FREE SETS AND REVERSE MATHEMATICS

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Abstract. Suppose that $f : [\mathbb{N}]^k \to \mathbb{N}$. A set $A \subseteq \mathbb{N}$ is *free* for f if for all $x_1, \dots, x_k \in A$ with $x_1 < x_2 < \cdots < x_k$, $f(x_1, \dots, x_k) \in A$ implies $f(x_1, \dots, x_k) \in \{x_1, \dots, x_k\}$. The free set theorem asserts that every function f has an infinite free set. This paper addresses the computability theoretic content and logical strength of the free set theorem. In particular, we prove that Ramsey's theorem for pairs implies the free set theorem for pairs, and show that every computable $f : [\mathbb{N}]^k \to \mathbb{N}$ has an infinite Π^0_k free set.

§1. Introduction. We will analyze the strength of the free set theorem using techniques from computability theory and reverse mathematics. A posting of H. Friedman in the FOM email list [5] and the section on open problems on free sets in [7] sparked our interest in this topic.

The purpose of Reverse Mathematics is to study the role of set existence axioms, trying to establish the weakest subsystem of second order arithmetic in which a theorem of ordinary mathematics can be proved. The basic reference for this program is Simpson's monograph [15]. While we assume familiarity with the development of mathematics within subsystems of second order arithmetic, we briefly recall the definition of RCA_0, WKL_0, and ACA_0.

RCA_0 includes some algebraic axioms, an induction scheme for Σ^0_1 formulas, and comprehension for sets defined by Δ^0_1 formulas, i.e. formulas which are equivalent both to a Σ^0_1 and to a Π^0_1 formula. WKL_0 extends RCA_0 by adding weak König's lemma, asserting that if T is a subtree of $2^{<\mathbb{N}}$ with no infinite path, then T is finite. ACA_0 consists of RCA_0 plus set comprehension for arbitrary arithmetical formulas.

Let X be a set equipped with a linear ordering (notice that, since we are working in subsystems of arithmetic, all sets have an underlying linear ordering). The expression $[X]^k$ denotes the set of all increasing k-tuples of elements of X. We are now ready to give the precise statement of the free set theorem, originally due to Friedman.

STATEMENT 1.1. (FS – free set theorem). *Let $k \in \mathbb{N}$ and let $f : [\mathbb{N}]^k \to \mathbb{N}$. Then there exists an infinite $A \subseteq \mathbb{N}$ such that for all $x_1, \dots, x_k \in A$ with $x_1 < x_2 < \cdots < x_k$, if $f(x_1, \dots, x_k) \in A$ then $f(x_1, \dots, x_k) \in \{x_1, \dots, x_k\}$. We use $FS(k)$ to denote the statement FS restricted to a fixed $k \geq 1$.*

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Natural analogs of free sets include sets of linearly independent elements in a vector space, sets of algebraically independent elements in a field, and sets of indiscernibles in any appropriate structure [12]. These analogs differ from Friedman's concept of free set in that they concern closure under operations as opposed to a single application of a function. The following definition and example[1] should help in pointing out this important difference.

There is a notion of a "free set" in a model theoretic setting ([4], [2], [3]). Let $\mathcal{M} = (M, R_i, f_j, c_k)$ be a structure and let $\emptyset \neq A \subseteq M$. By $\mathcal{M}(A)$ we denote the substructure of $\mathcal{M}$ generated by A. A is *free* for $\mathcal{M}$ if and only if for all $A' \subseteq A$, $\mathcal{M}(A') \cap A = A'$.

Let $\mathcal{M} = (\mathbb{N}, f)$ where $f(x) = x + 1$ for all x. Let A be the set of even numbers. A is free for f in the sense of Statement 1.1. But if we let A' be the set of numbers divisible by 4, then $\mathcal{M}(A') = \mathbb{N}$, hence $\mathcal{M}(A') \cap A = A \neq A'$. So A is not free for $\mathcal{M}$.

However we can show that if a set $A \subseteq M$ is free for $\mathcal{M}$ then, for all j, A is free for f_j in the sense of Statement 1.1. Let $f_j : M^{k_j} \to M$ and let A' be a subset of A of cardinality k_j. Since, by definition, $\mathcal{M}(A') \cap A = A'$, if $\overline{a}$ is a k_j-tuple of elements of A', we have that if $f_j(\overline{a}) \in A$, then $f_j(\overline{a}) \in \{a_1, \dots, a_{k_j}\}$. Hence A is free for f_j.

§2. Proof-theoretic results. In this section, we present some basic results about free sets. Some of them were already stated without explicit proof in [5] and [7].

THEOREM 2.1. RCA_0 *proves the following:*

(1) *If A is a free set for f then every subset of A is a free set for f.*
(2) *A is a free set for f if and only if any finite subset of A is a free set for f.*

PROOF. The proofs of the first item and the implication from left to right in the second item are immediate from the definitions. To prove the remaining implication, assume that every finite subset B of A is free. Pick any k-tuple $x_1, \dots, x_k \in A$. If $f(x_1, \dots, x_k) \notin A$ we are done. If $f(x_1, \dots, x_k) \in A$, take $B = \{x_1, \dots, x_k, f(x_1, \dots, x_k)\}$. Since B is free by hypothesis and $f(x_1, \dots, x_k) \in B$, we have $f(x_1, \dots, x_k) \in \{x_1, \dots, x_k\}$. Hence A is free. ⊣

THEOREM 2.2. [7]. RCA_0 *proves* $FS(1)$.

PROOF. Let $f : \mathbb{N} \to \mathbb{N}$. If f is bounded by some $k \in \mathbb{N}$, a free set for f is given by $A = \{n \mid n > k\}$.

Assume that f is unbounded, i.e. $\forall y \exists x \; f(x) > y$. We define the free set $A = \{x_0, x_1, \dots\}$ by induction. Let $x_0 = 0 \in A$. Inductively, for $n > 0$ define x_n to be the least natural number $z > x_{n-1}$ such that $z \notin \{f(x_0), f(x_1), \dots, f(x_{n-1})\}$ and $f(z) \notin \{x_0, x_1, \dots, x_{n-1}\}$. Such a z exists because f is unbounded. We claim that A is a free set for f. By construction $f(x_n) \neq x_i$ and $x_n \neq f(x_i)$ whenever $i < n$. Thus $x_i \neq f(x_j)$ whenever $i \neq j$. It follows that $A = \{x_0, x_1, \dots\}$ is free for f, and A is infinite because $x_n > x_{n-1}$ for all $n > 0$. ⊣

THEOREM 2.3. (RCA_0). *For any fixed k, $FS(k+1)$ implies $FS(k)$*

PROOF. Let $f : [\mathbb{N}]^k \to \mathbb{N}$ be given. We want to find a free set for f. Let us define $g : [\mathbb{N}]^{k+1} \to \mathbb{N}$ as $g(x_1, x_2, \dots, x_{k+1}) = f(x_2, \dots, x_{k+1})$. By hypothesis, g has a

[1] We would like to thank Friedman and the anonymous referee for this example.

free set, say B. Let $m = \min(B)$ and define $A = B \setminus \{m\}$. We prove that A is a free set for f. Let $x_1, \ldots, x_k \in A$. If $f(x_1, \ldots, x_k) \in A$, then also $g(m, x_1, \ldots, x_k) \in A \subseteq B$. Since B is free for g it follows $g(m, x_1, \ldots, x_k) \in \{m, x_1, \ldots, x_k\}$. But actually $g(m, x_1, \ldots, x_k) \neq m$ because $m \notin A$. Hence $g(m, x_1, \ldots, x_k) \in \{x_1, \ldots, x_k\}$ and therefore $f(x_1, \ldots, x_k) \in \{x_1, \ldots, x_k\}$ as required. $\dashv$

The following technical lemma shows that given FS, infinite free sets can be found within any infinite set. In this respect, free sets resemble the homogeneous sets of Ramsey's theorem.

LEMMA 2.4. (RCA_0). *For each $k \in \mathbb{N}$, the following are equivalent:*

(1) $FS(k)$.
(2) *Suppose that X is an infinite subset of $\mathbb{N}$ and $f : [X]^k \to \mathbb{N}$. Then X contains an infinite subset A which is free for f.*

PROOF. To prove that statement (2) implies statement (1), simply set $X = \mathbb{N}$ in Statement (2).

The proof of the converse is slightly more involved. Assume RCA_0 and $FS(k)$. Let X and f be as in the hypothesis and enumerate X, setting $X = \{x_1, x_2, \ldots\}$. Define a function $f^\star : [\mathbb{N}]^k \to \mathbb{N}$ by

$$f^\star(a_1, \ldots, a_k) = \begin{cases} a & \text{if } f(x_{a_1}, \ldots, x_{a_k}) = x_a \\ 0 & \text{if } f(x_{a_1}, \ldots, x_{a_k}) \notin X. \end{cases}$$

Let $A^\star$ be an infinite free set for $f^\star$. Since every subset of a free set is free, without loss of generality we may assume that $0 \notin A^\star$. Let $A = \{x_a \mid a \in A^\star\}$. A is obviously a subset of X. To complete the proof, we will show that that A is free for f. Suppose that $x_{a_1}, \ldots, x_{a_k} \in A$, and that $f(x_{a_1}, \ldots, x_{a_k}) \in A$. By the definition of A, there is an $a \in A^\star$ such that $f(x_{a_1}, \ldots, x_{a_k}) = x_a$. From the definition of $f^\star$, $f^\star(a_1, \ldots, a_k) = a$, and since $a \in A^\star$ and $A^\star$ is free for $f^\star$, we have $a \in \{a_1, \ldots, a_k\}$. Consequently $x_a \in \{x_{a_1}, \ldots, x_{a_k}\}$, completing the proof that A is free for f. $\dashv$

§3. A weak version of the free set theorem. The following weakened version of the free set theorem, known as the thin set theorem, was introduced by Friedman in [5].

STATEMENT 3.1. (TS – thin set theorem). *Let $k \in \mathbb{N}$ and let $f : [\mathbb{N}]^k \to \mathbb{N}$. Then there exists an infinite $A \subseteq \mathbb{N}$ such that $f([A]^k) \neq \mathbb{N}$. We denote by $TS(k)$ the statement TS for a fixed $k \geq 1$. We call a set A* thin *(for f) if $f([A]^k) \neq \mathbb{N}$.*

The next two results of Friedman show that TS is weak in the sense that it follows easily from FS. We conjecture that $TS(k)$ does not imply $FS(k)$.

THEOREM 3.2. (RCA_0). *For any $k \in \mathbb{N}$, $FS(k)$ implies $TS(k)$*

PROOF. Let $f : [\mathbb{N}]^k \to \mathbb{N}$. Let A be an infinite free set for f. Let B be a nonempty subset of A such that $A \setminus B$ is infinite. We show that $A \setminus B$ is a set which fulfills $TS(k)$. Assume, for a contradiction, that for all $n \in \mathbb{N}$ there exist $x_1, \ldots, x_k \in A \setminus B$ such that $f(x_1, \ldots, x_k) = n$. Take $n \in B$. In particular, we have also $n \in A$. Since A is free, it follows that $n \in \{x_1, \ldots, x_k\}$. Hence there is some $i \leq k$ such that $x_i = n \in B$, which contradicts $x_1, \ldots, x_k \in A \setminus B$ for all $i \leq k$. $\dashv$

Corollary 3.3 follows immediately from Theorem 3.2

COROLLARY 3.3. (RCA_0). *FS implies TS.*

Now we will show that an analog of Theorem 2.3 holds for the thin set theorem.

THEOREM 3.4. (RCA_0). *For any fixed k, $TS(k+1)$ implies $TS(k)$.*

PROOF. Let $f : [\mathbb{N}]^k \to \mathbb{N}$ be given. We want to find a set A such that $f([A]^k) \neq \mathbb{N}$ holds. Let us define $g : [\mathbb{N}]^{k+1} \to \mathbb{N}$ as $g(x_1, \dots, x_{k+1}) = f(x_1, \dots, x_k)$. Since $TS(k+1)$ holds, there exists an infinite set $A \subseteq \mathbb{N}$ such that $g([A]^{k+1}) \neq \mathbb{N}$. Because A is infinite and every increasing k-tuple from A is an initial segment of an increasing $(k+1)$-tuple from A, we have $f([A]^k) \subseteq g([A]^{k+1})$. Thus A is an infinite set which is thin for f, as needed. ⊣

TS asserts the existence of a set X such that the complement of $f([X]^k)$ is nonempty. Requiring that the complement of $f([X]^k)$ is infinite results in a statement of precisely the same logical strength. This result is implicit in [7].

THEOREM 3.5. (RCA_0). *For each $k \in \mathbb{N}$, the following are equivalent:*

(1) *$TS(k)$: If $f : [\mathbb{N}]^k \to \mathbb{N}$, then there is an infinite set X such that $f([X]^k) \neq \mathbb{N}$.*
(2) *$TS'(k)$: If $f : [\mathbb{N}]^k \to \mathbb{N}$, then there is an infinite set X such that $\mathbb{N} \setminus f([X]^k)$ is infinite. More formally, there are infinite sets X and Y such that for all $x_1, \dots, x_k \in X$ with $x_1 < x_2 < \dots < x_k$, $f(x_1, \dots, x_k) \notin Y$.*

PROOF. The proof that $TS'(k)$ implies $TS(k)$ follows immediately from the fact that when $\mathbb{N} \setminus f([X]^k)$ is infinite, $f([X]^k) \neq \mathbb{N}$. To prove the converse, assume RCA_0 and $TS(k)$, and let $f : [\mathbb{N}]^k \to \mathbb{N}$. Let p_i denote the i^{th} prime number, so that in particular we have $p_0 = 2$. Define a new coloring map by setting

$$g(x_1, \dots, x_k) = \begin{cases} i & \text{if } f(x_1, \dots, x_k) = p_i^n \text{ for some } n \geq 1, \\ f(x_1, \dots, x_k) & \text{otherwise.} \end{cases}$$

Applying $TS(k)$ to g, we can find an infinite set X and a $j \in \mathbb{N}$ such that $j \notin g([X]^k)$. If for some $(x_1, \dots, x_k) \in [X]^k$ and some $n \geq 1$ we have $f(x_1, \dots, x_k) = p_j^n$, then $g(x_1, \dots, x_k) = j$, contradicting $j \notin g([X]^k)$. Thus $\mathbb{N} \setminus f([X]^k)$ contains the infinite set $\{p_j^n \mid n \geq 1\}$, proving $TS'(k)$. ⊣

As a corollary, we can show that a relativized version of TS is provably equivalent to the original version.

COROLLARY 3.6. (RCA_0). *For each $k \in \mathbb{N}$, the following are equivalent:*

(1) *$TS(k)$.*
(2) *Let X be an infinite subset of $\mathbb{N}$. If the range of $f : [X]^k \to \mathbb{N}$ is unbounded in $\mathbb{N}$, then there exists an infinite set $A \subset X$ such that $f([X]^k) \setminus f([A]^k)$ is infinite.*

PROOF. Clearly the second statement implies $TS(k)$, taking $X = \mathbb{N}$. To show that $TS(k)$ implies the second statement, suppose that the range of $f : [X]^k \to \mathbb{N}$ is unbounded in $\mathbb{N}$. RCA_0 suffices to prove that there is an infinite set Y which is a subset of the range of f. Let $g : \mathbb{N} \to X$ and $h : Y \to \mathbb{N}$ be increasing, one-to-one and onto functions. Consider $g^* : [\mathbb{N}]^k \to \mathbb{N}$ defined as

$$g^*(x_1 \dots, x_k) = \begin{cases} h(f(g(x_1), \dots, g(x_k))) & \text{if } f(g(x_1), \dots, g(x_k)) \in Y \\ 0 & \text{otherwise.} \end{cases}$$

Using TS', which is equivalent to TS by Lemma 3.5, there is a set $A^* \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus g^*([A^*]^k)$ is infinite. Let $A = g(A^*)$. We claim that $f([X]^k) \setminus f([A]^k)$ is infinite. If this is not the case, $f([X]^k) \setminus f([g(A^*)]^k)$ should be finite and hence also $\mathbb{N} \setminus h(f([g(A^*)]^k))$ should be finite, which is a contradiction. $\dashv$

§4. Lower bounds on the strength of FS. In this section we show that, in contrast to Theorem 2.2, if $k \geq 2$ then neither RCA_0 nor WKL_0 is sufficiently strong to prove $FS(k)$. For RCA_0, this is immediate consequence of the following theorem. We will show in Proposition 5.5 that this result is best possible with respect to the arithmetic hierarchy.

THEOREM 4.1. *For each $k \geq 2$ there is a computable function $f : [\mathbb{N}]^k \to \mathbb{N}$ such that no infinite Σ^0_k set is thin for f.*

PROOF. The proof is very similar to the proof of the corresponding result for Ramsey's Theorem, i.e. Theorem 5.1 of [10]. This result is proved in relativized form by induction on k, starting at $k = 2$.

For the base step, we prove the result for $k = 2$ in unrelativized form for notational convenience, since relativization is routine. Since every infinite Σ^0_2 set has an infinite Δ^0_2 subset and every subset of a thin set is thin, it suffices to show that there is a computable function $f : [\mathbb{N}]^2 \to \mathbb{N}$ such that no infinite Δ^0_2 set is thin for f. By the proof of the Limit Lemma, there is a computable $\{0, 1\}$-valued function $h(e, n, s)$ such that for every Δ^0_2 set A there exists e with $A(n) = \lim_s h(e, n, s)$ for all n. Fix such an h, and for each e let A_e be the unique A with $A(n) = \lim_s h(e, n, s)$ for all n, provided that such a set A exists. If no such A exists (i.e. $\lim_s h(e, n, s)$ fails to exist for some n), let A_e be undefined. Thus the sets A_e with A_e defined are precisely the Δ^0_2 sets. It suffices to define a computable function $f : [\mathbb{N}]^2 \to \mathbb{N}$ which meets the following requirements $R_{\langle e,i\rangle}$ for all e and i.

$$R_{\langle e,i\rangle} : \text{ If } A_e \text{ is defined and infinite, then } i \in f([A_e]^2)$$

If A_e is defined and has more than $\langle e, i\rangle$ elements, let $F_{e,i}$ be the finite set consisting of the least $\langle e, i\rangle + 1$ elements of A_e. Let $F_{e,i,s}$ be the natural computable approximation to $F_{e,i}$ at stage s, i.e. if there are more than $\langle e, i\rangle$ numbers $n < s$ with $h(e, n, s) = 1$, let $F_{e,i,s}$ consist of the first $\langle e, i\rangle + 1$ such numbers, and otherwise let $F_{e,i,s}$ be undefined. Clearly, if $F_{e,i}$ is defined, then $F_{e,i,s} = F_{e,i}$ for all sufficiently large s.

The construction of f is carried out in stages, and $f(n, s)$ is defined at stage s for each $n < s$. Stage s has $s + 1$ substages, $0, 1, \ldots s$, and f is defined on at most one new argument at each substage $t < s$. The construction is as follows:

Stage s, substage $\langle e, i\rangle < s$. This substage is dedicated to meeting the requirement $R_{\langle e,i\rangle}$. If $F_{e,i,s}$ is not defined, proceed to the next substage $\langle e, i\rangle + 1$ without taking any action. If $F_{e,i,s}$ is defined, let $n_{e,i,s}$ be the least element n of $F_{e,i,s}$ with $f(n, s)$ not yet defined, and set $f(n_{e,i,s}, s) = i$. (Such a number n exists because $|F_{e,i,s}| = \langle e, i\rangle + 1$, and $f(k, s)$ has been defined for at most one value of k at each of the previous $\langle e, i\rangle$ substages. Note also that $n_{e,i,s} < s$ because $\max(F_{e,i,s}) < s$ by the definition of $F_{e,i,s}$.) Go to substage $\langle e, i\rangle + 1$.

At the final substage s of stage s, set $f(n, s) = 1$ for all $n < s$ such that $f(n, s)$ is as yet undefined. This completes the construction.

To see that each requirement $R_{\langle e,i\rangle}$ is met, assume that A_e is defined and infinite. Then $F_{e,i}$ is defined, and $F_{e,i,s} = F_{e,i} \subseteq A_e$ for all sufficiently large s. It follows by construction that $i \in f([F_{e,i} \cup \{s\}]^2)$ for all sufficiently large s. Since A_e is infinite, $i \in f([A_e]^2)$ as required. This completes the proof for $k = 2$.

(Note that we are not claiming that $\lim_s n_{e,i,s}$ exists for all e and i, as there might exist $\langle e', i'\rangle < \langle e, i\rangle$ such that $F_{e',i'}$ is not defined but $F_{e',i',s}$ is defined for infinitely many s.

Indeed, it is impossible that $\lim_s n_{e,i,s}$ exists for all e and i. To see this, assume that $\lim_s n_{e,i,s}$ exists for all e and i. Then one can easily show by induction on n that $\lim_s f(n, s)$ exists for all n, i.e. f is *stable*, as in [1], Definition 3.4. But if f is stable, there exists an infinite Δ^0_2 set which is thin for f, which is impossible by our construction.)

For the inductive step, assume that for each set $X \subseteq \omega$ there is an X-computable function $f : [\mathbb{N}]^k \to \mathbb{N}$ such that no infinite $\Sigma^{0,X}_k$ set is thin. To prove the corresponding result for $k+1$, let X be given. Using the inductive hypothesis, choose an X'-computable function $f : [\mathbb{N}]^k \to \mathbb{N}$ such that no infinite $\Sigma^{0,X'}_k$ set is thin and hence no infinite $\Sigma^{0,X}_{k+1}$ set is thin. By the Limit Lemma, there is an X-computable function $g : [\mathbb{N}]^{k+1} \to \mathbb{N}$ with $\lim_s g(x_1, \dots, x_k, s) = f(x_1, x_2, \dots, x_k)$ for all $(x_1, \dots x_k) \in [\mathbb{N}]^k$. Every set thin for g is thin for f, so no $\Sigma^{0,X}_{k+1}$ set is thin for g, as required to complete the induction. $\dashv$

COROLLARY 4.2. *Let $k \geq 2$. Then there is a computable function $f : [\mathbb{N}]^k \to \mathbb{N}$ with no infinite Σ^0_k free set.*

PROOF. Let $f : [\mathbb{N}]^k \to \mathbb{N}$ be a computable function with no infinite Σ^0_k thin set. If f had an infinite Σ^0_k free set A, then for any $a \in A$, $A \setminus \{a\}$ would be an infinite Σ^0_k thin set for f, by the proof of Theorem 3.2. $\dashv$

COROLLARY 4.3. [5]. *There is a computable function $f : [\mathbb{N}]^2 \to \mathbb{N}$ with no computable free set.*

Since $\mathbb{N}$ together with the computable sets form a model of RCA_0 the preceding corollary shows that there is a model of RCA_0 which is not a model of $FS(2)$. We can translate this into a proof theoretic result as follows.

COROLLARY 4.4. RCA_0 *does not prove $FS(2)$.*

The preceding result can be improved to show that WKL_0 does not prove $FS(2)$. This was announced by Friedman in [5] and proved in [6]. In Friedman's original proof, a computable function f is constructed such that no function that is primitive recursive in K can dominate any function that enumerates a solution to $TS(2)$ for f. This f witnesses the failure of $TS(2)$ in any ω-model of WKL_0 whose functions are all primitive recursive in K. Our proof uses iteration in a fashion similar to that of Friedman, but uses almost computable sets. A set A is *almost computable* if every function computable from A is majorized by a computable function.

LEMMA 4.5. *There is a computable function $g : [\mathbb{N}]^2 \to \mathbb{N}$ such that for any infinite set A, if A is almost computable, then $g([A]^2)$ is cofinite.*

PROOF. Let $\langle h_e\rangle_{e\in\omega}$ be a computable listing of the computable partial functions. We will write $h_{e,y}(n)$ to denote the value of $h_e(n)$ computed by stage y, and write $h_{e,y}(n) \downarrow$

if that value is defined. Define the function $\Delta(n,y):[\mathbb{N}]^2\to\mathbb{N}$ by

$$\Delta(n,y)=\max\left(\{h_{j,y}(m)\mid j\le n\wedge m\le n\wedge h_{j,y}(m)\downarrow\}\cup\{n+1\}\right).$$

Note that for each n and y, $\Delta(n,y)$ is defined, and $\lim_y \Delta(n,y)$ exists for each n. Let $\Delta^i(n,y)$ denote the i^{th} iteration of Δ calculated for a fixed y. For example, $\Delta^2(n,y)=\Delta(\Delta(n,y),y)$. Define $g:[\mathbb{N}]^2\to\mathbb{N}$ by setting

$$g(n,y)=\mu i\le n(\Delta^i(0,y)\ge n).$$

Note that $g(n,y)$ is computable, and for each n, $\lim_y g(n,y)$ exists. Furthermore, viewed as a function of n, $\lim_y g(n,y)$ is nondecreasing and unbounded in $\mathbb{N}$.

Suppose that A is an infinite almost computable set. Let $\langle a_i\rangle_{i\in\omega}$ be the enumeration of A in increasing order. This enumeration may not be computable, but since A is almost computable, we may fix a k so that for almost all $i\in\mathbb{N}$, $h_k(i)\ge a_i$. Our goal is to show that $g([A]^2)$ is cofinite.

Since $\lim_y g(n,y)$ is nondecreasing and unbounded, we can choose j so large that $\lim_y g(a_j,y)>k$. Let $\lim_y g(a_j,y)=t$. For any sufficiently large value of $y\in A$, $g(a_j,y)=t$, which implies that $\Delta^t(0,y)\ge a_j$. If y is also so large that $h_{k,y}(a_j)\downarrow$, then

$$\Delta^{t+1}(0,y)=\Delta(\Delta^t(0,y),y)\ge\Delta(a_j,y)\ge h_k(a_j)\ge a_{j+1},$$

so $\lim_y g(a_{j+1},y)\le t+1$. Because $\lim_y g(a_{j+1},y)\ge\lim_y g(a_j,y)=t$, we have $t\le\lim_y g(a_{j+1},y)\le t+1$. Indeed, for any $m\ge j$,

$$\lim_y g(a_m,y)\le\lim_y g(a_{m+1},y)\le\lim_y g(a_m,y)+1.$$

Since $\lim_y g(n,y)$ is unbounded, and for each i, $\lim_y g(a_i,y)\in g([A]^2)$, we have that $[t,\infty)\subset g([A]^2)$, showing that $g([A]^2)$ is cofinite. $\dashv$

THEOREM 4.6. (Friedman). *There is an ω-model of WKL_0 which is not a model of $TS(2)$.*

PROOF. Using Corollary VIII.2.22 of [15], select an ω-model M of WKL_0 such that for all $X\in M$, X is almost computable. The function g of the preceding lemma is in M, but for every infinite set $A\in M$, $g([A]^2)$ is cofinite, and hence not co-infinite. Thus $TS'(2)$ fails in M, and since RCA_0 proves that $TS(2)$ is equivalent to $TS'(2)$, $TS(2)$ also fails in M. Alternatively, this result can be proved by choosing an ω model M of WKL_0 such that every set $X\in M$ is low. Such an M exists by Corollary VIII.2.18 of [15]. Then M is not a model of $TS(2)$ by Theorem 4.1, since every low set is Σ^0_2. $\dashv$

Friedman has also found lower bounds for the strength of FS and TS. The article [7] contains a proof that ACA_0 does not imply TS, which by an application of Lemma 3.2 also shows that ACA_0 does not imply FS.

§5. Upper bounds on the strength of FS and the arithmetical complexity of free sets. In this section we will show that every computable coloring of k-tuples has an infinite Π^0_k free set. (By Corollary 4.2, this result is optimal with respect to the arithmetic hierarchy for $k\ge 2$.) The proof of this result will also show that Ramsey's theorem for

for 2-colorings of k-tuples (as formalized in the following definition) implies $FS(k)$ in RCA_0. This implication is due to Harvey Friedman [5] for $k \geq 3$, but is new for $k = 2$.

STATEMENT 5.1. (RT^k_n). *Given $f : [\mathbb{N}]^k \to n$, an n-coloring of the k-tuples of $\mathbb{N}$, there is an infinite set $X \subseteq \mathbb{N}$ such that f is constant on $[X]^k$. We use the notation $RT^k_{<\infty}$ to denote $(\forall n) RT^k_n$, and RT to denote $(\forall n)(\forall k) RT^k_n$.*

THEOREM 5.2. *Let $f : [\mathbb{N}]^k \to \mathbb{N}$ be computable. Then there is an infinite Π^0_k set C which is free for f.*

PROOF. If $\vec{w}$ is an ordered k-tuple and $1 \leq j \leq k$, we write $(\vec{w})_j$ for the jth component of $\vec{w}$.

Define

$$S = \{\vec{x} \in [\mathbb{N}]^k : f(\vec{x}) < (\vec{x})_k \quad \& \quad f(\vec{x}) \notin \{(\vec{x})_1, \ldots, (\vec{x})_k\}\}$$

For $\vec{x} \in S$, let $i(\vec{x})$ be the least j such that $f(\vec{x}) < (\vec{x})_j$. (Such a j exists because $f(\vec{x}) < (\vec{x})_k$.)

For $\vec{x} \in S$, let $h(\vec{x})$ be the increasing k-tuple which results from $\vec{x}$ by replacing $(\vec{x})_{i(\vec{x})}$ by $f(\vec{x})$. Hence, for $\vec{x} \in S$, $(h(\vec{x}))_{i(\vec{x})} < (\vec{x})_{i(\vec{x})}$.

For $\vec{x} \in S$, let $c(\vec{x})$ be the least $j \in \omega$ such that $h^{(j)}(\vec{x}) \notin S$ or $i(h^{(j)}(\vec{x})) \neq i(\vec{x})$. Here $h^{(j)}$ is the j-fold iteration of h. Note that $c(\vec{x})$ is defined for each $\vec{x} \in S$. (If not, then $(\vec{x})_{i(\vec{x})}, (h(\vec{x}))_{i(\vec{x})}, (h^{(2)}(\vec{x}))_{i(\vec{x})}, \ldots$ is an infinite descending chain of natural numbers by a remark in the previous paragraph.)

Define a computable function $g : [\mathbb{N}]^k \to 2k + 2$ as follows:

$$g(\vec{x}) = \begin{cases} 0 & \text{if } f(\vec{x}) \in \{(\vec{x})_1, (\vec{x})_2, \ldots, (\vec{x})_k\} \\ 1 & \text{if } f(\vec{x}) > (\vec{x})_k \\ 2i(\vec{x}) + j & \text{if } \vec{x} \in S,\ j \leq 1, \text{ and } c(\vec{x}) \equiv j \mod 2 \end{cases}$$

By [10], Theorem 5.5 there is an infinite Π^0_k set A which is homogeneous for g. We will show that there is an infinite set $B \leq_T A$ such that B is free for f. Of course, this suffices to prove that there is an infinite Δ^0_{k+1} set which is free for f. In order to obtain the stronger result that there is an infinite Π^0_k set which is free for f, we impose the additional requirement that A be retraceable by a total function $p \leq_T 0^{(k-1)}$. (This is shown to be possible for the case that g is a c.e. 2-coloring of $[\mathbb{N}]^k$ in [9], Theorem 3.1, and a similar argument works for computable colorings with any finite number of colors.)

Case 1. $g([A]^k) = \{0\}$. Then A is free for f.

Case 2. $g([A]^k) = \{1\}$. Define an increasing sequence $\{c_j\}$ of elements of A by recursion on j. Let c_0 be the least element of A. Given c_j, let c_{j+1} be the least $x \in A$ such that $x > c_j$ and $x \notin f([\{c_0, c_1, \ldots, c_j\}]^k)$. Then c_j is defined for every j because A is infinite. Let $C = \{c_j : j \in \omega\}$. Then C is infinite because $c_0 < c_1 < \ldots$. Also C is free for f. (If $\vec{x} \in [C]^k$, then $f(\vec{x}) > (\vec{x})_k$ because $\vec{x} \in [A]^k$ and $f([A]^k) = \{1\}$. But each element z of C is chosen so that it is not of the form $f(\vec{x})$ where $\vec{x}$ is any increasing k-tuple of elements of C, all smaller than z.)

To complete the proof in this case, it suffices to show that C is Π^0_k. This is proved by virtually the same argument as used in Theorem 3.1 of [9] to show that the set denoted C there is Π^0_k. We repeat the argument here for the convenience of the reader. By the retraceability hypothesis on A, there exists a function $q \leq_T 0^{(k-1)}$ such that, for all

$x \in A$, $D_{q(x)} = \{z \in A : z \leq x\}$, where D_z is the finite set with canonical index z. Now let T be the set of numbers x whose membership in C follows from the hypothesis that $\{z \in A : z \leq x\} = D_{q(x)}$. (That is, to determine whether $x \in T$ carry out the above recursive definition of $\{c_j\}$ using $D_{q(x)}$ in place of A until a j is found such that c_j is not defined. Then x is in T if and only if some c_j generated in this way is equal to x.) Note that $T \leq_T q \leq_T 0^{(k-1)}$ so T is Δ^0_k. Finally, observe that $C = A \cap T$, so C is Π^0_k, as needed to complete this case.

Case 3. $f([A]^k) = 2i + j$ where $i \geq 1$ and $j \leq 1$. We claim that in this case A itself is free, which suffices to complete the proof. Suppose not, and fix $\vec{x} \in [A]^k$ with $f(\vec{x}) \in A$. It follows from the case hypothesis that $\vec{x} \in S$. Also $h(\vec{x}) \in [A]^k$, by definition of h and the hypothesis that $f(\vec{x}) \in A$. Hence, by the case hypothesis, $f(h(\vec{x})) = 2i + j = f(\vec{x})$, so $c(\vec{x}) \equiv c(h(\vec{x}))$ mod 2. This is impossible because $c(\vec{x}) = c(h(\vec{x})) + 1$. To see this, recall that $c(\vec{x})$ is the number of times that h must be applied to $\vec{x}$ to obtain a vector $\vec{w}$ such that $\vec{w} \notin S$ or $i(\vec{w}) \neq i(\vec{x})$. As $i(\vec{x}) = i = i(h(\vec{x}))$, $c(h(\vec{x}))$ is computed in the same way, but starting with $h(\vec{x})$ instead of with $\vec{x}$, so one fewer iteration of h is required. Note also that $c(\vec{x}) \geq 1$ since $h(\vec{x}) \in S$ and $i(\vec{x}) = i = i(h(\vec{x}))$. ⊣

COROLLARY 5.3. *(H. Friedman [5] for $k \geq 3$).* (RCA_0). $(\forall k)[RT^k_{2k+2} \Longrightarrow FS(k)]$.

PROOF. Assuming RCA_0 and RT^k_{2k+2}, emulate the proof of Theorem 5.2 (omitting the second paragraph of case 2). The existence of the coloring g is provable in RCA_0, and the existence of a homogeneous set A follows from Ramsey's theorem. The proofs that A is free in cases 1 and 3 can be formalized in RCA_0. Finally, RCA_0 suffices to prove that the set C of case 2 exists and is free.

There is an alternative proof that works well for standard integers $k \geq 3$. Since RT^3_2 implies ACA_0 and ACA_0 implies $RT^{k+1}_{<\infty}$, it suffices to use $RT^{k+1}_{<\infty}$ to deduce $FS(k)$. Given $f : [\mathbb{N}]^k \to \mathbb{N}$, define $g : [\mathbb{N}]^{k+1} \to k+2$ by setting $g(x_1, \ldots, x_{k+1})$ equal to the least j such that $f(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k+1}) = x_j$, and equal to 0 if no such j exists. Any homogeneous set for g is free for f. ⊣

A number of corollaries follow immediately from the theorem via applications of the substantial body of results on the strength of RT^2_2. One immediate corollary is that every ω-model of RT^2_2 is an ω-model of $FS(2)$. This statement can also be proved by a forcing argument adapted from the proof in [1] that $\mathsf{RCA}_0 + RT^2_2 + \mathrm{I}\Sigma^0_2$ is Π^1_1-conservative over $\mathsf{RCA}_0 + I\Sigma^0_2$.

COROLLARY 5.4. (1) *For each k, it is provable in RCA_0 that RT^k_2 implies $FS(k)$ and that $FS(k)$ implies $TS(k)$.*
(2) *Over RCA_0, $FS(2)$ does not imply ACA_0.*
(3) *Over RCA_0, $FS(2)$ does not imply $RT^2_{<\infty}$.*
(4) *Suppose $k \geq 2$ and let $f : [\mathbb{N}]^k \to \mathbb{N}$ be a computable function. Then f has an infinite free set A with $A'' \leq_T 0^{(k)}$.*
(5) *$FS(2)$ is Π^1_1-conservative over $\mathsf{RCA}_0 + I\Sigma^0_2$.*

PROOF. The first statement in Part (1) follows from Corollary 5.3 and the fact that, for each n and k, RT^k_2 implies RT^k_n over RCA_0. The second statement is immediate from the proof of Theorem 3.2.

The next two parts follow from the facts that RT^2_2 cannot prove ACA_0 [14], and that $RT^2_{<\infty}$ is strictly stronger than RT^2_2 [1].

Part (4) follows by applying the existence of infinite homogeneous sets A with $A'' \leq_T 0^{(k)}$ for computable colorings of $[\mathbb{N}]^k$ with finitely many colors [1], Theorem 3.1, and the remark in the proof of Theorem 5.2 that for each infinite set A homogeneous for g there is an infinite set C free for f with $C \leq_T A$.

The last part follows from the fact that $\mathsf{RCA}_0 + RT^2_2 + \mathrm{I}\Sigma^0_2$ is Π^1_1-conservative over $\mathsf{RCA}_0 + I\Sigma^0_2$ [1]. $\dashv$

The following result shows that Theorem 4.1 is optimal with respect to the arithmetical hierarchy.

PROPOSITION 5.5. *Let $f : [\mathbb{N}]^k \to \mathbb{N}$ be computable. Then there is an infinite Π^0_k set A which is thin for f.*

PROOF. Define $g : [\mathbb{N}]^k \to \{0, 1\}$ by $g(\vec{x}) = 0$ if $f(\vec{x}) = 0$ and $g(\vec{x}) = 1$ otherwise. Then g is a computable 2-coloring of $[\mathbb{N}]^k$ so by [10], Theorem 5.5, there is an infinite Π^0_k set which is homogeneous for g and hence thin for f. Alternatively, the proposition follows from Theorem 5.2. $\dashv$

Since for each standard natural number k, ACA_0 proves Ramsey's theorem for k-tuples, we have the following corollary which appears in [5].

COROLLARY 5.6. [5]. *For each $k \in \omega$, ACA_0 proves $FS(k)$.*

COROLLARY 5.7. [5]. *Every arithmetical function f has an arithmetical infinite free set.*

PROOF. This is immediate from a relativized form of Theorem 5.2. For a different proof, note that the model of second order arithmetic consisting of ω together with the arithmetical sets is a model of ACA_0. By Corollary 5.6, this is also a model of $FS(k)$ for each standard number k. Every function in the model must have a free set in the model. $\dashv$

The previous corollary led us to conjecture and to prove that the degrees of the free sets are closed upwards. The proof uses a result of Jockusch [11] that we recall here for the reader.

THEOREM 5.8. [11]. *If $\mathcal{P}$ is a property of infinite sets which is hereditary under inclusion and enjoyed by some arithmetical set, then the class of $\mathcal{P}$-degrees is closed upwards.*

COROLLARY 5.9. *For every arithmetical function f, the degrees of the free sets for f are closed upwards.*

PROOF. Since every infinite subset of a free set is free (Remark 2.1) and Corollary 5.7 witnesses that there exist arithmetical free sets, the result immediately follows from Theorem 5.8. $\dashv$

It is known that RT is equivalent to $\mathsf{ACA}_0{}'$ over RCA_0, where the system $\mathsf{ACA}_0{}'$ is defined as $\mathsf{ACA}_0 + \forall n\ \forall X\ (\mathit{the}\ n^{th}\ \mathit{Turing\ jump\ of}\ X\ \mathit{exists})$. From Corollary 5.3 we know that RT implies FS and consequently we have the following corollary, which appears in [7].

COROLLARY 5.10. [7]. *$\mathsf{ACA}_0{}'$ implies FS.*

We now consider freeness for *partial* functions, which is defined in the obvious way below. This will lead to a proof that a certain result holds relative to $0'$ whereas the result itself remains open.

Definition 5.11. A set A is *free* for a partial function ψ on $[\mathbb{N}]^k$ if there do not exist $x_1 < x_2 < \cdots < x_k$ with each x_i in A, and $\psi(x_1, \ldots, x_k) \downarrow \in A - \{x_1, \ldots, x_k\}$.

Corollary 5.12. *The following result* $(*)$ *holds* when relativized to $0'$*:*
$(*)$ *For every computable partial function* ψ *on* $[\mathbb{N}]^2$ *there is an infinite* Π^0_2 *free set.*

Proof. Suppose ψ is a $0'$-computable partial function defined on $[\mathbb{N}]^2$. Let g be a 3-place computable function so that $\psi(a, b) = \lim_s g(a, b, s)$ for all (a, b) in the domain of ψ. (Such a g exists by the proof of the Limit Lemma.) Let A be an infinite Π^0_3 free set for g. Then A is also an infinite $\Pi^{0,K}_2$ free set for ψ. $\dashv$

It follows from the above corollary that $(*)$ cannot be refuted by a relativizable argument. On the other hand, we have not been able to prove $(*)$. In particular, the proof of Theorem 5.2 does not seem to adapt to partial functions, and an independent unpublished proof of Theorem 5.2 for the case $k = 2$ (not based on Ramsey's theorem) does not seem to adapt to partial functions either.

We close this section with a version of FS that is equivalent to Ramsey's theorem. The reader may wish to compare the following theorem to Corollary 3.6.

Theorem 5.13. *For all* $k \in \omega$*,* RCA_0 *proves that the following are equivalent:*

(1) RT^k_2

(2) *If* $f : [\mathbb{N}]^k \to \mathbb{N}$ *is not constant, then there exists an infinite* $A \subseteq \mathbb{N}$ *such that* $f([A]^k) \neq f([\mathbb{N}]^k)$.

Proof. (1) $\Longrightarrow$ (2). Let $f : [\mathbb{N}]^k \to \mathbb{N}$. Fix $n_0 \in f([\mathbb{N}]^k)$. Define a 2-coloring

$$g(x_1, \ldots, x_k) = \begin{cases} \text{red} & \text{if } f(x_1, \ldots, x_k) = n_0 \\ \text{blue} & \text{otherwise} \end{cases}$$

By RT^k_2 there exists a homogeneous set H. Define $A = H$. If H is red, we have $g([H]^k)$ red if and only if $f([H]^k) = n_0$, and hence $f([H]^k) \neq f([\mathbb{N}]^k)$. If H is blue we have $f([H]^k) \subseteq \mathbb{N} \setminus \{n_0\} \neq f([\mathbb{N}]^k)$. Therefore in both cases we are done.

(2) $\Longrightarrow$ (1). Let $f : [\mathbb{N}]^k \to 2$ be a 2-coloring. By the statement (2), there exists an infinite set $A \subseteq \mathbb{N}$ such that $f([A]^k) \neq \{0, 1\}$. Hence A is a homogeneous set for f. $\dashv$

§6. FS for subsets. In this section, we will prove a variation of the free set theorem in which finite sets play the role previously played by k-tuples. We will need the following definitions. A sequence $X = \langle X_i \rangle_{i \in \mathbb{N}}$ of finite subsets of $\mathbb{N}$ is said to be *increasing* if for every i the maximum element of X_i is less than the minimum element of X_{i+1}. When the maximum element of X_i is less than the minimum element of X_{i+1}, we write $X_i < X_{i+1}$. The subsystem $\mathsf{ACA}_0{}^+$ consists of ACA_0 together with an axiom that asserts that $A^{(\omega)}$ exists for each set A. This system is strictly stronger than $\mathsf{ACA}_0{}'$.

Theorem 6.1. ($\mathsf{ACA}_0{}^+$). Suppose $F : [\mathbb{N}]^{<\omega} \to \mathbb{N}$. There is an infinite increasing sequence $X = \langle X_i \rangle_{i \in \mathbb{N}}$, of subsets of $\mathbb{N}$ such that whenever Y is a finite union of elements of X, if $F(Y) \in \cup X$, then $F(Y) \in Y$.

We will postpone the proof of Theorem 6.1 until after the statement of the following result on Milliken's theorem.

THEOREM 6.2. ($\mathsf{ACA_0}^+$). Milliken's Theorem: Suppose that $F : [[\mathbb{N}]^{<\omega}]^3 \to k$ is a finite coloring of increasing triples of finite subsets. Then there is a value c and an infinite sequence X of increasing subsets of $\mathbb{N}$ such that whenever Y_0, Y_1, Y_2 is an increasing triple consisting of finite unions of elements from X, then $F(Y_0, Y_1, Y_2) = c$.

COMMENT: Milliken's theorem first appears in [13]. A proof of Milliken's theorem (for n-tuples) in $\mathsf{ACA_0}^+$ appears as corollary 7.24 in [8]. The basic idea is that Milliken's theorem is equivalent to a version of Hindman's theorem for countable collections of colorings. $\dashv$

PROOF OF THEOREM 6.1: Suppose $F : [\mathbb{N}]^{<\omega} \to \mathbb{N}$. We will use the following cases to define an auxiliary function.

Case 1: $F(Y_0) \in Y_1$.	Case 3: $F(Y_0 \cup Y_2) \in Y_1$.
Case 2: $F(Y_1) \in Y_0$.	Case 4: None of the above.

Define the function $G : [[\mathbb{N}]^{<\omega}]^3 \to \{1, 2, 3, 4\}$ on increasing triples of finite subsets by setting $G(Y_0, Y_1, Y_2)$ to the number of the least case that holds. As noted above, within $\mathsf{ACA_0}^+$ we may apply Milliken's theorem, and find a c between 1 and 4 and an infinite sequence of increasing sets $X = \langle X_i \rangle_{i \in \mathbb{N}}$ such that whenever Y_0, Y_1 and Y_2 form an increasing sequence of finite unions of elements of X, then $G(Y_0, Y_1, Y_2) = c$.

Let X_0, X_1, X_2, and X_3 be the least elements of X. If $c = 1$, then $G(X_0, X_1, X_3) = G(X_0, X_2, X_3) = 1$, so $F(X_0)$ is in both X_1 and X_2. But X is an increasing sequence, so X_1 and X_2 are disjoint. Thus $c \neq 1$. A similar argument shows that $c \neq 2$.

If $c = 3$, then $G(X_0, X_1, X_3) = G(X_0, X_2, X_3) = 3$, so $F(X_0 \cup X_3)$ is in both X_1 and X_2. These sets are disjoint, so $c \neq 3$. Thus $c = 4$.

Let $Y_0, \ldots Y_n$ be any increasing list of elements of X, and let $Y = \cup_{i \le n} Y_i$. Assume that $F(Y) \in \cup X$. Suppose by way of contradiction that $F(Y) \notin Y$. Then there is a set $T \in X$ such that $T \cap Y = \emptyset$ and $F(Y) \in T$. Let Z be an element of X such that $Y_n < Z$ and $T < Z$. If $T < Y_0$, then $G(T, Y, Z) = 2$, contradicting the claim that $c = 4$. If $Y_n < T$, then $G(Y, T, Z) = 1$, yielding another contradiction. Finally, if for some $j < n$ we have $Y_j < T < Y_{j+1}$, then $G(\cup_{i \le j} Y_i, T, \cup_{j < i \le n} Y_i) = 3$, contradicting $c = 4$. This eliminates all possible locations for T, proving that $F(Y) \in Y$. Summarizing, we have shown that if $F(Y) \in \cup X$, then $F(Y) \in Y$. $\dashv$

§7. Questions.

The preceding work leads us to a number of questions. It was already mentioned in Section 5 that the statement $(*)$ in Corollary 5.12 is open. Additional questions follow.

QUESTION 7.1. (1) *Does $FS(2)$ imply RT^2_2?*
(2) *Does $FS(2)$+$\mathsf{WKL_0}$ imply RT^2_2?*
(3) *Does $FS(2)$ imply $B\Sigma^0_2$ (or equivalently $RT^1_{<\infty}$)?*

Notice that Hirst proved that RT^2_2 implies $B\Sigma^0_2$ in [8]. Since $\mathsf{WKL_0}$ is Π^1_1-conservative over $\mathsf{RCA_0}$, a positive answer to either 7.1(1) or 7.1(3) would give another proof of Friedman's result that $FS(2)$ fails in an ω-model of $\mathsf{WKL_0}$ (see Theorem 4.6).

Recall now the statements known as CAC and COH.

STATEMENT 7.2. (CAC – Chain or Anti-chain Condition). *Every infinite partial order has an infinite chain or an infinite anti-chain.*

STATEMENT 7.3. (COH). *For any sequence of sets $(R_i)_{i\in\mathbb{N}}$ there is an infinite set A such that for all i, either $A \subseteq^* R_i$ or $A \subseteq^* \overline{R_i}$.*

Such a set A is called $\overrightarrow{R}$-cohesive. $X \subseteq^* Y$ means that there is a k such that for all x, if $x \in X$ then either $x \in Y$ or $x \leq k$. (For more about COH see [1]).

QUESTION 7.4. (1) *Does $FS(2) + CAC$ imply RT^2_2?*
(2) *Does $FS(2) + COH$ imply RT^2_2?*

QUESTION 7.5. *What happens in all the above questions if we replace $FS(2)$ by $FS(k)$, where $k > 2$? by $TS(k)$, where $k \geq 2$?*

QUESTION 7.6. *Does $FS(k)$ (or $TS(k)$ or FS or TS) imply ACA_0 for $k \geq 3$?*

QUESTION 7.7. *Does $TS(k+1)$ imply $FS(k)$?*

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