subset f$ and β believes there are

$d, l < |\beta|$ such that $[l, \infty) \subseteq W_{\varphi_e(\langle h(\alpha),d\rangle)}$. We will assume that the $\mathcal{Q}_i$ are assigned in increasing order modulo finite injury along the true path. The finite injury along the true path will be discussed below.

Assume that β believes there are $d, l < |\beta|$ such that $[l, \infty) \subseteq W_{\varphi_e(\langle h(\alpha),d\rangle)}$. Since $d < |\beta|$ there is a $\gamma \subset \beta$ with $|\gamma| = d$. Furthermore, since we will continue to meet $\mathcal{N}_\gamma$, either $W_d \subseteq^* \bigsqcup_{\delta\subseteq\gamma} F_\delta$ or $W_d \cup \bigsqcup_{\delta\subseteq\gamma} F_\delta =^* \omega$. By Lemma 7.5, if $W_d \cup \bigsqcup_{\delta\subseteq\gamma} F_\delta =^* \omega$ then $W_d \cap A \neq \emptyset$ and we have met $\mathcal{Q}_e$. If $W_d \subseteq^* \bigsqcup_{\delta\subseteq\alpha} F_\delta$ then we have met $\mathcal{Q}_e$. Hence the only case where we must take action to meet $\mathcal{Q}_e$ is when $W_d \subseteq^* \bigsqcup_{\alpha\subset\delta\subseteq\gamma} F_\delta$. In this case we will force $\bigsqcup_{\alpha\subset\delta\subseteq\gamma} F_\delta$ to be computable and hence, by Lemma 7.2 (5), W_d is not A-special. This means we will have to later reconsider how we form the A-special list.

Assume that β must take action to meet $\mathcal{Q}_e$. β will take action by changing how we meet $\mathcal{R}_{\gamma,e}$, for all $\alpha \subseteq \gamma \subseteq \beta$. Let $\alpha \subseteq \gamma \subseteq \beta$. The action taken for $\mathcal{R}_{\gamma,e}$ is revised as follows: if $\gamma \subseteq f_s$, $W_{e,s} \cap F_{\gamma,s} = \emptyset$, and there is an x such that $\beta \not\subseteq \alpha(x,s)$, $|\alpha(x,s)| \geq |\gamma| + e$, $x \in W_{e,s}$ and $x \notin \bigsqcup_{\delta\in T} F_\delta$, then add x to F_γ at stage s. Now to help with the creation of an A-special list we must injure all $\mathcal{Q}_i$ assigned to some γ between α and β. We will assign then in increasing order to some δ where $\beta \subset \delta$. This is finite injury along the true path.

If no $\alpha \subset \beta \subset f$ believes that it must take action to meet $\mathcal{Q}_e$ then the above argument for the verification of $\mathcal{R}_{\gamma,e}$ still holds and F_γ is not computable.

Assume that some $\beta \subset f$ believes that it must take action to meet $\mathcal{Q}_e$. From the above verification, we know that almost all x either enter $\bigsqcup_{\delta\subseteq\beta} F_\delta$ or are β-allowed. By the above modification of the action for $\mathcal{R}_{\gamma,e}$ once a ball either enters $\bigsqcup_{\delta\subseteq\beta} F_\delta$ or is β-allowed it cannot be used to meet $\mathcal{R}_{\gamma,e}$. Hence F_γ is computable and $\mathcal{Q}_e$ is met.

The issue of an A-special list remains. Using the true path f and $\mathbf{0}'''$ we will inductively show how to construct an A-special list. Assume that we have built the list up to i and have used $\alpha_i \subset f$. Let α^+ be such that $\alpha \subset \alpha^+ \subset f$ and $|\alpha^+| = |\alpha| + 1$. Assume that $\mathcal{Q}_e$ is assigned to α^+ and by induction $\mathcal{Q}_e$ is not injured from below. Use $\mathbf{0}'''$ to see if some $\beta \subset f$ takes action to meet $\mathcal{Q}_e$. If no $\beta \subset f$ must take action to meet $\mathcal{Q}_e$ then $F_{i+1} = F_{\alpha^+}$ is not computable and let $\alpha_{i+1} = \alpha^+$. Otherwise there is a $\beta \subset f$ which takes action to meet $\mathcal{Q}_e$. In this case F_β is not computable and let $F_{i+1} = \bigsqcup_{\alpha_i\subset\gamma\subseteq\beta} F_\gamma$ and $\alpha_{i+1} = \beta$. In either case there is no injury from below above α_{i+1}.

7.2.6. *The tree T and related definitions.* We will define one tree which can be used for both lemmas. We will define T, the true path f, and the

approximation to the true path at stage s, f_s via induction on the length of γ.

We have to code a few items into T. At a node β we must code whether W_β is infinite and whether there exists an $\alpha \subset \beta$ and $e, d, l, s < \beta$ such that $\mathcal{Q}_e$ is assigned by α, α has not been injured by any γ with $\alpha \subset \gamma \subseteq \beta$, $\varphi_e(\langle h(\alpha), d\rangle \downarrow) = w$, $[l, \infty) \subseteq W_w$, and $W_d \subseteq \bigsqcup_{\alpha \subset \delta \subseteq \beta} F_\delta$. Since $F_\delta = W_{g(\delta)}$, all this information is Δ^0_3 and hence can be easily coded into a tree. In the interest of space and energy we are not going to go into the details of the definition of the tree. Similar constructions with all the details can be found in Section 7.2.6, Cholak [3], Cholak [2], and Weber [20]. There is one added twisted that there is finite injury along the true path. But that kink was discussed above and is implemented in the standard fashion. □

7.3. **Reflecting on $\varphi(A)$ and Theorem 7.10.** Theorem 7.10 implies that $\mathcal{O}$ is different than any other known orbit. But it might be worthwhile to reflect on $\mathcal{O}$'s similarity to the orbit formed by the maximal sets or the orbit formed by the Herrman sets (for a definition of Herrmann sets, see Cholak et al. [4]). This reflection will also impact how we approach the proof of Theorem 7.10.

Definition 7.16. $\mathcal{D}(A)$ is the ideal generated by the sets F such that either $F \cap A = \emptyset$ or $F \subseteq^* A$. $\mathcal{D}(A)$ is a Σ^0_3 ideal of $\mathcal{E}$. Let $\mathcal{E}_{\mathcal{D}(A)}$ be $\mathcal{E}$ modulo $\mathcal{D}(A)$. We write $X \subseteq_{\mathcal{D}(A)} Y$ if X is contained in Y modulo $\mathcal{D}(A)$. If A is understood from the context we drop the "(A)".

The last clause of $\varphi(A)$ implies that $\mathcal{E}_{\mathcal{D}}$ is the two element Boolean algebra. This is also the case with maximal sets and Herrmann sets. When this is the case we say that A is $\mathcal{D}$-maximal. It is also possible to consider A where $\mathcal{E}_{\mathcal{D}}$ is a Boolean algebra, in which case, A is called $\mathcal{D}$-hhsimple. (For more on $\mathcal{D}$-hhsimple sets, see Cholak et al. [4], Herrmann and Kummer [12], and Kummer [13] in that order.)

Assume that A is $\mathcal{D}$-hhsimple. Furthermore assume that $W \neq_{\mathcal{D}} A$. Then there is a $\tilde{W}$ such that $W \cap \tilde{W} =_{\mathcal{D}} \emptyset$ and $W \cup \tilde{W} =_{\mathcal{D}} \omega$. So there is a set $F \in \mathcal{D}$ such that $W \cap \tilde{W} \subseteq F$ and $W \cup \tilde{W} \cup F = \omega$. Therefore there is a computable set R such that $R \cap \overline{F} = W \cap \overline{F}$.

Let $\widetilde{\mathcal{L}}(A)$ be the definable (in $\mathcal{E}$) quotient substructure of $\mathcal{S}_{\mathcal{R}}(A)$ given by $\{R \cap H : R \text{ is computable}\}$ modulo $\mathcal{R}(A)$. Given the above paragraph, it is straightforward to verify that $\widetilde{\mathcal{L}}(A)$ and $\mathcal{E}_{\mathcal{D}}$ are Δ^0_3-isomorphic.

Assume A and $\hat{A}$ are automorphic by Φ. By Theorem 6.1, $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\hat{A})$ are Δ^0_3-isomorphic via an isomorphism induced by Φ. So $\widetilde{\mathcal{L}}(A)$ and $\widetilde{\mathcal{L}}(\hat{A})$ are Δ^0_3-isomorphic via an isomorphism induced by Φ. Hence $\mathcal{E}_{\mathcal{D}(A)}$

and $\mathcal{E}_{\mathcal{D}(\hat{A})}$ are Δ^0_3-isomorphic. (A similar argument appeared in Section 11 of Cholak and Harrington [6].) Hence we have the following theorem.

Theorem 7.17. *Assume that A is $\mathcal{D}$-hhsimple. If A and $\hat{A}$ are automorphic via Φ then $\mathcal{E}_{\mathcal{D}(A)}$ and $\mathcal{E}_{\mathcal{D}(\hat{A})}$ are Δ^0_3-isomorphic via an isomorphism induced by Φ.*

One should compare this theorem to Theorem 8.7 where the hypothesis that A be $\mathcal{D}$-hhsimple is removed but the complexity of the isomorphism increases to Δ^0_6.

Soare showed that the maximal sets, M, do not form an effective orbit by exploiting the fact that deciding if $W \subseteq^* M$ or $W \cup M =^* \omega$ is Δ^0_3. Soare built maximal sets M and $\hat{M}$ such that for each computable function f there is an e with $W_e \subseteq^* M$ iff $W_{f(e)} \cup \hat{M} =^* \omega$. (For more details, see Soare [17] and Cholak [1].)

But Theorem 7.17 implies that we cannot exploit the fact of deciding if $W \subseteq_{\mathcal{D}} A$ or $W =_{\mathcal{D}} \omega$ is Δ^0_3 to show there are A and $\hat{A}$ in $\mathcal{O}$ which are not Δ^0_3-isomorphic. Hence the proposed approach of Cholak and Downey (thankfully unpublished) to the proof of Theorem 7.10 just cannot work. To show Theorem 7.10 we exploited the fact that given a set W disjoint from A we cannot always computably in $\mathbf{0}''$ find an A-special set disjoint from W.

8. On the complexity of orbits of $\mathcal{E}$

The goal of this section is to improve Theorem 7.17 and add to our comments from Section 7.3. We are going to do this by coding where W, for $W \neq_{\mathcal{D}} A$, must go under an arbitrary automorphism of $\mathcal{E}$, using various splits of A. We will break this into two subsections: the first subsection will focus on the coding and the second subsection will present the results which use this coding.

8.1. **Maximal supports.**

Fix a computably enumerable set A. A definition of $\mathcal{D}(A)$ can be found in Definition 7.16.

Definition 8.1. M is *maximally supported* by S if M is supported by S (so S is a split of A, $S \subseteq M$ and $(M - A) \sqcup S$ is a computably enumerable set) and for all W, if W is supported by S, then $W \subseteq_{\mathcal{D}} M \cup A$.

Lemma 8.2. *Whether S maximally supports M is Π^0_4.*

Proof. By Lemma 4.2, whether T supports X is Σ^0_3. □

If S is a maximal support of W and $T =_{\mathcal{R}} S$ then T is a maximal support of W.

Lemma 8.3. *If $Y \not\subseteq_{\mathcal{D}} X$, S is a maximal support for X and T is a support for Y then $T \not\subseteq_{\mathcal{R}} S$.*

Proof. Since S maximally supports X, S cannot support Y. So T is not a subset of S. The same holds modulo $\mathcal{R}(A)$. □

Note it is possible that S and T maximally support W but $S \neq_{\mathcal{R}} T$. But this will not cause a problem.

Recall A is promptly simple iff there is a computable function p such that for all W, if W is infinite, then there is an x and s such that $x \in W_{\text{at } s} \cap A_{p(s)}$. Also if A is simple then $W \subseteq_{\mathcal{D}} M$ iff $W \subseteq^* M \cup A$.

Lemma 8.4. *Assume that A is promptly simple. Let $A \subseteq M$. There is an S such that M is maximally supported by S.*

Furthermore $S = M \searrow A$ using $\{A_{p(s)}\}_{s\in\omega}$ as the enumeration of A; i.e., S is the set of x such that x enters M at stage s and x is not in $A_{p(s)}$ but x is in A.

Proof. M is supported by the S defined above; $(M - A) \sqcup S$ is the set of x such that x enters M at stage s and x is not in $A_{p(s)}$.

To ensure M is maximally supported by S it is enough to show the following conditions are met:

$\mathcal{N}_{e,i}$: either $W_e \subseteq^* M \cup A$ or $W_i \neq (W_e - A) \sqcup S$.

Assume $W_e \not\subseteq^* M \cup A$ and $W_i = (W_e - A) \sqcup S$ (i.e., that we fail to meet $\mathcal{N}_{e,i}$). Then $W = (W_e \cap W_i)\backslash(M \cup A)$ is infinite. Then there is an x and s such that $x \in W_{\text{at } s} \cap A_{p(s)}$. Now x is in W_i and thus in one of $W_e - A$ or S. But x cannot be in either of these two sets. Contradiction. □

It would be nice if we could prove the above lemma for all A but the above proof heavily relies on the assumption that A was promptly simple. However we do have the following lemma.

Lemma 8.5. *For all W, $\tilde{W}$, if $W \neq_{\mathcal{D}} \tilde{W}$ then there is an M such that M is maximally supported by $S = M \searrow A$, $M \subseteq W$, and $M \not\subseteq_{\mathcal{D}} \tilde{W}$.*

Proof. Fix W and $\tilde{W}$. Clearly $M \searrow S$ supports M. So we must build M to meet the following requirements:

$\mathcal{N}_{e,i}$: either $W_e \subseteq_{\mathcal{D}(A)} M \cup A$ or $W_i \neq (W_e - A) \sqcup S$.

(i.e., either W_e is contained in $M \cup A$ modulo $\mathcal{D}(A)$ or S does not support it) and

$\mathcal{P}_{e,i}$: either $W_e \cap A \neq \emptyset$, or $W_i \cap A \neq \emptyset$, or $M \cup W_i \cup A \not\subseteq \tilde{W} \cup W_e \cup A$

(so M is not contained modulo $\mathcal{D}(A)$ in $\tilde{W}$). Assume that these requirements are linearly ordered.

To meet $\mathcal{N}_{e,i}$ we will hold everything in $X = (W_e \cap W_i)\backslash(M \cup A)$ out of M until there is an $x \in X \cap A$ and hence $W_i \neq (W_e - A) \sqcup S$. Assume this fails. Then X is disjoint from A. So if $W_i = (W_e - A) \sqcup S$ then $W_e \subseteq M \cup A \cup X$. And hence we still meet $\mathcal{N}_{e,i}$.

To meet $\mathcal{P}_{e,i}$ we need to first define a length of agreement function (to measure a Π^0_2 fact). Let $l(s)$ be the greatest x such that $(W_{e,s} \cap A_s) \upharpoonright x = \emptyset$, $(M_s \cup W_{i,s} \cup A_s) \upharpoonright x = (\tilde{W}_s \cup W_{e,s} \cup A_s) \upharpoonright x$, and $(W_{i,s} \cap A_s) \upharpoonright x = \emptyset$. Let $m(0) = 0$. If $l(s) > m(s-1)$ then s is *expansionary* (for $\mathcal{P}_{e,i}$) and $m(s) = l(s)$; otherwise $m(s) = m(s-1)$.

If there are infinitely many expansionary stages we must take some action to ensure $\mathcal{P}_{e,i}$ is met. At expansionary stages we will dump everything in W which is not restricted by higher priority requirements into M and reset all lower priority requirements.

As we argued above, the set X of x which is restrained by high priority requirements is disjoint from A. Therefore if there are infinitely many expansionary stages then $M \cup Z \cup A = W \cup Z \cup A$, where Z is the union of finitely many Xs from the higher priority negative requirements. Hence $W =_{\mathcal{D}} M =_{\mathcal{D}} \tilde{W}$. Hence, under the above hypothesis, there cannot be infinitely many expansionary stages and $\mathcal{P}_{e,i}$ is met. □

8.2. **Coding with maximal supports.**

Theorem 8.6. *Assume that A and $\hat{A}$ are promptly simple. Then A and $\hat{A}$ are automorphic iff A and $\hat{A}$ are Δ^0_3 automorphic.*

Proof. Assume that A and $\hat{A}$ are automorphic via Φ. We can assume that $\Phi \upharpoonright \mathcal{E}^*(A)$ is Δ^0_3. We must show that $\Phi \upharpoonright \mathcal{L}^*(A)$ is Δ^0_3. We know that $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\hat{A})$ are Δ^0_3 isomorphic via an isomorphism Θ induced by Φ.

Given W, look for a support S of W, a set $\tilde{W} \subseteq \hat{\omega}$, and a support $\tilde{S}$ of $\tilde{W}$ such that $S \subseteq_{\mathcal{R}} \Theta^{-1}(\tilde{W} \searrow \hat{A})$ and $\tilde{S} \subseteq_{\mathcal{R}} \Theta(W \searrow A)$. Such sets exist; consider $\tilde{W} = \Phi(W)$, $S = \Phi^{-1}(\Phi(W) \searrow \hat{A})$, and $\tilde{S} = \Phi(W \searrow A)$. Since such sets exist, we can find them using $\mathbf{0}''$ as an oracle.

Since Θ is induced by the automorphism Φ, by Lemma 8.4, $\Theta^{-1}(\tilde{W} \searrow \hat{A})$ maximally supports $\Phi^{-1}(\tilde{W})$. Therefore, by Lemma 8.3 and the fact that for simple sets, A, $=^*$, and $=_{\mathcal{D}}$ agree, $W \subseteq^* \Phi^{-1}(\tilde{W})$. Similarly $\tilde{W} \subseteq^* \Phi(W)$. So $W =^* \Phi^{-1}(\tilde{W})$ and $\tilde{W} =^* \Phi(W)$ and hence $\tilde{W} =^* \Phi(W)$. □

Theorem 8.7. *If A and $\hat{A}$ are automorphic via Φ then $\mathcal{E}_{\mathcal{D}(A)}$ and $\mathcal{E}_{\mathcal{D}(\hat{A})}$ are Δ^0_6-isomorphic via an isomorphism induced by Φ.*

Proof. Assume that A and $\hat{A}$ are automorphic via Φ. We can assume that $\Phi \upharpoonright \mathcal{E}^*(A)$ is Δ^0_3. We know that $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\hat{A})$ are Δ^0_3 isomorphic via

an isomorphism Θ induced by Φ. Given W we must find a $\tilde{W}$, in a Δ^0_6 way, such that $\tilde{W} =_{\mathcal{D}} \Phi(W)$.

By Lemma 8.5, $Y \subseteq_{\mathcal{D}} \tilde{Y}$ iff, for all M and X, if $M \subseteq Y$, M is maximally supported by $S = M \searrow A$, and S supports X, then $X \subseteq_{\mathcal{D}} \tilde{Y}$. Since Θ is induced by the automorphism Φ, $\tilde{W} \subseteq_{\mathcal{D}} \Phi(W)$ iff for all $\tilde{M}$ and X, if $\tilde{M} \subseteq \tilde{W}$, $\tilde{M}$ is maximally supported by $\tilde{S} = \tilde{M} \searrow \hat{A}$, and $\Theta^{-1}(\tilde{S})$ supports X, then $X \subseteq_{\mathcal{D}} W$, a Π^0_5-statement. And similarly, $\Phi(W) \subseteq_{\mathcal{D}} \tilde{W}$ iff for all M and $\tilde{X}$, if $M \subseteq W$, M is maximally supported by $S = M \searrow A$, and $\Theta(S)$ supports $\tilde{X}$, then $\tilde{X} \subseteq_{\mathcal{D}} \tilde{W}$, a Π^0_5-statement.

Therefore whether $\tilde{W} =_{\mathcal{D}} \Phi(W)$ is Π^0_5. Since such a $\tilde{W}$ exists, it can be found using $\mathbf{0}^{(5)}$ as an oracle. □

Corollary 8.8. *If A is simple, then A and $\hat{A}$ are automorphic iff A and $\hat{A}$ are Δ^0_6-automorphic.*

Proof. Assume that A and $\hat{A}$ are automorphic by Φ where $\Phi \upharpoonright \mathcal{E}^*(A)$ is Δ^0_3. Since A is simple, if $W \subseteq \overline{A}$ then W is finite. Therefore $\mathcal{L}^*(A)$ and $\mathcal{E}_{\mathcal{D}(A)}$ are isomorphic, by the identity map. Therefore $\Phi \upharpoonright \mathcal{L}^*(A)$ is Δ^0_6. So Φ is Δ^0_6. □

If A is simple and $A \subset W$ then where an automorphism of $\mathcal{E}$ takes W is completely determined by certain splits of A, the maximal supports. Hence the following is a corollary of the proofs of Theorem 8.7 (8.6) and Theorem 6.4.

Theorem 8.9. *The (promptly) simple sets A and $\hat{A}$ are automorphic iff there are Ψ, $\mathcal{B}$, $\hat{\mathcal{B}}$, and Θ such that*

(1) *$\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$ are Δ^0_6-isomorphic (Δ^0_3-isomorphic) via Ψ,*
(2) *$\mathcal{B}$ and $\hat{\mathcal{B}}$ are extendible algebras which are extendibly Δ^0_3 isomorphic via Θ,*
(3) *$\mathcal{B}$ supports $\mathcal{L}^*(A)$,*
(4) *$\hat{\mathcal{B}}$ supports $\mathcal{L}^*(\hat{A})$,*
(5) *the isomorphisms Ψ and Θ preserve supports.*

The r-maximal sets are simple. So r-maximal sets are automorphic iff they are Δ^0_6-automorphic. But this is not a "nice" algebraic classification, at least for r-maximal sets. It is possible that the $\mathcal{L}^*$s of r-maximal sets have a nice structure. So we might be able to replace Condition 1 of Theorem 8.9 with something more algebraic and easier to understand, like the other conditions. The reader is directed to the last section of Cholak and Nies [9] for some suggestions. We should point out that Lempp et al. [14] have shown that there is no Δ^0_3 classification ("nice" or otherwise) of the $\mathcal{L}^*$s of

r-maximal sets. But this does not rule out a "nice" arithmetic classification of the $\mathcal{L}^*$s.

The results in this section and that of Section 7.3 drive home the point that to build sets whose orbits are complex we are forced to use techniques like those described in Sections 7.1.1 and 7.2.5. In a forthcoming paper we will do just that.

REFERENCES

[1] P. Cholak. Boolean algebras and orbits of the lattice of r. e. sets modulo the finite sets. *J. Symbolic Logic*, 55:744–760, 1990. 7.3

[2] P. Cholak. The translation theorem. *Arch. Math. Logic*, 33:87–108, 1994. 1, 3, 3.1, 3.2, 6.1, 7.2.6

[3] Peter Cholak. Automorphisms of the lattice of recursively enumerable sets. *Mem. Amer. Math. Soc.*, 113(541):viii+151, 1995. ISSN 0065-9266. 1, 6.1, 7.2.6

[4] Peter Cholak, Rod Downey, and Eberhard Herrmann. Some orbits for $\mathcal{E}$. *Ann. Pure Appl. Logic*, 107(1-3):193–226, 2001. ISSN 0168-0072. MR 2001k:03086. 4, 5, 7.3, 7.3

[5] Peter Cholak and Leo A. Harrington. On the definability of the double jump in the computably enumerable sets. *J. Math. Log.*, 2(2):261–296, 2002. ISSN 0219-0613. Pdf. MR 2003h:03063. 1

[6] Peter Cholak and Leo A. Harrington. Isomorphisms of splits of computably enumerable sets. *J. Symbolic Logic*, 68(3):1044–1064, 2003. ISSN 0022-4812. Pdf. MR 2004f:03077. 1, 2.2, 2.3, 5.6, 6, 6.1, 6, 7.3

[7] Peter A. Cholak. The global structure of computably enumerable sets. In *Computability theory and its applications (Boulder, CO, 1999)*, volume 257 of *Contemp. Math.*, pages 61–72, Providence, RI, 2000. Amer. Math. Soc. MR 2001d03099. 1

[8] Peter A. Cholak and Leo A. Harrington. Definable encodings in the computably enumerable sets. *Bull. Symbolic Logic*, 6(2):185–196, 2000. A copy can be found at `http://www.nd.edu/~cholak`. 1, 2.1

[9] Peter A. Cholak and André Nies. Atomless r-maximal sets. *Israel J. Math.*, 113:305–322, 1999. ISSN 0021-2172. MR 2001a:03087. 8.2

[10] R. G. Downey and M. Stob. Automorphisms of the lattice of recursively enumerable sets: Orbits. *Adv. in Math.*, 92:237–265, 1992. 1

[11] Leo A. Harrington and Robert I. Soare. The Δ^0_3-automorphism method and noninvariant classes of degrees. *J. Amer. Math. Soc.*, 9(3):617–666, 1996. ISSN 0894-0347. 1

[12] Eberhard Herrmann and Martin Kummer. Diagonals and $\mathcal{D}$-maximal sets. *J. Symbolic Logic*, 59(1):60–72, 1994. MR 95f:03066. 7.3

[13] Martin Kummer. Diagonal and semihyperhypersimple sets. *J. Symbolic Logic*, 56:1068–1074, 1991. 7.3

[14] Steffen Lempp, André Nies, and D. Reed Solomon. On the filter of computably enumerable supersets of an r-maximal set. *Arch. Math. Logic*, 40(6):415–423, 2001. ISSN 0933-5846. MR 2002h:03093. 8.2

[15] W. Maass. On the orbit of hyperhypersimple sets. *J. Symbolic Logic*, 49:51–62, 1984. 1

[16] Robert I. Soare. The new extension theorem. To appear. 1

[17] Robert I. Soare. Automorphisms of the lattice of recursively enumerable sets I: maximal sets. *Ann. of Math. (2)*, 100:80–120, 1974. 1, 5.3, 5, 7.3

[18] Robert I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic, Omega Series. Springer–Verlag, Heidelberg, 1987. 2.1, 2.3, 2.3, 2.4, 3.4, 5

[19] Robert I. Soare. Extensions, automorphisms, and definability. In *Computability theory and its applications (Boulder, CO, 1999*, volume 257 of *Contemp. Math.*, pages 279–307. Amer. Math. Soc., 2000. 1

[20] Rebecca Weber. *A definable relation between c.e. sets and ideals*. PhD thesis, University of Notre Dame, 2004. 6.1, 7.2.6

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556-5683

E-mail address: Peter.Cholak.1@nd.edu

URL: http://www.nd.edu/~cholak

Department of Mathematics, University of California, Berkeley, CA 94720-3840

E-mail address: leo@math.berkeley.edu