Some Orbits for $\mathcal{E}$*

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Abstract

In this article we establish the existence of a number of new orbits in the automorphism group of the computably enumerable sets. The degree theoretical aspects of these orbits also are examined.

1 Introduction

Despite significant recent advances in our understanding of the automorphism group of $\mathcal{E}$, the lattice of computably enumerable sets, we are still far from understanding the extent to which algebraic properties of a computably enumerable set determine its degree. One of the reasons for this lack of understanding is the lack of known orbits for $\mathcal{E}$. So far, the only externally defined orbits we have are the maximal sets (Soare [24]) and variations such as quasimaximal, the hemimaximal sets (Downey-Stob [9]), and the creative sets (Harrington-Myhill see [25]). The goal of the present paper is to extend this collection, extending and giving proofs of some, more or less, unpublished claims of Herrmann [16].

Our first new orbit is a class of sets we call *Herrmann sets*, based on the fact that Herrmann was the first to claim that they formed an orbit. Here we give *two* proofs.

The first proof is based on a modification of the automorphism machinery. The second proof follows ideas of Herrmann and is in some sense based on nonuniform "static" methods. We discuss exactly what this notion of "static" means, and believe that it is highly useful to have two apparently differing proofs side by side.

We remark that the proofs give insight into other arguments based on the automorphism machinery introduced by Soare [24], since the so-called $\overline{A}$ to $\overline{B}$ part is based on a modification of the order-preserving enumeration theorem. But this uses the congruence $=_{\mathcal{D}}$ in place of $=^*$. That is, our sets are $\mathcal{D}$-maximal, rather than maximal. Here we say that A is $\mathcal{D}$-maximal iff for all c.e. $C \supseteq A$, either there is a c.e. set D disjoint from A with $C = A \cup D$ or there is a c.e. set D disjoint from A with $C \cup D = \omega$.

$\mathcal{D}$-maximal and, more generally, $\mathcal{D}$-hyperhypersimple sets are of independent interest since they are precisely the sets that are not versions of K, (the standard code of the halting set), under some Friedberg enumeration (see Kummer [18]). These sets and their degrees have been previously examined by Kummer [18] and Herrmann and Kummer [17]. Herrmann sets are ones that are $\mathcal{D}$-maximal and have an additional property (*strong R-separability*).

Our proof of the fact that Herrmann sets form an obit, admits some further modifications. For instance, we are able to prove orbits for various variations such as "quasi-"Herrmann, and analogs of hhs-sets, as well as "hemi"-Herrmann sets. Again, this gives insight into how the machinery works and points towards understanding its limitations.

We also look at various degree-theoretical aspects of Herrmann sets, such as jump inversion, highness and the like, and relate their degrees to various known classes such as prompt sets and the hemimaximal sets as well as the degrees analyzed by Downey and Harrington [8] in the "no fat orbit" result. We also obtain additional results on the possible tardiness of Downey-Harrington sets. Our most interesting result in this vein is to show that there is a c.e. degree which contain Herrmann sets but not hemimaximal sets. Hence the orbit of Herrmann sets is different degree theoretically from the orbit of hemimaxinal sets. The idea is to try to understand how the information content as measured by degree, and the dynamical content, as measured by promptness, relate to its algebraic aspects. We hope that this material will contribute to the longstanding program of trying to understand the automorphism group of $\mathcal{E}$.

2 Preliminaries

For a computably enumerable set A, let $\mathcal{L}(A)$ denote the lattice of c.e. supersets of A. Analogously, we let

$$\mathcal{D}(A) = \{B : B \in \mathcal{L}(A) \wedge B - A \text{ is computably enumerable}\}.$$

We obtain a quotient structure for $\mathcal{L}(A)$ via $[B]_{\mathcal{D}(A)} = \{C \in \mathcal{L}(A) : B \equiv C \bmod \mathcal{D}(A)\}$. Herrmann and Kummer [17] called a set A $\mathcal{D}$-hyperhypersimple iff $\mathcal{L}(A)/\mathcal{D}(A)$ is a boolean algebra and a set A $\mathcal{D}$-maximal iff $\mathcal{L}(A)/\mathcal{D}(A)$ is the 2 element boolean algebra.

While such sets seem at first glance bizarre, in fact they are related to a central topic in computability theory. Harrington proved that $K = \{e; \varphi_e(e) \downarrow\}$, and hence any creative set, is definable. Kummer was led to investigate *diagonal* sets which are the sets of the form $\{e : \psi_e(e) \downarrow\}$ for some computable enumeration $\{\psi_e : e \in \omega\}$ of the partial computable functions. Kummer [18] and Herrmann-Kummer [17] proved the amazing result that a computably enumerable set A is not diagonal iff it is computable, or is $\mathcal{D}$-hhs. As a consequence, being diagonal is elementary lattice theoretic. We also remark that the structure of $\mathcal{L}(A)/\mathcal{D}(A)$ is used extensively in all known undecidability proofs of the first order theory of $\mathcal{E}$. As best we can tell, the first author to study $\mathcal{L}(A)/\mathcal{D}(A)$, and define $\mathcal{D}$-maximal and $\mathcal{D}$-quasimaximal in terms of this structure, even in passing, was Degtev [7], but the first systematic study can be found in Herrmann [15].

We remark that an equivalent formulation of $\mathcal{D}$-quasimaximality is that $\mathcal{L}(A)/\mathcal{D}(A)$ is finite.

Theorem 2.1 *$\mathcal{L}(A)/\mathcal{D}(A)$ is finite iff $\mathcal{L}(A)/\mathcal{D}(A)$ is a finite boolean algebra.*

Proof. Suppose that $\mathcal{L}(A)/\mathcal{D}(A)$ is finite. It is a distributive lattice as $\mathcal{E}$ is, and hence we need only prove that it is complemented. So suppose that $A \subseteq B$ with $[B]_{\mathcal{D}(A)}$ non-complemented in $\mathcal{L}(A)$. By the fact that $\mathcal{L}(A)/\mathcal{D}(A)$ is finite we can suppose that $[B]$ is minimal with this property. We dig the contradiction from the proof of Herrmann and Kummer [17], Theorem 2.4. They prove that if B has the property that (everything c.e.)

$$(\forall C \supseteq A)(\forall X \subseteq \overline{A})[(B \cap C) - A \text{ not c.e.} \vee B \cup C \cup X \neq \omega],$$

then B can be split into a pair of sets $B_1 \sqcup B_2 = B$, both of which have the same property as B. (An analog of Owings splitting theorem [22].) This is enough for our purposes since we claim that $[B_1] <_{\mathcal{D}} [B]$ and is also non-complemented, contradicting the minimality of $[B]$. Since $[B_1] \leq_{\mathcal{D}} [B]$ it can only be that $[B] \equiv_{\mathcal{D}} [B_1]$. Suppose that $[B] \equiv_{\mathcal{D}} [B_1]$. Then there exists a computably enumerable set E disjoint from A with $B_1 \cup E = B \cup E$. This implies that $B - B_1 \subseteq E$. That is, $B_2 \subseteq E$. But this is a contradiction since it implies that $B_2 \equiv_{\mathcal{D}} A$. $\square$

The following provides an equivalent but very useful formulation of a set being $\mathcal{D}$-maximal.

Lemma 2.2 *A c.e. noncomputable set A is $\mathcal{D}$-maximal iff for all c.e. $W \supseteq A$, either $W - A$ is c.e. or there is a computable R such that $A \subseteq R \subset W$.*

Proof. First suppose that A is noncomputable and $\mathcal{D}$-maximal. Let $C \supseteq A$, with $C - A$ not c.e.. Then for some c.e. Q disjoint from A we have $C \cup A \cup Q = \omega$. Now there exist disjoint $\widehat{C} \subseteq C \cup A$ and $\widehat{Q} \subseteq Q$ such that $\widehat{Q} \cup \widehat{C} = \omega$. But these sets are computable and we see that if $R = \overline{\widehat{Q}}$ then $A \subseteq R \subseteq C \cup A$.

The other direction is similarly easy. Suppose that A is noncomputable and satisfies the condition of the lemma. Since A is noncomputable, $\mathcal{L}(A)/\mathcal{D}(A)$ contains at least 2 elements. So suppose that $C \supseteq A$ and $C - A$ is not c.e.. Then there is a computable R such that $A \subseteq R \subseteq C$. But then $\overline{R} \cup C = \omega$ and $\overline{R}$ is disjoint from A. $\square$

Recall that a set A is called *hemimaximal* (Downey-Stob [9]) if there is a non-computable c.e. set B disjoint from A with $A \sqcup B$ maximal. Under our definition, maximal and hemimaximal sets are $\mathcal{D}$-maximal. Indeed Kummer observed that if a simple set is $\mathcal{D}$-maximal then it is maximal, and if a quasi-simple set is $\mathcal{D}$-maximal then it is hemimaximal. This paper is concerned with orbits that are far away from those generated by the simple sets, and we will need a new guessing procedure for the superset lattice.

Definition 2.3 (i) We say that a c.e. set A is *r-separable* if, for all c.e. sets B disjoint from A, there is a computable set C such that $B \subset C$ and $A \subset \overline{C}$.

(ii) We say that A is *strongly* r-separable if, additionally, we can choose C so that $C - B$ is infinite.

(iii) We say that a set A is *Herrmann* if it is both $\mathcal{D}$-maximal and strongly r-separable.

For instance, a maximal set is r-separable, but not strongly so. Actually Herrmann was concerned with $\mathcal{D}$-maximal, r-separable sets that were additionally *pseudo-creative*. Recall that a set A is pseudo-creative iff for all B disjoint from A, there exists an infinite c.e. set C disjoint from $A \cup B$. We remark that if A is a simple set then $A \times \omega$ is pseudo-creative and r-separable. (See Rogers [26], Exercise 8.36). As we will see, not every degree contains Herrmann sets. It is not difficult to prove that a set A is Herrmann iff it is $\mathcal{D}$-maximal, r-separable and pseudo-creative.

Lemma 2.4 *A is strongly r-separable iff A is r-separable and pseudo-creative.*

Proof. To see this first suppose that A is strongly r-separable. Let B be disjoint from A. Let C be computable with $A \subset C$ and $B \subseteq \overline{C}$, and $B \neq^* \overline{C}$. Now apply strong r-separability to $\overline{C}$. There must exist computable D with $A \subset D$ and $\overline{C} \subset \overline{D}$ and $D \neq^* C$. Then $C - D$ is a computable infinite set disjoint from $B \cup A$, and hence A is pseudo-creative. Conversely, suppose that A is r-separable and pseudo-creative. Suppose that B is disjoint from A. As A is r-separable, choose computable C with $B \subset C$ and $A \subset \overline{C}$. Now apply pseudo-creativity to C and A to get an infinite computable D, a subset of $\overline{C}$ disjoint from $A \cup C$. Then $\widehat{C} = C \cup D \neq^* B$ and contains B with $\overline{\widehat{C}} \supset A$. $\square$

We remark that Degtev (in [7], Theorem 2) was the first to construct a Herrmann set. Since this proof is not widely known, and we will be proving various things about Herrmann sets, We sketch a proof below.

Theorem 2.5 (Degtev [7]) *Herrmann sets exist.*

Proof. We build A in stages. At each stage s, we let $a_{0,s}, a_{1,s}, \dots$ list $\overline{A_s}$ in order. We must meet the two types of requirements below.

$$\mathcal{K}_e : W_e \cap A \neq \emptyset \vee (\exists C_e)[C_e \text{ computable } \wedge W_e \subseteq C_e \wedge A \subseteq \overline{C_e}].$$
$$\mathcal{G}_e : W_e \cap A \neq \emptyset \vee (\exists X_e)(|X_e| = \infty \wedge X_e \cap (A \cup W_e) = \emptyset).$$
$$\mathcal{N}_e : W_e \supseteq A \to [(\exists Q_e)(Q_e \cap A = \emptyset) \wedge (Q_e \cup A = W_e \vee W_e \cup Q_e = \omega))].$$

There is a great deal of flexibility in this construction. This flexibility will be exploited later.

The basic strategy for $\mathcal{K}_0$ is simple and finitary. The basic strategy for meeting $\mathcal{K}_0$ is to divide the universe ω into two computable pieces, say 2ω and $C_e = 2\omega + 1$. We build A only in $\overline{C_0} = 2\omega$. Additionally, if W_0 enumerates some number into $\overline{C_0}$ we will put such a number into A, causing $W_0 \cap A \neq \emptyset$.

The basic strategy for $\mathcal{G}_0$ is equally simple and similarly finitary. In the presence of $\mathcal{K}_0$ above, of higher priority, say, it divides the $\mathcal{G}_0$ universe, $\overline{C_0} = 2\omega$, where A is being built, into two computable pieces. For instance, we choose $4\omega, 4\omega + 2$, and let $X_0 = (4\omega + 2) \cup C_0$. The idea is very similar. We build A

only in $\overline{X_0} = 4\omega$, *unless* W_e enumerates some number x into X_0. In that case we enumerate such an x into A meeting $\mathcal{G}_0$ by causing $W_0 \cap A \neq \emptyset$.

The basic strategy for $\mathcal{N}_0$ is to cohere with ($\mathcal{K}_0$ and) $\mathcal{G}_0$ above, and monitor $W_{0,s} \cap \overline{A}_s$ *but only on* $\overline{X_0}$. The idea is relatively simple. We will assign *states* to elements of $\overline{X}_0 \cap W_{0,s} \cap \overline{A}_s$ rather like a maximal set construction. However there are *two* ways $\mathcal{N}_0$ can be met. Suppose initially $W_{0,t} \cap \overline{A}_t \cap \overline{X}_0 = \emptyset$. When we see some element $x_{0,0} \in W_{0,s} \cap \overline{A}_s \cap \overline{X_0}$ at some stage $s > t$, give the element $x_{0,0}$ a high 0-state.

Now we must process the elements $\{y : y \in \overline{X}_0 \cap \overline{A}_s \ \& \ y < x_{0,0}\}$. There are two things we can do with them:

(i) enumerate them into $A_{s+1} - A_s$, as with a maximal set, or

(ii) enumerate them into a set $Q_{0,s+1} - Q_s$ we are building for the sake of $\mathcal{N}_0$, *promising to keep* $Q_0 \cap A = \emptyset$.

It makes no difference which strategy we choose for $\mathcal{N}_0$ in the basic construction, but it will make a significant difference when we wish to control the degrees of Hermann sets.

Of course once we have $x_{0,0}$ of the high 0-state we look for $x_{0,1} > x_{0,0}$ $x_{0,1}$ in $W_{0,u} \cap \overline{X}_0 \cap \overline{A}_u$ and then process the elements of $y \in \overline{A}_u \cup \overline{X}_0$ with $x_{0,1} > y > x_{0,0}$ either into A or into Q_0.

This strategy has two outcomes, the finite outcome f, and the infinite outcome ∞, with $\infty <_L f$ in the usual Π_2 way. If there are not infinitely integers in the high state then we say this strategy has the finite outcome, in which case we can found a Q such that $W_0 = A \cap Q$ and Q is disjoint from A ($Q =^* X_0 \cap W_0$). Otherwise, this strategy has the infinite outcome and we can find a Q such that $\omega = A \cup W_0 \cup Q$ and Q is disjoint from A ($Q =^* Q_0 \cup X_0$).

Since this strategy has two outcomes, there will be two versions of $\mathcal{G}_1$. The first possibility is that the outcome is f and the requirement simply divides $\overline{X}_0$ into two pieces, calling one X_1^f and building A in $\overline{X}_1^f \cap \overline{X}_0$. For instance, $X_1^f = 4\omega + 2$ could be used, initially.

The second version of $\mathcal{G}_1$ waits for a supply of high state elements; $S = \{x_{0,0}, x_{0,1}, \ldots\}$, and uses the set S in the place of $\overline{X}_0$. That is, it divides this computable set into two computable pieces X_1^∞ and $\overline{X}_1^\infty$.

The construction then proceeds in the usual e-state Π_2-guessing fashion for the inductive strategies. $\square$

Remark 2.6 *The reader should note that if we choose the option of building Q_e and not enumerating into A, the only requirements that puts anything into A are the $\mathcal{G}_e$ and $\mathcal{K}_e$.*

3 Cholak's Modified Extension Lemma

The beautiful *extension theorem* of Soare [24] occupies a justifiably central place in the study of the automorphism group of $\mathcal{E}$. Cholak [2] proved a very useful variation on this result which allows us to prove results about automorphisms of $\mathcal{E}$ without having to construct *effective* skeletons[1] (as the original extension lemma needs) nor to apply the whole tree methodology as the Cholak [1] or Harrington-Soare [13] machinery needs. In the next section, we will use Cholak's version to establish that Herrmann sets form an orbit[2]. First we need some notation and terminology.

Definition 3.1 $\{X_n\}_{n<\omega}$ is an *uniformly computable collection* of c.e. sets if there is a computable function h and for all n, $X_n = W_{h(n)}$. $\{X_n\}_{n<\omega}$ is an *uniformly* $\mathbf{0}''$*-computable collection* of c.e. sets if there is a function h such that $h \leq_T \mathbf{0}''$ and for all n, $X_n = W_{h(n)}$. $\{X_{n,s}\}_{n,s<\omega}$, is an *uniformly* $\mathbf{0}''$*-computable enumeration* if there is a function h such that $h \leq_T \mathbf{0}''$ and for all n, $X_{n,s} = W_{h(n),s}$.

Definition 3.2 For any e, if we are given uniformly computable enumerations $\{X_{n,s}\}_{n\leq e,s<\omega}$ and $\{Y_{n,s}\}_{n\leq e,s<\omega}$ of c.e. sets $\{X_n\}_{n\leq e}$ and $\{Y_n\}_{n\leq e}$, define the *full e-state of x at stage s, $\nu(e, x, s)$* with respect to (w.r.t.) $\{X_{n,s}\}_{n,s<\omega}$ and $\{Y_{n,s}\}_{n,s<\omega}$

[1]Recall that a collection $\{X_e : e \in \omega\}$ is called a *skeleton* iff for all e there is a i such that $W_e =^* X_i$. The skeleton is *effective* if there is a computable function f such that $X_i = W_{f(i)}$ for all i, as enumerations.

[2]It also would be possible to use Soare's forthcoming New Extension Theorem.

to be the triple

$$\nu(e, x, s) = \langle e, \sigma(e, x, s), \tau(e, x, s)\rangle$$

where

$$\sigma(e, x, s) = \{i : i \leq e \wedge x \in X_{i,s}\}$$

and

$$\tau(e, x, s) = \{i : i \leq e \wedge x \in Y_{i,s}\}.$$

Definition 3.3 Given any collection of c.e. sets $\{X_n\}_{n<\omega}$ and $\{Y_n\}_{n<\omega}$, define the *final e-state of x,* $\nu(e, x)$ with respect to $\{X_n\}_{n<\omega}$ and $\{Y_n\}_{n<\omega}$ to be the triple

$$\nu(e, x) = \langle e, \sigma(e, x), \tau(e, x)\rangle$$

where

$$\sigma(e, x) = \{i : i \leq e \wedge x \in X_i\}$$

and

$$\tau(e, x) = \{i : i \leq e \wedge x \in Y_i\}.$$

Definition 3.4 Given computable enumerations $\{X_s\}_{x<\omega}$ and $\{Y_s\}_{x<\omega}$ of X and Y, we define

(i) $X \backslash Y = \{z : (\exists s)(z \in X_s - Y_s)\}$,

(ii) $X \searrow Y = (X \backslash Y) \cap Y$.

Definition 3.5 Given states $\nu = \langle e, \sigma, \tau\rangle$ and $\nu' = \langle e', \sigma', \tau'\rangle$, we define

(i) ν is an *initial segment* of ν' ($\nu \preccurlyeq \nu'$) iff $e \leq e'$, $\sigma = \sigma' \cap \{0, 1, \ldots, e\}$, and $\tau = \tau' \cap \{0, 1, \ldots, e\}$.

(ii) The *length* of ν, $|\nu|$, is e.

(iii) $\nu = \nu' \upharpoonright e$ iff $\upsilon \preccurlyeq \upsilon'$ and $|\upsilon| = e$.

(iv) ν *covers* ν' ($\nu \geq \nu'$) iff $e = e'$, $\sigma \supseteq \sigma'$ and $\tau \subseteq \tau'$.

Definition 3.6 Assume $\{T_s\}_{s<\omega}$ is a uniformly computable enumeration of T, an infinite c.e. set. For any e, if we are given uniformly computable enumerations $\{X_{n,s}\}_{n\leq e,s<\omega}$ and $\{T_{n,s}\}_{n\leq e,s<\omega}$ of c.e. sets $\{X_n\}_{n\leq e}$ and $\{Y_n\}_{n\leq e}$, for each full e-state ν, define the c.e. set

$$D^T_\nu = \{x : \exists t \text{ such that } x \in T_t - T_{t-1} \wedge \nu = \nu(e, x, t) \text{ w.r.t } \{X_{n,s}\}_{n\leq e,s<\omega} \text{ and } \{Y_{n,s}\}_{n\leq e,s<\omega}\} \quad (3.1)$$

If $x \in D^T_\nu$, we say that ν is the *entry e-state* of x w.r.t. $\{X_{n,s}\}_{n\leq e,s<\omega}$ and $\{Y_{n,s}\}_{n\leq e,s<\omega}$ into T. We say that D^T_ν is measured w.r.t. $\{X_{n,s}\}_{n\leq e,s<\omega}$ and $\{Y_{n,s}\}_{n\leq e,s<\omega}$.

Theorem 3.7 (Cholak's Modified Extension Theorem [2]) *Assume* $\{T_s\}_{s<\omega}$, $\{\hat{T}_s\}_{s<\omega}$, $\{U_{n,s}\}_{n,s<\omega}$, $\{\breve{V}_{n,s}\}_{n,s<\omega}$, $\{\breve{U}_{n,s}\}_{n,s<\omega}$, *and* $\{V_{n,s}\}_{n,s<\omega}$ *are uniformly* $\mathbf{0}''$*-computable enumerations of the infinite c.e. sets* T *and* $\hat{T}$ *and the uniformly* $\mathbf{0}''$*-computable collection of c.e. sets* $\{U_n\}_{n<\omega}$, $\{\breve{V}_n\}_{n<\omega}$, $\{\breve{U}_n\}_{n<\omega}$ *and* $\{V_n\}_{n<\omega}$ *satisfying the following Conditions:*

$$\forall n[T \searrow \breve{U}_n = \hat{T} \searrow \breve{V}_n = \emptyset], \quad (3.2)$$

$$(\forall\nu)[D^{\hat{T}}_\nu \text{ is infinite } \Rightarrow (\exists\nu' \geq \nu)[D^T_{\nu'} \text{ is infinite}]], \text{ and} \quad (3.3)$$

$$(\forall\nu)[D^T_\nu \text{ is infinite } \Rightarrow (\exists\nu' \leq \nu)[D^{\hat{T}}_{\nu'} \text{ is infinite}]], \quad (3.4)$$

where for all e-states ν, D^T_ν *is measured w.r.t.* $\{U_{n,s}\}_{n\leq e,s<\omega}$ *and* $\{\breve{V}_{n,s}\}_{n\leq e,s<\omega}$ *and* $D^{\hat{T}}_\nu$ *is measured w.r.t.* $\{\breve{U}_{n,s}\}_{n\leq e,s<\omega}$ *and* $\{V_{n,s}\}_{n\leq e,s<\omega}$. *Then there is an uniformly* $\mathbf{0}''$*-computable collection of c.e. sets* $\{\breve{U}_n\}_{n\in\omega}$ *and* $\{\breve{V}_n\}_{n\in\omega}$ *such that*

$$\breve{U}_n \cap \overline{\hat{T}} =^* \hat{U}_n \cap \overline{\hat{T}}, \ \breve{V}_n \cap \overline{T} =^* \hat{V}_n \cap \overline{T}, \text{ and} \quad (3.5)$$

$$\begin{gathered}\exists^\infty x \in T \text{ with final e-state } \nu \text{ w.r.t } \{U_n\}_{n<\omega} \text{ and } \{\hat{V}_n\}_{n<\omega} \\ \text{iff } \exists^\infty \hat{x} \in \hat{T} \text{ with final e-state } \nu \text{ w.r.t. } \{\hat{U}_n\}_{n<\omega} \text{ and } \{V_n\}_{n<\omega}.\end{gathered} \quad (3.6)$$

We remark that the statement of Soare's Extension Theorem [24] is the same as the statement of Cholak's Modified Extension Theorem except the first two occurrences of "uniformly $\mathbf{0}''$-computable" are replaced with "uniformly computable".

As Cholak [2] notes, since the array of sets constructed in the Cholak's Modified Extension Theorem is an uniformly $\mathbf{0}''$-computable collection of c.e. sets, the automorphism produced is a Δ_3-automorphism.

4 Herrmann sets form an orbit; a dynamic proof

Theorem 4.1 *Herrmann sets form an orbit.*

Let M_1 and M_2 be Herrmann sets. We show that M_1 and M_2 are automorphic in the lattice of computably enumerable sets. We base this proof on a modification of Cholak's proof [2] that Maximal Sets form a orbit.

Since M_i is Herrmann we know that there is a c.e. D disjoint from M_i such that either $W_e \cup M_i \cup D = \omega$ or $W_e = M_i \cup D$ and furthermore deciding whether $W_e \cup M_i \cup D = \omega$ or $W_e = M_i \cup D$ can be done computably in $\mathbf{0}''$. As always we will consider $\hat{\omega}$ as a copy of ω; integers from $\hat{\omega}$ will always wear hats; M_1 is a subset of ω; and M_2 is a subset of $\hat{\omega}$.

Since we are using Cholak's Modified Extension Theorem it is enough to find uniformly $\mathbf{0}''$-enumerations $\{M_{1,s}\}_{s<\omega}$, $\{M_{2,s}\}_{s<\omega}$, $\{U_{n,s}\}_{n,s<\omega}$, $\{\breve{V}_{n,s}\}_{n,s<\omega}$, $\{\breve{U}_{n,s}\}_{n,s<\omega}$, and $\{V_{n,s}\}_{n,s<\omega}$ of the (hopefully) uniformly $\mathbf{0}''$-computable collection of c.e. sets M_1, M_2, $\{U_n\}_{n<\omega}$, $\{\breve{V}_n\}_{n<\omega}$, $\{\breve{U}_n\}_{n<\omega}$, and $\{V_n\}_{n<\omega}$ satisfying the following Conditions:

$$\forall n[M_1 \searrow \breve{U}_n = M_2 \searrow \breve{V}_n = \emptyset] \tag{4.1}$$

$$(\forall \nu)[D^{M_2}_{\nu} \text{ is infinite } \Rightarrow (\exists \nu' \geq \nu)[D^{M_1}_{\nu'} \text{ is infinite}]], \text{ and} \tag{4.2}$$

$$(\forall \nu)[D^{M_1}_{\nu} \text{ is infinite } \Rightarrow (\exists \nu' \geq \nu)[D^{M_2}_{\nu'} \text{ is infinite}]], \tag{4.3}$$

$$\begin{gathered}\text{if } n = 2m \text{ then } U_n =^* W_m \text{ and } V_n = \emptyset \text{ and}\\ \text{if } n = 2m+1 \text{ then } V_n =^* W_m \text{ and } U_n = \emptyset,\end{gathered} \tag{4.4}$$

$$\begin{gathered}\exists^{\infty} x \in \overline{M_1} \text{ with final } e\text{-state } \nu \text{ w.r.t. } \{U_n\}_{n<\omega} \text{ and } \{\hat{V}_n\}_{n<\omega}\\ \text{iff } \exists^{\infty} \hat{x} \in \overline{M_2} \text{ with final } e\text{-state } \nu \text{ w.r.t. } \{\hat{U}_n\}_{n<\omega} \text{ and } \{V_n\}_{n<\omega},\end{gathered} \tag{4.5}$$

where for all e-states ν, $D^{M_1}_{\nu}$ is measured w.r.t. $\{U_{n,s}\}_{n\leq e,s<\omega}$ and $\{\breve{V}_{n,s}\}_{n\leq e,s<\omega}$ and $D^{M_2}_{\nu}$ is measured w.r.t $\{\breve{U}_{n,s}\}_{n\leq e,s<\omega}$ and $\{V_{n,s}\}_{n\leq e,s<\omega}$ (in this section M_1

will play the role of T and M_2 that of $\hat{T}$).

Before we construct this enumeration, we will show that this is enough to conclude that these sets are automorphic. First, by the Modified Extension Theorem, there is as uniformly $\mathbf{0}''$-computable collection of c.e. sets $\{\hat{U}_n\}_{n\in\omega}$ and $\{\hat{V}_n\}_{n\in\omega}$ such that

$$\breve{U}_n \cap \overline{M_2} =^* \hat{U}_n \cap \overline{M_2}, \;\; \breve{V}_n \cap \overline{M_1} =^* \hat{V}_n \cap \overline{M_1}, \tag{4.6}$$

and

$$\begin{aligned} &\exists^\infty x \in M_1 \text{ with final } e\text{-state } \nu \text{ w.r.t. } \{U_n\}_{n<\omega} \text{ and } \{\hat{V}_n\}_{n<\omega} \\ &\text{iff } \exists^\infty \hat{x} \in M_2 \text{ with final } e\text{-state } \nu \text{ w.r.t. } \{\hat{U}_n\}_{n<\omega} \text{ and } \{V_n\}_{n<\omega}. \end{aligned} \tag{4.7}$$

From (4.5), (4.6), and (4.7), we have that

$$\begin{aligned} &\exists^\infty x \in \omega \text{ with final } e\text{-state } \nu \text{ w.r.t. } \{U_n\}_{n<\omega} \text{ and } \{\hat{V}_n\}_{n<\omega} \\ &\text{iff } \exists^\infty \hat{x} \in \hat{\omega} \text{ with final } e\text{-state } \nu \text{ w.r.t. } \{\hat{U}_n\}_{n<\omega} \text{ and } \{V_n\}_{n<\omega}. \end{aligned} \tag{4.8}$$

By (4.4), it is easy to see

$$\begin{aligned} &\exists^\infty x \in \omega \text{ with final } e\text{-state } \nu \text{ w.r.t. } \{W_e\}_{e<\omega} \text{ and } \{\hat{V}_{2e+1}\}_{e<\omega} \\ &\text{iff } \exists^\infty \hat{x} \in \hat{\omega} \text{ with final } e\text{-state } \nu \text{ w.r.t. } \{\hat{U}_{2e}\}_{e<\omega} \text{ and } \{W_e\}_{e<\omega}, \end{aligned} \tag{4.9}$$

and hence $\Phi(W_e) = \hat{U}_{2e}$ and $\Phi^{-1}(W_e) = \hat{V}_{2e+1}$ defines an automorphism of the lattice of the computably enumerable sets modulo the finite sets such that $\Phi(M_1) =^* M_2$. Φ can be easily converted into an automorphism Ψ of the lattice of the computably enumerable sets such that $\Psi(M_1) = M_2$ (see [24, 25]).

We will now focus on meeting (4.1) through (4.5). We will just pick any enumeration of M_1 and M_2. To meet (4.1), we will not enumerate integers into $\breve{U}_n$ ($\breve{V}_n$) once they have entered M_2 (M_1). Since we will meet (4.4), we can let $\breve{U}_{2e+1} = \breve{V}_{2e} = \emptyset$.

To meet (4.5), the basic idea is the following. Consider U_{2e}, e.g. U_0. As M_1 is $\mathcal{D}$-maximal, there is a set Q_{2e} disjoint from M_1 such that either $U_{2e} \cup M_1 = M_1 \cup Q_{2e}$ or $U_{2e} \cup M_1 \cup Q_{2e} = \omega$.

In the first case, by r-separability, there is a computable set X_{2e} such that $M_1 \subset X_{2e}$ and $Q_{2e} \subset \overline{X_{2e}}$. In this case, the idea is to use the strong r-separability of M_2 to choose a computable $\widehat{X_{2e}}$ from $\widehat{\omega}$ with $M_2 \subset \widehat{X_{2e}}$, and $\widehat{\omega} - \overline{\widehat{X_{2e}}}$ infinite. We would then map $\overline{X_{2e}} \mapsto \overline{\widehat{X_{2e}}}$, making the isomorphism g_{2e} computable, hence carrying Q_{2e} to a $\widehat{Q_{2e}} =_{\mathrm{def}} g_{2e}(Q_{2e})$. Hence, in the final automorphism, we will have $U_{2e} \mapsto C_{2e} \sqcup \widehat{Q_{2e}}$ where C_{2e} is a subset of M_2 determined by the extension machinery.

The second case is similar. This time $U_{2e} \cup M_1 \cup Q_{2e} = \omega$. Hence, by r-separability, there is a computable X_{2e} with $Q_{2e} \subset \overline{X_{2e}}$. Again we can use strong r-separability to get a corresponding $\widehat{X_{2e}}$ on the $\widehat{\omega}$ side, and map $\overline{X_{2e}} \mapsto \overline{\widehat{X_{2e}}}$, making the isomorphism g_{2e} computable, hence carrying $U_{2e} \cap \overline{X_{2e}}$ to a $\widehat{E_{2e}} =_{\mathrm{def}} g_{2e}(U_{2e} \cap \overline{X_{2e}})$. This time the automorphism maps $U_{2e} \mapsto C_{2e} \sqcup \widehat{E_{2e}}$ where $C_{2e} = (\widehat{X_{2e}} - M_2) \cup J_{2e}$ where J_{2e} is determined by the extension machinery.

Actually, of course, the above is more than just a basic module for $2e = 0$, in the sense that once we have processed U_0 we will have fixed the automorphism on $\overline{X_0}$, where $\overline{X_0} \cap M_1$ is empty. Therefore the inductive strategy will not use ω and $\widehat{\omega}$ as its universes but those provided by earlier strategies. For instance, U_{2e} would, in fact, use the computable universe X_{2e-1} in place of ω, and $\widehat{X_{2e-1}}$ in place of $\widehat{\omega}$. This refinement makes no real difference but should be kept in mind in the ensuing discussion.

A first (failed) attempt to meet (4.5) would be to naively implement the above without caring for the enumerations. For instance, if $U_{2e} \cup M_1 \cup Q_{2e} = \omega$ then take some X_{2e} with $\overline{X_{2e}} \supset Q_{2e}$, let $\breve{U}_{2e} = X_{2e}$, otherwise let $\breve{U}_{2e} = \emptyset$, and if $V_{2e+1} \cup M_2 \cup \overline{Y_{2e}} = \omega$ then let $\breve{V}_{2e+1} = Y_{2e}$, otherwise let $\breve{V}_{2e+1} = \emptyset$ (without choosing any enumeration of these sets). Since M_1 and M_2 are both Herrmann, this will meet (4.5) but as we will see this fails to meet the entry Conditions (4.2) and (4.3). Assume that $U_0 \cup M_1 \cup Q_0 = \omega$ (even if, e.g. $Q_0 = \emptyset$) and we have the bad luck to enumerate U_0, $\breve{V}_0$, $\breve{U}_0$, and V_0 such that when we only consider 0-states $D_\nu^{M_1}$ is infinite (measured w.r.t. the bad enumeration of U_0 and $\breve{V}_0$) iff $\nu \in \{\langle 0, \emptyset, \emptyset\rangle, \langle 0, \{0\}, \emptyset\rangle\}$ and $D_\nu^{M_2}$ is infinite iff $\nu \in \langle 0, \{0\}, \emptyset\rangle$ (measured w.r.t. the enumeration of $\breve{U}_0$ and V_0). Hence (4.3) is not met if $\nu = \langle 0, \emptyset, \emptyset\rangle$. We must ensure that our entry states cohere; this will be done by carefully controlling the enumerations of the desired sets.

We will do this by induction on $e \in \omega \cup \{-1\}$. Assume that we have enumerations $\{U_{n,s}\}_{n\leq e, s<\omega}$, $\{\breve{V}_{n,s}\}_{n\leq e, s<\omega}$, $\{\breve{U}_{n,s}\}_{n\leq e, s<\omega}$, and $\{V_{n,s}\}_{n\leq e, s<\omega}$, such that Conditions (4.1) through (4.5) are satisfied when restricted to e-states and

$n \leq e$. Assume that we have computable sets $X_{2n} : 2n \leq e$, $\widehat{X_{2n}} : 2n \leq e$, $Y_{2n+1} : 2n+1 \leq e$ and $\widehat{Y_{2n+1}} : 2n+1 \leq e$ with $X_{-1} = \omega$ and $\widehat{X_{-1}} = \widehat{\omega}$, such that

$$X_{-1} \supset X_0 \supset \widehat{Y_1} \supset X_2 \supset \widehat{Y_3} ... \supset M_1 \text{ and,}$$

$$\widehat{X_{-1}} \supset \widehat{X_0} \supset Y_1 \supset \widehat{X_2} \supset Y_3 ... \supset M_2 \text{ and such that}$$

the following hold. In the following we let Z_n denote X_n if n is even and denote $\widehat{Y_n}$ if n is odd and similarly for $\widehat{Z_{2n}}$.

Additionally, we will have constructed a computable partial bijection g_m with domain $\overline{Z_m}$ and range $\overline{\widehat{Z_m}}$.

We assume that for all $n \leq e$, we have sets $\mathcal{F}$ and $\mathcal{P}$ of n-states

$$\nu \in \mathcal{F}_n \text{ iff } D_\nu^{M_1} \text{ is infinite iff } D_\nu^{M_2} \text{ is infinite, and} \tag{4.10}$$

$$\begin{aligned} \nu \in \mathcal{P}_n \text{ iff } \exists^\infty x \in \overline{M_1} \cap Z_n, \nu(n,x) = \nu \text{ iff } \exists^\infty \hat{x} \in \overline{M_2} \cap \widehat{Z_n}, \hat{\nu}(n,\hat{x}) = \nu \text{ iff} \\ \text{for all } x \in \overline{M_1} \cap Z_n, \text{ if there exists a stage } s \text{ such that } \nu(n,x,s) = \nu, \\ \text{then } \nu(n,x) = \nu \\ \text{for all } \hat{x} \in \overline{M_2} \cap \widehat{Z_n}, \text{ if there exists a stage } s \text{ such that } \hat{\nu}(n,\hat{x},s) = \nu, \\ \text{then } \hat{\nu}(n,\hat{x}) = \nu \end{aligned} \tag{4.11}$$

(where $D_\nu^{M_1}$ and $\nu(n,x,s)$ are measured w.r.t. $\{U_{n,s}\}_{n \leq e, s < w}$ and $\{\breve{V}_{n,s}\}_{n \leq e, s < \omega}$, $\nu(n,x)$ w.r.t. $\{U_n\}_{n \leq e}$ and $\{\breve{V}_n\}_{n \leq e}$, $D_\nu^{M_2}$ and $\breve{\nu}(n,\hat{x},s)$ w.r.t. $\{\breve{U}_{n,s}\}_{n \leq e, s < \omega}$ and $\{V_{n,s}\}_{n \leq e, s < \omega}$, and $\breve{\nu}(n,\hat{x})$ w.r.t. $\{U_n\}_{n \leq e}$ and $\{\breve{V}_n\}_{n \leq e}$. If $n = -1$, let $\mathcal{F}_{-1} = \mathcal{P}_{-1} = \{\langle -1, \emptyset, \emptyset \rangle\}$. Given this we will define the enumeration of U_{e+1}, $\breve{V}_{e+1}$, $\breve{U}_{e+1}$, and V_{e+1}, as follows:

Assume that $e+1 = 2m$. Hence we must ensure that $U_{e+1} =^* W_m$. For all s, let $\breve{V}_{e+1,s} = V_{e+1,s} = \emptyset$. Let $\mathcal{F}_{e+1}^* = \{\langle e+1, \sigma, \tau \rangle : \langle e, \sigma, \tau \rangle \in \mathcal{F}_e\}$ and $\mathcal{P}_{e+1}^* = \{\langle e+1, \sigma, \tau \rangle : \langle e, \sigma, \tau \rangle \in \mathcal{P}_e\}$. There is some Q disjoint from M_1 such that either $W_m \cup M_1 \cup Q \supseteq Z_e$ or $(W_m \cup M_1) \cap Z_e = M_1 \cup Q$.

If $(W_m \cup M_1) \cap Z_e = M_1 \cup Q$, then we can computably separate Q from M_1 via some L_1. That is, $M_1 \subset L_1$ and $Q \subseteq \overline{L_1}$. Then we will define Z_{e+1} and

X_{e+1} to be $L_1 \cap Z_e$. In the $\widehat{\omega}$ side, we take an infinite computable subset S of $\widehat{Z_e}$ disjoint from M_2 and let $\widehat{Z_{e+1}} = \widehat{Z_e} - S$ and $\widehat{X_{e+1}} = \widehat{Z_{e+1}}$. We let $\mathcal{F}_{e+1} = \mathcal{F}^*_{e+1}$, $\mathcal{P}_{e+1} = \mathcal{P}^*_{e+1}$, $U_{e+1,s+1} = (W_{m,s+1} \cap \overline{Z_{e+1}}) \sqcup (W_{m,s+1} \cap M_{1,s})$, and $\breve{U}_{e+1,s+1} = \emptyset$.

Now assume $W_m \cup M_1 \cup Q \supseteq Z_e$. Again we can computably separate Q from M_1 via some L_1, so that $M_1 \subset L_1$ and $Q \subseteq \overline{L_1}$. Then again we will define $Z_{e+1} = X_e$ to be $L_1 \cap Z_e$. In the $\widehat{\omega}$ side, we take an infinite computable subset S of $\widehat{Z_e}$ disjoint from M_2 and let $\widehat{Z_{e+1}} = \widehat{Z_e} - S$.

For all x, $\hat{x}$ and stages s do the following: Assume $x \notin U_{e+1,s}$. We will add x to U_{e+1} at stage $s+1$ iff $x \in W_{m,s+1}$ and one of the following three conditions holds.

- $x \in \overline{Z_{e+1}}$,
- $x \in M_{1,s}$ or
- $\nu(e+1, x, s) \in \mathcal{P}^*_{e+1}$ and for all $\nu \in \mathcal{F}^*_{e+1}$, $|D^{M_1}_{\nu,s+1}| \geq x$.

Assume $\hat{x} \notin \breve{U}_{e+1,s}$.

We will add $\hat{x}$ to $\breve{U}_{e+1}$ at stage $s+1$ iff $\hat{x} \notin M_{2,s}$,and either

- for some $z \in \overline{Z_{e+1,s+1}}$, $g_{e+1}(z) = \hat{x}$, or
- $\breve{\nu}(e+1, \hat{x}, s) \in \mathcal{P}^*_{e+1}$, and for all $\nu \in \mathcal{F}^*_{e+1}$, $|D^{M_2}_{\nu,s+1}| \geq \hat{x}$. (Where $\nu(e+1, x, s)$ and $D^{M_1}_{\nu}$ are measured w.r.t. $\{U_{n,s}\}_{n \leq e+1, s<\omega}$ and $\{\breve{V}_{n,s}\}_{n \leq e+1, s<\omega}$, and $\breve{\nu}(e+1, \hat{x}, s)$ and $D^{M_2}_{\nu}$ are measured w.r.t. $\{\breve{U}_{n,s}\}_{n \leq e+1, s<\omega}$ and $\{V_{n,s}\}_{n \leq e+1, s<\omega}$.)

Let $\mathcal{P}_{e+1} = \{\langle e+1, \sigma \cup \{e+1\}, \tau\rangle : \langle e, \sigma, \tau\rangle \in \mathcal{P}_e\}$ and $\mathcal{F}_{e+1} = \mathcal{F}^*_{e+1} \cup \mathcal{P}_{e+1}$.

By our enumeration if $\nu \in \mathcal{F}^*_{e+1}$ then $D^{M_1}_{\nu}$ and $D^{M_2}_{\nu}$ are infinite. Since $\mathcal{P}_e$ is the set of maximal e-states and M_1 and M_2 are Herrmann sets, $\mathcal{P}_{e+1}$ is the set of maximal $(e+1)$-states within Z_{e+1}, and hence (4.11) holds. Since M_1 and M_2 are $\mathcal{D}$-maximal, if $\nu \in \mathcal{P}_{e+1}$ then $D^{M_1}_{\nu}$ and $D^{M_2}_{\nu}$ are infinite. Since for an integer x to be raised into a maximal $(e+1)$-state, x must be in a maximal e-state, (4.10) holds for $\mathcal{F}_{e+1}$. From (4.10) and (4.11) it is easy to see that the rest of the induction hypothesis holds. The case where $e+1$ is odd is done in a similar fashion. Hence the enumeration of $\{U_n\}_{n<\omega}$, $\{\check{V}_n\}_{n<\omega}$, $\{\check{U}_n\}_{n<\omega}$, and $\{V_n\}_{n<\omega}$ constructed in this manner will satisfy Conditions (4.1) through (4.5). Conditions (4.10) and (4.11) are *exactly* the special properties of Herrmann sets which allow us to conclude that all Herrmann sets are automorphic.

Finally in either case, we will extend g_e to g_{e+1} by defining it upon L_1 in any obvious way. For instance, we can map the n-th element of $L_1 \cap Z_e$ to the n-th element of S.

As with Cholak [2], it remains to observe that this enumeration can be represented as an uniformly $\mathbf{0}''$-enumeration? It is clear that there are functions q_0, q_1, q_2, and q_3 computable in $\mathbf{0}''$ such that for all e and s, $U_{e,s} = W_{q_0(e,s)}$, $\check{V}_{e,s} = W_{q_1(e,s)}$, $\check{U}_{e,s} = W_{q_2(e,s)}$, and $V_{e,s} = W_{q_3(e,s)}$. We need functions q_0, q_1, q_2, and q_3 computable in $\mathbf{0}''$ such that for all e and s, $U_{e,s} = W_{q_0(e),s}$, $\check{V}_{e,s} = W_{q_1(e),s}$, $\check{U}_{e,s} = W_{q_2(e),s}$, and $V_{e,s} = W_{q_3(e),s}$. Following the maximal set case, to find such a function we must do the above construction on a tree and use the Recursion Theorem as follows:

Let $Tr = 2^{<\omega}$. At $\alpha \in Tr$, we will construct c.e. sets U_α, $\check{V}_\alpha$, $\check{U}_\alpha$, V_α, $\widehat{Z_\alpha}$ and Z_α, and an enumeration of these sets (we build U_α and its enumeration in a similar manner to the way we built U_{e+1} and its enumeration). The details of this construction are as follows: We will do this by induction on $\alpha \in Tr$. If $\alpha = \lambda$, let $\mathcal{F}_\alpha = \mathcal{P}_\alpha = \{\langle -1, \emptyset, \emptyset\rangle\}$ and for all s, $U_{\alpha,s} = \check{V}_{\alpha,s} = \check{U}_{\alpha,s} = V_{\alpha,s} = \emptyset$. Assume that we have enumerations $\{U_{\beta,s}\}_{\beta\subset\alpha,s<\omega}$, $\{\check{V}_{\beta,s}\}_{\beta\subset\alpha,s<\omega}$, $\{\check{U}_{\beta,s}\}_{\beta\subset\alpha,s<\omega}$, $\{V_{\beta,s}\}_{\beta\subset\alpha,s<\omega}$, $\{\widehat{Z_{\beta,s}}\}_{\beta\subset\alpha,s<\omega}$, $\{Z_{\beta,s}\}_{\beta\subset\alpha,s<\omega}$, functions $\{g_{\beta,s}\}_{\beta\subset\alpha,s<\omega}$, and sets $\mathcal{F}_\beta$ and $\mathcal{P}_\beta$ of $|\beta|$-states. Assume that $|\alpha| - 1 = 2m$. We will ensure that $U_\alpha =^* W_m$. For all s, let $\check{V}_{\alpha,s} = V_{\alpha,s} = \emptyset$. Let $\mathcal{F}^*_\alpha = \{\langle |\alpha|, \sigma, \tau\rangle : \langle e, \sigma, \tau\rangle \in \mathcal{F}_{\alpha^-}\}$ and $\mathcal{P}_\alpha = \{\langle |\alpha|, \sigma, \tau\rangle : \langle e, \sigma, \tau\rangle \in \mathcal{P}_{\alpha^-}\}$. There are $\omega^* + \omega^*$ many cases: either $\alpha = \alpha^-\hat{}(0, i, j)$ or $\alpha = \alpha^-\hat{}(1, i, j)$ (this will be used to code whether $W_m \cup M_1 =_{\mathcal{D}} \omega$ or $W_m \cup M_1 =_{\mathcal{D}} M_1$). Here i is the guess as to the witnessing set W_i with (e.g. $W_m \cup M_1 = M_1 \sqcup W_i$) and similarly the j is the index for the witnessing computable Z_α.) In either case we can generate the appropriate Z_α and g_α as above. If $\alpha = \alpha^-\hat{}(0, i, j)$, then let $\mathcal{F}_\alpha = \mathcal{F}^*_\alpha$, $\mathcal{P}_\alpha = \mathcal{P}^*_\alpha$, $U_{\alpha,s+1} = (W_{m,s+1} \cap M_{1,s} \cap Z_{\alpha,s}) \sqcup (W_{m,s+1} \cap \overline{Z_{\alpha,s}})$, and $\check{U}_{\alpha,s+1} = g_{\alpha,s}(W_{m,s+1} \cap \overline{Z_{\alpha,s}})$.

Assume $\alpha = \alpha^{-}\hat{}(1, i, j)$. For all x, $\hat{x}$, and stages s, do the following: Assume $x \notin U_{\alpha,s}$. We will add x to U_α at stage $s+1$ iff $x \in W_{m,s+1}$ and either $x \in \overline{Z_{\alpha,s}}$; or $x \in M_{1,s}$ or $x \in Z_{\alpha,s}$ and $\nu(|\alpha|, x, s) \in \mathcal{P}^*_\alpha$ and for all $\nu \in \mathcal{F}^*_\alpha$, $|D^{M_1}_{\nu,s+1}| \geq x$. Assume $\hat{x} \notin \breve{U}_{\alpha,s}$. We will add $\hat{x}$ to $\breve{U}_\alpha$ at stage $s+1$ iff $\hat{x} \notin M_{2,s}$, and either $\hat{x} \in g_{\alpha,s}(\overline{Z_{\alpha,s}} \cap W_{m,s})$, or $\hat{x} \in \widehat{Z_{\alpha,s}}$, and $\breve{\nu}(|\alpha|, \hat{x}, s) \in \mathcal{P}^*_\alpha$, and for all $\nu \in \mathcal{F}^*_\alpha$, $|D^{M_2}_{\nu,s+1}| \geq \hat{x}$. (Where $D^{M_1}_\nu$ and $\nu(|\alpha|, x, s)$ are measured w.r.t. $\{U_{\beta,s}\}_{\beta\subseteq\alpha,s<\omega}$ and $\{\breve{U}_{\beta,s}\}_{\beta\subseteq\alpha,s<\omega}$, and $D^{M_2}_\nu$ and $\breve{\nu}(|\alpha|, \hat{x}, s)$ are measured w.r.t. $\{\breve{U}_{\beta,s}\}_{\beta\subseteq\alpha,s<\omega}$ and $\{V_{\beta,s}\}_{\beta\subseteq\alpha,s<\omega}$.). Let $\mathcal{P}_\alpha = \{\langle |\alpha|, \sigma \cup \{|\alpha|\}, \tau\rangle : \langle e, \sigma, \tau\rangle \in \mathcal{P}_\alpha\}$ and $\mathcal{F}^*_\alpha \cup \mathcal{P}_\alpha$.

By the Recursion Theorem there are computable functions h_0, h_1, h_2, and h_3 from Tr into ω such that $U_{\alpha,s} = W_{h_0(\alpha),s}$, $\breve{V}_{\alpha,s} = W_{h_1(\alpha),s}$, $\breve{U}_{\alpha,s} = W_{h_2(\alpha),s}$, and $V_{\alpha,s} = W_{h_3(\alpha),s}$. (And similarly for the auxiliary sets Z_α and functions g_α.) Using $\mathbf{0}''$ choose an infinite branch f through Tr as follows: $\lambda \subseteq f$, if $\alpha \subseteq f$ and $|\alpha| = 2m$ then $\alpha\hat{}(1, i, j) \subseteq f$ iff $W_m \cup M_2 =_{\mathcal{D}} \omega$ with least witnesses i, j. (And similarly for the $(0, i, j)$ option.) If $\alpha \subset f$ and $|\alpha| = e+1$ then $U_{e,s} = W_{h_0(\alpha),s}$, $\breve{V}_{e,s} = W_{h_1(\alpha),s}$, $\breve{U}_{e,s} = W_{h_2(\alpha),s}$, and $V_{e,s} = W_{h_3(\alpha),s}$. Hence we have found an uniformly $\mathbf{0}''$-enumeration of $\{M_{1,s}\}_{s<\omega}$, $\{M_{2,s}\}_{s<\omega}$, $\{U_{n,s}\}_{n,s<\omega}$, $\{\breve{V}_{n,s}\}_{n,s<\omega}$, $\{\breve{U}_{n,s}\}_{n,s<\omega}$ and $\{V_{n,s}\}_{n,s<\omega}$ satisfying Conditions (4.1) through (4.5).

Therefore M_1 and M_2 are automorphic sets since, as noted earlier, the $\overline{M_1}$ to $\overline{M_2}$ part is done by piecing together with the Z_α and this can be extended to an automorphism since we have satisfied conditions (4.1) through (4.5). □

5 Herrmann sets form an orbit; a "static" proof

We also will give a second proof or the fact that the Herrmann sets form an orbit which is "static" in the sense that we do not dynamically satisfy the hypotheses of the extension lemma, but piece the automorphism together. The proof is based on unpublished material of the last author. It is interesting to compare the two proofs. We remark that in some sense the proofs are kind of the same since there is a hidden use of the extension machinery in the following theorem of Soare.

Lemma 5.1 (Soare's Lemma, Soare [25]) *Let Z be c.e. and let $\mathcal{S}(Z)$ denote the structure $\{Y : Y$ is c.e. and either $Y \cup Z = \omega$ or $Y \subseteq^* Z\}$ with inclusion relation. Let $\mathcal{S}^*(Z)$ be $\mathcal{S}(Z)$ modulo the finite sets. Then for any two infinite c.e.*

noncomputable sets, A and B,

$$\mathcal{S}^*(A) \cong_{\Delta^0_3} \mathcal{S}^*(B).$$

We remark that the lemma above is *actually* what the proof of the maximal set automorphism theorem achieves. The point is that the order preserving enumeration theorem gives the isomorphism on the $\overline{A}$ to $\overline{B}$ part for those sets in $\mathcal{S}(A)$ and $\mathcal{S}(B)$, and then the extension theorem allows the extension to an isomorphism. The fact that the sets used by Soare are maximal means that the isomorphism is in fact an automorphism.

For a c.e. set Z let

$$(\mathcal{S}(Z), \mathcal{R}_Z)$$

denote expansion of $\mathcal{S}(Z)$ by the unary predicate satisfied by the collection of computable subsets of Z. Let $(\mathcal{S}^*(Z), \mathcal{R}^*_Z)$ be $(\mathcal{S}(Z), \mathcal{R}_Z)$ modulo the finite sets. The following provides us with a strengthing of Soare's lemma.

Lemma 5.2 $(\mathcal{S}^*(A), \mathcal{R}^*_A) \cong (\mathcal{S}^*(B), \mathcal{R}^*_B)$ *iff* $\mathcal{S}^*(A) \cong \mathcal{S}^*(B)$.

Proof. The implication from left to right is clear. Assume that ψ is an isomorphism between $\mathcal{S}^*(A) \cong \mathcal{S}^*(B)$. Let R be a computable subset of A. Then $A \cup \overline{R} = \omega$. Hence $\psi(\overline{R}^*)$ exists. Now since $R \cup \overline{R} = \omega$ and $R \cap \overline{R} = \emptyset$,$\psi(R^*) \cup \psi(\overline{R}^*) =^* \omega$ and $\psi(R^*) \cap \psi(\overline{R}^*) =^* \emptyset$. Similarly for ψ^{-1}. Hence ψ is an isomorphism between $(\mathcal{S}^*(A), \mathcal{R}^*_A) \cong (\mathcal{S}^*(B), \mathcal{R}^*_B)$. $\square$

Let $\mathcal{D}_1(A) = \{B : B \text{ is an infinite c.e. set and } B \cap A = \emptyset\}$ ordered by the inclusion relation. Let $\mathcal{D}^*_1(A)$ be $\mathcal{D}_1(A)$ modulo the finite sets. This is almost an automorphism invariant for $\mathcal{D}$-maximal sets, but not quite (we will later prove this; see Lemma 6.4). Let

$$(\mathcal{D}_1(A), \mathcal{R}_{\overline{A}})$$

be an expansion of $\mathcal{D}_1(A)$ by a unary predicate satisfied by all the computable sets in $\mathcal{D}_1(A)$. (Note that is is the same as the above except that $\overline{Z}$ replaces Z.) Let $(\mathcal{D}^*_1(A), \mathcal{R}^*_{\overline{A}})$ be $(\mathcal{D}_1(A), \mathcal{R}_{\overline{A}})$ modulo the finite sets. The next lemma shows that this structure characterizes precisely when $\mathcal{D}$-maximal sets are automorphic.

Lemma 5.3 (Herrmann) *Suppose that A and B are $\mathcal{D}$-maximal. Then A and B are automorphic iff*

$$(\mathcal{D}^*_1(A), \mathcal{R}^*_{\overline{A}}) \cong (\mathcal{D}^*_1(B), \mathcal{R}^*_{\overline{B}}).$$

Proof. One direction is clear. So suppose that $(\mathcal{D}_1^*(A), \mathcal{R}^*_{\overline{A}}) \cong (\mathcal{D}_1^*(B), \mathcal{R}^*_{\overline{B}})$ via the isomorphism φ. Let ψ denote the isomorphism from $(\mathcal{S}^*(A), \mathcal{R}^*_A)$ to $(\mathcal{S}^*(B), \mathcal{R}^*_B)$. We show that the isomorphism φ taking $(\mathcal{D}_1^*(A), \mathcal{R}^*_{\overline{A}})$ to $(\mathcal{D}_1^*(B), \mathcal{R}^*_{\overline{B}})$ can be extended with ψ to an automorphism Φ.

For notation ease in what follows we will drop the "*" (or modulo the finite sets). Hence we consider ψ as an isomorphism from $(\mathcal{S}(A), \mathcal{R}_A)$ to $(\mathcal{S}(B), \mathcal{R}_B)$, φ as an isomorphism from $(\mathcal{D}_1(A), \mathcal{R}_{\overline{A}})$ to $(\mathcal{D}_1(B), \mathcal{R}_{\overline{B}})$ and we will build an automorphism of $\mathcal{E}$, Φ. Once this simplification is understood, it is easy to add back in the "*".

Let C be a computably enumerable. There are two cases: one if $C - A$ is c.e. and one otherwise. Define

$$\Phi(C) = \psi(C \cap A) \sqcup \varphi(C - A) \text{ if } C - A \text{ is c.e..}$$

Now assume that $C - A$ is infinite and not c.e.. Since A is $\mathcal{D}$-maximal, there is a computable set R with $A \subseteq R \subset A \cup C$. There are sets $C' \subset C$ and $A' \subset A$ such that $A' \cap C' = \emptyset$, $A' \cup C' = A \cup C$ and $A' \cap R = A'$. Hence,

$$(A' \cup C') \cap R = R = (A' \cap R) \cup (C' \cap R).$$

Thus A' is a computable subset of A. Therefore,

$$C = (C - R) \sqcup (\overline{A'} \cap R) \sqcup (C \cap A').$$

We define

$$\Phi(C) = \varphi(C - R) \sqcup (\overline{\psi(A')} \cap \overline{\varphi(\overline{R})}) \sqcup \psi(C \cap A').$$

We must show that this definition is well defined. Clearly it is well-defined for C such that $C - A$ is c.e.. Let R_0 and A'_0 another pair of sets that satisfy the above equations. WLOG we can assume that $R_0 \subset R$ and $A'_0 \subset A'$ (if not take the intersection of R_0 and R and similarly for the A''s and note that these intersections also satisfy the above equations). We must show

Claim 5.4

$$\varphi(C-R)\sqcup(\overline{\psi(A')}\cap\overline{\varphi(\overline{R})})\sqcup\psi(C\cap A') = \varphi(C-R_0)\sqcup(\overline{\psi(A'_0)}\cap\overline{\varphi(\overline{R_0})})\sqcup\psi(C\cap A'_0).$$

Proof. $\varphi(C-R) \subset \varphi(C-R_0)$ and their difference $\varphi(C-R_0)-\varphi(C-R)$ is $\varphi(R-R_0)$ which is a subset of $(\overline{\psi(A')} \cap \varphi(\overline{R}))$ (this uses that φ is an isomorphism and that $\overline{B} \subset \overline{\psi(A')}$).

We break the second clause into two pieces: the part in $\overline{B}$ and the part in B.

$(\overline{\psi(A')} \cap \overline{\varphi(\overline{R})} \cap \overline{B})$ contains $(\overline{\psi(A'_0)} \cap \overline{\varphi(\overline{R_0})} \cap \overline{B})$ and their difference $\varphi(R-R_0)$ which is a subset of $\varphi(C-R_0)$ (again this uses that φ is an isomorphism, $\overline{B} \subset \overline{\psi(A')}$ and $\overline{B} \subset \overline{\psi(A'_0)}$).

$(\overline{\psi(A')} \cap \overline{\varphi(\overline{R})} \cap B)$ is contained in $(\overline{\psi(A'_0)} \cap \overline{\varphi(\overline{R_0})} \cap B)$ and their difference $\psi(A'-A'_0)$ which is a subset of $\psi(C \cap A')$ (this uses that ψ is an isomorphism, $B \subset \overline{\varphi(\overline{R})}$ and $B \subset \overline{\varphi(\overline{R_0})}$).

$\psi(C \cap A'_0) \subset \psi(C \cap A')$ and their difference is $\psi(A'-A'_0)$ which is contained in $(\overline{\psi(A'_0)} \cap \overline{\varphi(\overline{R_0})} \cap B)$ (this uses that ψ is an isomorphism). $\square$

So Φ is well defined. In a similar fashion, we can show Φ^{-1} is well defined.

It remains to show that Φ and Φ^{-1} preserves inclusion. Assume that $X \subsetneq Y$. If $Y-A$ is c.e. then $\Phi(X) = \psi(X \cap A) \sqcup \varphi(X-A) \subsetneq \Phi(Y) = \psi(Y \cap A) \sqcup \varphi(Y-A)$ since ψ and φ are isomorphisms which preserve inclusion. Otherwise $Y = (Y-R) \sqcup (\overline{A'} \cap R) \sqcup (Y \cap A')$ for some R and A'. If $X-R \subsetneq Y-R$ then, since φ is an isomorphism of $(\mathcal{D}_1(A), \mathcal{R}_{\overline{A}})$ to $(\mathcal{D}_1(B), \mathcal{R}_{\overline{B}})$, $\Phi(X) \subsetneq \Phi(Y)$. Similarly if $X \cap A' \subsetneq Y \cap A'$. If either of these two cases fail then $X' = X \cap (\overline{A'} \cap R) \subsetneq (\overline{A'} \cap R) \sqsubset Y$. If $X'-A$ is c.e. then $\Phi(X') \subsetneq (\overline{\psi(A')} \cap \overline{\varphi(\overline{R})}) \sqsubset \Phi(Y)$. Otherwise $X' = (X'-R_1) \sqcup (\overline{A'_1} \cap R_1) \sqcup (X' \cap A'_1)$ for some R_1 and A'_1. $R_1 \subset R$ and $A' \subset A'_1$. So either $(X'-R_1) \subsetneq Y$ or $(X' \cap A'_1) \subsetneq Y$. In either case, $\Phi(X) \subsetneq \Phi(Y)$. In a similar fashion, we can show Φ^{-1} preserves inclusion.

Hence we have defined an automorphism of $\mathcal{E}$. $\square$

Now we can give the alternative proof of theorem 4.1. It relies on the following characterization of $\mathcal{D}_1(A)$ for A Herrmann.

Lemma 5.5 *Suppose that A is Herrmann. Then $\mathcal{D}_1(A)$ is isomorphic to the weak sum, $\oplus_{i\in\omega}\mathcal{E}$, of countably many computable copies of $\mathcal{E}$. (That is, the lattice of c.e. sets which can be presented as functions $\{f : f : \omega \mapsto \mathcal{E}$ such that $(a.a.n)[f(n) =$*

$\emptyset$] *and where* $f \subseteq g$ *iff* $\forall n(f(n) \subseteq g(n))$.) *Let* $\Phi : \mathcal{D}_1(A) \mapsto \oplus_{i\in\omega}\mathcal{E}$ *denote the isomorphism above. Let* f_X *be the image of* X *under* Φ. *Then* $X \in \mathcal{R}_{\overline{A}}$ *holds iff* $(\forall n)[f_X(n)$ *is computable*]. *Consequently,* $(\mathcal{D}_1(A), \mathcal{R}_{\overline{A}}) \cong (\mathcal{D}_1(B), \mathcal{R}_{\overline{B}})$.

Proof. The proof is straightforward. Let $V_i : i \in \omega$ list the c.e. sets in $\mathcal{D}_1(A)$. Now, as above, V_0 is a subset of an infinite computable set R_0 disjoint from A. We map R_0 to ω by a computable bijection g_0. Now the set V_1 can be split into two sets $V_1 - R_0$ and $V_1 \cap R_0$. We can separate $V_1 - R_0$ by putting it into a computable set R_1 disjoint from A and R_0, and then mapping R_1 by a computable bijection g_1 to ω, so that the image of V_1 is given by $g_1(V_1 \cap R_1) \cup g_0(R_0 \cap V_1)$, etc. This does the job. Note the isomorphism is Δ^0_3. $\square$

6 Variations on the Theme

As with the maximal set case, the above allows one to manufacture further orbits. For instance, call a set A *quasi-Herrmann* of rank n if $\mathcal{L}(A)/\mathcal{D}(A)$ is a finite boolean algebra with n atoms.

Theorem 6.1 *Suppose that* A *and* B *are quasi-Herrmann of rank* n. *Then* A *is automorphic to* B.

Proof: We do the case $n = 2$, the others being entirely analogous. There are two Herrmann sets C, D with $C \neq_{\mathcal{D}(A)} D$, $C \cap D =_{\mathcal{D}(A)} A$ and $C \cup D =_{\mathcal{D}(A)} \omega$.

There exists a c.e. set W such that $C \cup D \cup W = \omega$ and $W \cap A = \emptyset$. By the separation principle, there are disjoint computable sets C', D', W' such that $C' \sqcup D' \sqcup W' = \omega$. Now observe that $C' \cap A$ is Herrmann inside C' and similarly $D' \cap A$ inside D'. Similarly find Herrmann $\widehat{C}$ and $\widehat{D}$ on the hatted side and from these sets define $\widehat{C}'$, $\widehat{D}'$, and $\widehat{W}'$. Then map $C' \mapsto \widehat{C}'$, $D' \mapsto \widehat{D}'$ and $W' \mapsto \widehat{W}'$. Use Theorem 4.1 to map $C' \cap A \mapsto \widehat{C}' \cap B$ and similarly D. make the map from W to $\widehat{W}'$ computable. This clearly induces an automorphism of $\mathcal{E}$ taking A to B. $\square$

One can with a lot of work also prove the analog of Maass's Theorem on hhs-sets with Σ^0_3 isomorphic lattices of supersets. The proof is entirely analogous to the proof of Theorem 4.1 and the corresponding result of Maass. We omit this.

Theorem 6.2 *Suppose that A and B are $\mathcal{D}$-hhs, and there is a Σ_3^0 isomorphism from $\mathcal{L}(A)/\mathcal{D}(A)$ to $\mathcal{L}(B)/\mathcal{D}(B)$. The A and B are automorphic.*

Another variation is obtained via the ideas of Downey and Stob [9]. Recall that for a property P of c.e. sets, we say that a noncomputable c.e. set A is *hemi-P* if there a noncomputable c.e. set B disjoint from A such that $A \sqcup B$ has property P. Also a set A is *half-P* if there is a splitting $A_1 \sqcup A_2 = A$ such that A_1 has property P. Downey and Stob proved that hemimaximal sets formed an orbit. There are two proofs that hemimaximal sets form an orbit. One is due to Downey and Stob [9], and is based on modifying the extension lemma for pairs. The other is due to Herrmann and can be found in Downey-Stob [11], is much shorter, and relieves on Theorem 5.3. Since it is very short and provides an interesting reflection on Lemma 5.3, we will provide another version here.

Theorem 6.3 *The hemimaximal sets form an orbit.*

Proof. Let A and $\widehat{A}$ be hemimaximal sets. They are $\mathcal{D}$-maximal. Hence it is enough to show $(\mathcal{D}_1^*(A), \mathcal{R}^*_{\overline{A}}) \cong (\mathcal{D}_1^*(\widehat{A}), \mathcal{R}^*_{\overline{\widehat{A}}})$. Let M and B be c.e. sets such that M is maximal and $A \sqcup B = M$ (similarly for $\widehat{A}$). Let $W \in \mathcal{D}_1$. So W is disjoint from A. Either $W \cup B \cup A =^* \omega$ or $W \subset^* B$. If $W \cup B \cup A =^* \omega$ then A is computable (if $x \in W \cup B$ then $x \notin A$). Hence $W \subset^* B$. Let ψ be an one-to-one onto computable function from B to $\widehat{B}$. ψ is an isomorphism from $(\mathcal{D}_1^*(A), \mathcal{R}^*_{\overline{A}})$ to $(\mathcal{D}_1^*(\widehat{A}), \mathcal{R}^*_{\overline{\widehat{A}}})$. □

Lemma 6.4 *$\mathcal{D}_1^*(A)$ is not automorphism invariant. That is there are sets A and $\widehat{A}$ such that $\mathcal{D}_1^*(A)$ is isomorphic to $\mathcal{D}_1^*(\widehat{A})$ but A is not automorphic to $\widehat{A}$.*

Proof. Let A be hemimaximal. Let R be a coinfinite infinite computable set. Let $\widehat{A}$ be maximal in R. A and $\widehat{A}$ are not automorphic. Let M and B be c.e. sets such that M is maximal and $A \sqcup B = M$. Let ψ be an one-to-one onto computable function from B to R. ψ is an isomorphism from $\mathcal{D}_1^*(A)$ to $\mathcal{D}_1^*(\widehat{A})$. □

Theorem 6.5 *Hemi-Herrmann sets form an orbit.*

Proof. We will provide two proofs: The first is shorter and is based on the above work of Herrmann. The second is based on work due to Downey and Stob [9].

Let A and $\widehat{A}$ be hemi-Herrmann sets. They are $\mathcal{D}$-maximal. Hence it is enough to show $(\mathcal{D}_1^*(A), \mathcal{R}^*_{\overline{A}}) \cong (\mathcal{D}_1^*(\widehat{A}), \mathcal{R}^*_{\overline{\widehat{A}}})$. Let H and B be c.e. sets such that H is Herrmann and $A \sqcup B = H$ (similarly for $\widehat{A}$). Let ψ be an one-to-one onto computable function from B to $\widehat{B}$. Using Lemma 5.5, let φ be the isomorphism between $(\mathcal{D}_1^*(H), \mathcal{R}^*_{\overline{H}})$ and $(\mathcal{D}_1^*(\widehat{H}), \mathcal{R}^*_{\overline{\widehat{H}}})$. Let W be in $\mathcal{D}_1(A)$. So W is disjoint from A. Assume there is a computable set R such that $A \sqcup B = H \subset R \subset W \sqcup A$. Therefore $A \cup B \cup W \cup \overline{R} = \omega$ and A is computable (if $x \in B \cup W \cup \overline{R}$ then $x \notin A$). Since A is not computable and H is Herrmann, $W - B = W - H$ is c.e.. Let $\Phi(W^*) = \varphi((W - B)^*) \sqcup \psi(W \cap B)$. Clearly this is well-defined. (Similarly for the hatted side and Φ^{-1}.) Φ is the desired isomorphism from $(\mathcal{D}_1^*(A), \mathcal{R}^*_{\overline{A}})$ to $(\mathcal{D}_1^*(\widehat{A}), \mathcal{R}^*_{\overline{\widehat{A}}})$.

Now we turn to the proof based on Downey and Stob [9]. Two applications of Cholak's extension lemma (first to A_0, $\hat{A}_0$ and second to A_1, $\hat{A}_1$) yields the following two set version of Cholak's extension lemma, directly analogous to Downey-Stob [9], Lemma 2.

Lemma 6.6 (Modified Downey-Stob extension lemma) Let A and $\hat{A}$ be infinite c.e. sets with splittings $A = A_0 \sqcup A_1$ and $\hat{A} = \hat{A}_0 \sqcup \hat{A}_1$. Assume $\{A_{i,s}\}_{s<\omega}$, $\{\hat{A}_{i,s}\}_{s<\omega}$, $\{U_{n,s}\}_{n,s<\omega}$, $\{\breve{V}_{n,s}\}_{n,s<\omega}$, $\{\breve{U}_{n,s}\}_{n,s<\omega}$, and $\{V_{n,s}\}_{n,s<\omega}$ are uniformly $\mathbf{0}''$-computable enumerations of the infinite c.e. sets $A_{i,s}$ and $\hat{A}_{i,s}$ and the uniformly $\mathbf{0}''$-computable collection of c.e. sets $\{U_n\}_{n<\omega}$, $\{\breve{V}_n\}_{n<\omega}$, $\{\breve{U}_n\}_{n<\omega}$ and $\{V_n\}_{n<\omega}$ satisfying the following Conditions:

$$\forall n, i[A_i \searrow \breve{U}_n = \hat{A}_i \searrow \breve{V}_n = \emptyset], \tag{6.1}$$

$$(\forall \nu)[D_\nu^{\hat{A}_i} \text{ is infinite } \Rightarrow (\exists \nu' \geq \nu)[D_{\nu'}^{A_i} \text{ is infinite}]], \text{ and} \tag{6.2}$$

$$(\forall \nu)[D_\nu^{A_i} \text{ is infinite } \Rightarrow (\exists \nu' \leq \nu)[D_{\nu'}^{\hat{A}_i} \text{ is infinite}]], \tag{6.3}$$

where for all e-states ν, $D_\nu^{A_i}$ is measured w.r.t. $\{U_{n,s}\}_{n\leq e,s<\omega}$ and $\{\breve{V}_{n,s}\}_{n\leq e,s<\omega}$ and $D_\nu^{\hat{A}_i}$ is measured w.r.t. $\{\breve{U}_{n,s}\}_{n\leq e,s<\omega}$ and $\{V_{n,s}\}_{n\leq e,s<\omega}$. Then there is an uniformly $\mathbf{0}''$-computable collection of c.e. sets $\{\hat{U}_n\}_{n\in\omega}$ and $\{\hat{V}_n\}_{n\in\omega}$ such that for $i = 0, 1$,

$$\breve{U}_n \cap \overline{\hat{A}_i} =^* \hat{U}_n \cap \overline{\hat{A}_i}, \; \breve{V}_n \cap \overline{A_i} =^* \hat{V}_n \cap \overline{A_i}, \text{ and} \tag{6.4}$$

$$\begin{gathered}\exists^\infty x \in A_i \text{ with final } e\text{-state } \nu \text{ w.r.t } \{U_n\}_{n<\omega} \text{ and } \{\hat{V}_n\}_{n<\omega} \\ \text{iff } \exists^\infty \hat{x} \in \hat{A}_i \text{ with final } e\text{-state } \nu \text{ w.r.t. } \{\hat{U}_n\}_{n<\omega} \text{ and } \{V_n\}_{n<\omega}.\end{gathered} \tag{6.5}$$

To complete the proof that hemi-Herrmann sets form an orbit, need only describe how to fulfill the requirements of the modified Downey-Stob extension lemma. This is entirely analogous to the proof of theorem 4.1, and the analogous result in Downey-Stob ([9], Lemma 4) In view of this we will only sketch the main idea, the formal details being left to the reader. So suppose that $M_1 = A_0 \sqcup A_1$ and $M_2 = \hat{A}_0 \sqcup \hat{A}_1$ with M_i Herrmann and the splittings c.e.noncomputable.

To meet the analog of (4.5), the basic idea is is modified as follows. Consider U_{2e}, e.g. U_0. As M_1 is $\mathcal{D}$-maximal, there is a set Q_0 disjoint from M_1 such that either $U_0 = M_1 \cup Q_0$ or $U_0 \cup M_1 \cup Q_0 = \omega$.

In the first case, by r-separability, there is a computable set X_0 such that $M_1 \subset X_0$ and $Q_0 \subset \overline{X_0}$. Again use the strong r-separability of M_2 to choose a computable $\widehat{X_0}$ from $\widehat{\omega}$ with $M_2 \subset \widehat{X_0}$, and $\widehat{\omega} - \widehat{\overline{X_0}}$ infinite. We would then map $\overline{X_0} \mapsto \widehat{\overline{X_0}}$, making the isomorphism g_0 computable, hence carrying Q_0 to a $\widehat{Q_0} =_{\text{def}} g_0(Q_0)$. Hence, in the final automorphism, we will have $U_0 \mapsto C_0^0 \sqcup C_0^1 \sqcup \widehat{Q_0}$ where C_0^i are subsets of $\hat{A}_i$ determined by the modified Downey-Stob extension machinery. (So the strategy will be to have $\breve{U}_0 \cap X_0 = \emptyset$, as in the proof of Theorem 4.1.)

As with Theorem 4.1, in the second case we have $U_0 \cup M_1 \cup Q_0 = \omega$. Hence, by r-separability, there is a computable X_0 with $Q_0 \subset \overline{X_0}$. Again we can use strong r-separability to get a corresponding $\widehat{X_0}$ on the $\widehat{\omega}$ side, and map $\overline{X_0} \mapsto \widehat{\overline{X_0}}$, making the isomorphism g_0 computable, hence carrying $U_0 \cap \overline{X_0}$ to a $\widehat{E_0} =_{\text{def}} g_0(U_0 \cap \overline{X_0})$. This time the automorphism maps $U_0 \mapsto C_0 \sqcup E_0$ where $C_0 = (\widehat{X_0} - M_2) \cup J_0^0 \cup J_0^1$ where J_0^i is a subset of $\hat{A}_i$ is determined by the modified Downey-Stob extension machinery. The crucial thing we need to notice is that since $X_0 \supset M_1$ is computable and the A_i's are a non-trivial splitting of M_1, $X_0 \searrow A_i$ is infinite. Hence we can cause, in the $\mathbf{0}''$-enumerations, infinitely many elements to enter both A_0 and A_1 from X_0. (This is the main trick for the hemimaximal set case.) Thus again we can make sure that all state flows are covered, by splitting all flows from $\overline{M_1}$ entering M_1, as described. The remaining details are entirely analogous to the proof of Theorem 4.1 □

We remark that hemi-Herrmann sets obey all the degree results for Hermann sets. The proofs are analogous to the hemimaximal case and the Herrmann proofs of the next section.

Theorem 6.7 *(i) All high c.e. degrees are hemi-Herrmann.*

(ii) All jump classes contain hemi-Herrmann sets.

(iii) Below each nonzero c.e. degree there is a hemi-Herrmann degree.

There are other ways Herrmann sets resemble hemi-Herrmann sets.

Theorem 6.8 *Suppose that A is $\mathcal{D}$-maximal, and $A_0 \sqcup A_1 = A$ is a nontrivial splitting of A. Then A_i is $\mathcal{D}$-maximal.*

Proof. Suppose that $B \supseteq A_0$. Then there is a W disjoint from A such that either $A \cup B \cup W = \omega$ or $A \cup B = A \cup W$. In the first case $B \cup (A_1 \cup W) = \omega$ and in the second, $B = ((A_1 \cap B) \cup W) \sqcup A_0$. □

However, the notions are separate since r-separability is not inherited by splittings.

Theorem 6.9 *Suppose that $A_0 \sqcup A_1 = A$ is a Friedberg splitting of a noncomputable set. Then A_i's are not r-separable.*

Proof. Assume that $R \supseteq A_1$ with R computable and disjoint from A_0. Then $R - A$ is not computably enumerable. Since the splitting is a Friedberg splitting, $R \cap A_0 \neq \emptyset$. So we cannot computably separate the A_i's. □

The original use of hemimaximal sets by Downey and Stob was to demonstrate that certain classes of c.e. sets were automorphic to complete sets. Since each hemimaximal set is automorphic to a complete one, any *half*-hemimaximal c.e. set would be automorphic to a complete set. A consequence of this result was that (Downey and Stob [9], Theorem 12) the following classes of c.e. sets were half-hemimaximal and hence automorphic to complete sets:

(i) all low$_2$ simple sets.

(ii) all semilow$_{1.5}$ simple sets.

(iii) every d-simple set with a maximal superset.

Naturally we could use the same reasoning for both half-Herrmann and half-hemi-Herrmann sets. It would be interesting to know if there was a wide class of sets to which the reasoning for hemimaximal sets could not be applied but the Herrmann reasoning could be applied. The intuition is that Herrmann sets are in some sense *small* whereas hemimaximal sets are *large*. However the *dynamics* of their respective constructions are very similar.

7 Degrees of Herrmann Sets

In this section we will look at the possible degrees of Hermann sets, and compare them with the invariant classes realized by a single orbit. First, given the nature of one strategy (non-enumeration) for the $\mathcal{N}_e$ requirements of the proof of Theorem 2.5, it is perhaps not surprising that the Herrmann degrees are downward dense.

Theorem 7.1 *Let* $\mathbf{a} \neq \mathbf{0}$ *be computable enumerable. Then there exists a Herrmann degree* $\mathbf{b} \leq \mathbf{a}$.

Proof. We modify the proof of Theorem 2.5 by adding simple permitting. That is, we choose the option of always putting the y in $\overline{A}$ of the low state between $x_{i,s}$ and $x_{i+1,s}$ into $Q_{e,s+1} - Q_{e,s}$, instead of enumerating into A. In that case, as we noted in Remark 2.6, the only requirements which with to enumerate elements into A are the $\mathcal{G}_e$, and $\mathcal{K}_e$. These wish to enumerate a single element into A, from a computable list of potential winning candidates. Adding simple permitting causes no problem and gives the result at hand. □

The other operation (enumeration) allows for a lot of coding.

Theorem 7.2 *Let* $\mathbf{a}$ *be any high computably enumerable degree. Then* $\mathbf{a}$ *is Herrmann.*

Proof. Now we choose the "enumeration" option, in the satisfaction of the $\mathcal{N}_e$. This is very similar to a maximal set construction, and naturally combines with high permitting and coding. (See also Theorem 8.10.) □

Despite the fact that the Herrmann degrees and the hemimaximal degrees are different, they do exhibit a number of similarities. Up to the present paper, the hemimaximal degrees were the only known elementary definable orbit realizing all possible jumps (Downey and Stob [10]). Herrmann sets share this property.

Theorem 7.3 *Let S be c.e. in and above $\emptyset'$. Then there is a Herrmann set A with $A' \equiv_T S$.*

Proof. The argument is similar to Downey-Stob [10], Theorem 2.1. It is a combination of ideas from the Sacks jump theorem and from Theorem 2.5.

Let B be an r.e. set with $B^{(y)}$ an initial segment of $\omega^{(y)}$ ($= \{\langle x, y\rangle : x \in \omega\}$) such that

$$y \in S \Leftrightarrow |B^{(y)}| < \infty \text{ and } y \notin S \Leftrightarrow |B^{(y)}| = \infty \Leftrightarrow B^{(y)} = \omega^{(y)}$$

At each stage s, let $\{a_{i,s} : i \in \omega\}$ enumerate $\overline{A_s}$ in order of magnitude. We need to meet the requirements $\mathcal{K}_e$, $\mathcal{N}_e$ and $\mathcal{G}_e$ from Theorem 2.5, and the requirements below.

$$\begin{aligned} P_e &: \text{code } B^{(e)} \text{ into } A, \\ F_e &: \lim_s a_{e,s} = a_e \text{ exists.} \end{aligned}$$

And we try to meet the "pseudo-requirement" (terminology of Soare [25])

$$\hat{F}_e : \exists^\infty s(\Phi_{e,s}(A_s; e) \downarrow \Leftrightarrow \Phi_e(A; e) \downarrow). \tag{7.1}$$

Of course strictly speaking we need only ensure that we meet $\hat{F}_e$ along the true path of the construction; that is where the computations are B-correct. We let the hat convention apply and let $\hat{r}(e, s)$ be the usual restraint preserving the left hand side of 7.1. The basic idea is to define a A-computable functional $\lambda(A, x, y)$ in such a way that A' can code $B^{(y)}$ via the limit lemma (in relativized form).

The idea is that at some stage s, we enumerate an axiom saying "$\lambda(A, x, y) = 0$" for $\langle x, y\rangle \notin B_s$. We set the use of this to be $a_{n(x,y),s}$ for some large n. If we later see $\langle x, y\rangle \in B_t$ we can correct the computation by putting some $z \le a_{n(x,y),s}$ into $A_{t+1} - A_t$ and hence set $\lambda(A, s, y) = 1$. Provided that we succeed for almost all y, the limit lemma will make the value of $\lim_y \lambda(A, x, y)$ computable in A' and hence S computable in A'. All of this is subject to the restraints $\hat{R}(y, s) = \max\{\hat{r}(p, s) : p \le y\}$ (to keep the jump down). Our construction is rather different from the jump theorem in that we have rather less control over the "markers" $a_{n,s}$ that we use for coding.

The basic coding module

For a single P_e the basic idea is to assign some $a_{n,s} = a_{\langle s,e\rangle,s}$ ($> \hat{R}(e,s)$) to the least $\langle x,e\rangle \notin B_s$ for the coding of $B^{(e)}$. For this $a_{n,s}$, until we see $\langle x,e\rangle \in B_t$ every time the j-state machinery (for $j < e$) causes us to make $a_{n,s+1} \neq a_{n,t}$ we enumerate some $z \leq a_{n,t+1}$ into A_{t+1}. For instance, for $\mathcal{N}_j$ from Theorem 2.5, we would await several numbers in the high j-state, then P_e would request that the low state numbers not be put into Q_e, but at least one enter A_{t+1} so that A can comprehend the marker movement and be able to (later) correct $\lambda(A,x,e)$. Of course, here we will not reset $\lambda(A,x,e)$ unless $\langle x,e\rangle \in B_t$, but simply change the use. For any $y > n$, if $a_{y,s}$ is in the low state and is being ejected, we can put $a_{y,s}$ into Q_e (or A as desired). Here, in particular, we would be concerned with those y associated with P_e via some $z > x$. It is then easy to see that if $\lim_s \hat{R}(e,s)$ exists, then for $|B^{(e)}| < \infty$ we will get stuck on some $a_n = \lim_s a_{n,s}$ with $n = n(q,e)$ (for some q) since any particular $a_{n,s}$ is only reset finitely often by j-state machinery. Once $a_{n,s}$ reaches the high j-state, the only reason for change is due to the action of P_{2k} of higher priority or due to P_e.

The α-module

Of course, there are a couple of problems with the above in the actual construction, since we only know that $\liminf_s \hat{R}(e,s) < \infty$ and $\lim_s \hat{R}(e,s)$ may not exist. We will modify the above to give the "α-correct" version allowing all the requirements to cohere with one another. This is of course the heart of a Π_2 argument.

The first problem is that initially we might assign $a_{n,s}$ to P_e as the coding maker of $B^{(e)}$ as above. Implicitly note that P_e is guessing the value of $\lim(\inf)\hat{R}(e,s)$. While we await $\langle x,e\rangle$ to occur in $B^{(e)}$ we may see a stage where $\hat{R}(e,t) > \hat{R}(e,s)$ and $\hat{R}(e,t) > a_{n,s}$. Since this may be the correct value of $\lim_s \hat{R}(e,s)$ we must begin a new P_e strategy based on this new guess. Hence at stage t we would pick a new $\langle \hat{x},e\rangle(\hat{x} > x)$ and a new $a_{n(\hat{x},e),t}$ and by the way we choose makers, we would choose $n(\hat{x},e) = \langle t,e\rangle$. If the $\hat{R}(e,u)$ drops down to (e.g.) $\hat{R}(e,s)$ at some stage $u > t$ we would abandon $a_{n(\hat{x},e),t}$ (forever) and hopefully return to $a_{n(x,e),s}$. Here we say "hopefully" since we are ignoring the effect of the $\mathcal{N}_j$. Assuming that the $\mathcal{N}_j$ does not affect things, the idea above will mean that, as in the jump theorem, we will return to some maker via the non-deficiency window infinitely often, and hence succeed in meeting P_e.

We finally have to consider how this all interacts with the $\mathcal{N}_j$. If $j \geq e$, then $\hat{R}(e, s)$ can assert control over $\mathcal{N}_j$ and ask that only the k-states ($k < e$) change for those $a_{m,s} < \hat{R}(e, s)$. Again at non-deficiency stages we get to maximize j-states for those $a_m > \liminf_s \hat{R}(e, s)$.

The problem is that the j-states for $j < e$ are more or less out of control for $\hat{R}(e, s)$ and cannot be "guessed" (consider making A low – this is not really a tree argument). That is, we cannot "wait to act until the correct j-state" since B' should control $\hat{R}(y, s)$. Therefore if a j-state requirement $\mathcal{N}_j$ requests us to declare the A position of some $z < \hat{R}(e, s)$ (because it is in the low state), by putting it into A or Q_j, we really must put it into Q_j for the sake of $\hat{F}_e$.

Now we come to the crux of the whole construction; consider $j < e < f$. Suppose that P_f has a maker $a_{n,s}$ devoted to coding $\langle f, x\rangle$. We are faced with the following timing problem. At stage s, suppose $a_{n,s} > \hat{R}(e, s)$ which is, say, equal to $\liminf_s \hat{R}(e, s)$. At stage $s_1 > s$, $\langle f, x\rangle \notin B_{s_1}$, but $\hat{R}(e, s_1) > a_{n,s} = a_{n,s_1}$. At stage $s_2 > s_1$ $\hat{R}(e, s_2) = \hat{R}(e, s_1)$ but some $z \leq a_{n,s}$ enters $A_{s_2+1} \cup Q_{j,s_2+1}$ for the sake of the $\mathcal{N}_e$ (the e-state machinery). Say $z = a_{n,s}$. As $e < f$, we must put $a_{n,s}$ into Q_{j,s_2+1} and not A_{s_2+1}. But at some stage $s_3 > s_2$ we see $\hat{R}(e, s_3) = \hat{R}(e, s)$. Now suppose we have $\langle f, x\rangle \in B - B_s$. Obviously we want to code this fact into A. But alas, $a_{n,s} \in Q_j$ and hence not available for A.

Our solution to the problem above is to make it possible for A to realize $a_{n,s} \neq a_{n,t}$ at any stage t during the window when $\hat{R}(e, t)$ drops down. Since $a_{n,s}$ may have been enumerated (e.g. at stage s_2 above), our only recourse is to enumerate the largest $a_{i,t} \leq a_{n,s}$ from $\overline{X}_{e,s} \cup \overline{Q}_{e,s}$, into A_{t+1} for the sake of P_e. Because of this it is our duty to ensure that the following does not occur: $a_{n,s} \in \overline{X}_{e,s} \cup \overline{Q}_{e,s}$ and, say, $a_{n-1,s} = a_{n-1,s_2}$ as above and a_{n,s_2} is in the high j-state. Now at stage t (=s_3) we enumerate $a_{n-1,t}$. In turn, this process might cause us to drop the j-state of $a_{n-1,t}$ or $a_{n,t}$. We must be careful to ensure that this process does not repeat itself infinitely often causing $A \cup X_e \cup Q_e = \omega$.

The obvious solution to the dilemma above is to delay the enumeration of $a_{n-1,t}$ (or more generally $a_{n,s}$) until we see a window stage $u > t$ where all of $a_{n,u}, \ldots, a_{n+t,u}$ have the same f-state as $a_{n-1,t}$ and many are in $\overline{X}_f$. In that case there is no injury to $\mathcal{N}_j$. It will be then clear that the α-correct version of P_f will succeed in meeting the requirement since almost all of $\overline{A}$ in $\overline{X}_e$ has the same

f-state.

The remaining details are more or less routine Π_2 implementation and are left to the reader. □

However, not every degree is Herrmann. This is an immediate consequence of the following theorem of Downey and Harrington:

Theorem 7.4 (Downey and Harrington [8]) *Let $S(A)$ denote the property below: (everything c.e.)*

$$(\exists C)(\forall X \subseteq C)[A \cup X \supseteq C \rightarrow$$

$$(\exists B \subseteq X)[A \cup B \supseteq X \wedge (\forall Y)(A \cup Y = \omega \rightarrow |(X \cap Y) - B| = \infty)]].$$

Then

(i) there exists a high$_2$ degree $\mathbf{e}$ such that if B has degree $\leq \mathbf{e}$ then $\neg S(B)$

(ii) there exists a low degree $\mathbf{c}$ and high$_2$ degree $\mathbf{b}$ with $\mathbf{c} < \mathbf{b}$ such that if B has degree between $\mathbf{c}$ and $\mathbf{b}$ then $S(B)$.

In particular, note that the Downey-Harrington result above implies that no member of the high$_n$–low$_n$ hierarchy[3] is definable by a single orbit except high$_1$. For our purposes, Theorem 7.1 implies that if A is Herrmann then $\neg S(A)$. In fact one does not need all of the properties defining Herrmann sets.

Theorem 7.5 *Suppose that A satisfies $S(A)$. Then A is not $\mathcal{D}$-maximal.*

Proof. Let C be the relevant set in property $S(A)$. Consider $A \cup C$.

Claim 7.6 *$(A \cup C) - A$ not computably enumerable.*

Proof of Claim. To see claim 7.6, suppose otherwise. Let $X = (A \cup C) - A$. Then $X \subseteq C$ and $A \cup X \supseteq C$. Hence, since $S(A)$, there exists $B \subseteq X$ such that

$$A \cup B \supseteq X \wedge (\forall Y)(A \cup Y = \omega \rightarrow |(X \cap Y) - B| = \infty.$$

[3] That is, no degree whose n-th jump is either $\mathbf{0}^n$ or $\mathbf{0}^{n+1}$.

Since $A \cup B \supseteq X$ and X is disjoint from A, yet $B \subseteq X$, it can only be that $X = B$. But then $(\forall Y)(Y \supseteq \overline{A} \Rightarrow (X \cap Y) = (B \cap Y)))$, a contradiction. $\square$

Therefore, if A is $\mathcal{D}$-maximal, then there must exist W such that

(i) $W \cap A = \emptyset$, and

(ii) $(C \cup A) \cup W = \omega$.

Let $X = C$. Then $(\exists B)(B \subseteq X(= C)\&A \cup B = A \cup C)$. Hence $B \supset W \cap C$ and note $(W \cap C) \cap A = \emptyset$ as $W \cap A = \emptyset$.

Let $Y = W$. Since $X \cap Y = C \cap W \subseteq B$, we see $X \cap Y \subseteq B$, a contradiction to $S(A)$. $\square$

Corollary 7.7 There is a low set **a** and a high$_2$ set **b** with $\mathbf{a} < \mathbf{b}$ such that no set in the interval $[a, b]$ is $\mathcal{D}$-maximal (and hence neither Herrmann nor hemimaximal).

Actually, $S(A)$ is an extremely interesting property in the following sense. It is a program of Harrington and Soare to try to understand computably enumerable sets in terms of their dynamical properties, perhaps relative to some skeleton. For instance, they show [13] that every "almost prompt" (more later) set is automorphic to a complete set. Sets satisfying $S(A)$ exhibit a number of similar properties and we explore their degrees in the next section.

Returning to Herrmann sets, as we have seen the result above implies a number of restrictions on their possible degrees. We have the following theorem of Kummer [18].

Theorem 7.8 (Kummer [18], Theorem 6) *Let* $\mathbf{c} < \mathbf{d}$ *be low. Then there are* $\mathbf{a}, \mathbf{b}$ *with* $\mathbf{c} \leq \mathbf{a} < \mathbf{b} \leq \mathbf{d}$ *such that no set of degree in* $[\mathbf{a}, \mathbf{b}]$ *is* $\mathcal{D}$*-hhs.*

One possible conjecture is that the hemimaximal degrees and the Herrmann degrees coincide. This is not true.

Theorem 7.9 *There is a non-hemimaximal (non hemi-hhs) degree containing a Herrmann set.*

Proof. The argument is finite injury. We build A. To make its degree not hemi-hhs we next need the requirements

$$H_e : (\Gamma_e^A = V_e \wedge \Delta^{V_e} = A \wedge W_e \cap V_e = \emptyset) \Rightarrow W_e \sqcup V_e \text{ is not hhs.}$$

To meet H_e we use some auxiliary sets $\{D_{e,i} : i \in \omega\}$, a weak array, and must ensure that if

$$\Gamma^A = V_e \wedge \Delta^{V_e} = A \wedge W_e \cap V_e = \emptyset$$

then we need meet $H_{e,i}$ for some i:

$$H_{e,i} : D_{e,i} \not\subseteq V_e.$$

Let $\ell(e, s)$ denote the A-controllable length of agreement for H_e. That is

$$\ell(e, s) = \max\{x : (\forall y < x)(\Delta_e^{V_e} \restriction y = A \restriction y \wedge \Gamma_e^A \restriction \delta_e(y) = V_e \restriction \delta_e(y))[s]\}$$

where $\delta_e(y)$ denotes the Δ_e-use and we have used the usual convention of appending $[s]$ to denote the state s situation.

¿From Downey and Stob [9], the fundamental idea used to meet H_e is the following. First assume $W_e \cap V_e = \emptyset$, because as soon as some z enters $W_{e,s} \cap V_{es}$ we get a global win on H_e. To meet $H_{e,i}$ (assuming $\ell(e, s) \to \infty$), when $H_{e,i}$ becomes "active"' say at stage u, we will wait for a stage t where $\ell(e, t) > u$ and initialize all lower priority requirements. We pick a follower $x = x(e, i, t) > t$ and wait until $\ell(e, s) > x$. At this stage we initialize all lower priority requirements so, once $H_{e,i}$ has priority, we can know that the only number below s which can enter A will be $x(e, i, s)$. Now, the idea is to put into $D_{e,i,s+1}$ all numbers z with $\delta_e(u)[s] \leq z \leq \delta_e(x)[s]$.

The claim is that this is enough. For suppose $D_{e,i,s+1} \subseteq W_e \sqcup V_e$, and $H_{e.i}$ is not initialized. Then the stage s situation is unchanged for all $s' > s$. So if $D_{e,i,s+1} \subseteq W_e \sqcup V_e$ we will see a stage $s_0 > s$ where $D_{e,i} \subseteq W_e \sqcup V_e[s_0]$. At such a stage s_0, we can get a *global win* on H_e by enumerating x into A and otherwise restraining A with priority e.

The point is that if $\ell(e, s_1) > x$ for some $s_1 > s_0$, $W_e \restriction \delta_e(x)[s] \neq W_e \restriction \delta_e(x)[s]$, so W_{e,s_1} must have changed on some z with $\delta_e(u)[s] \leq z \leq \delta_e(x)$. This is impossible if $W_e \cap V_e = \emptyset$ since all such z are in $D_{e.i} \subseteq W_e \cup V_e$.

Additionally, we must make A Herrmann. We meet the requirements of Theorem 2.5

$$\mathcal{K}_e : W_e \cap A \neq \emptyset \vee (\exists C_e)[C_e \text{ computable } \wedge W_e \subseteq C_e \wedge A \subseteq \overline{C_e}].$$
$$\mathcal{G}_e : W_e \cap A \neq \emptyset \vee (\exists X_e)(|X_e| = \infty \wedge X_e \cap (A \cup W_e) = \emptyset).$$
$$\mathcal{N}_e : W_e \supseteq A \rightarrow [(\exists Q_e)(Q_e \cup A = W_e \vee W_e \cup Q_e = \omega)].$$

Naturally we will be using the non-enumeration strategy for meeting the $\mathcal{N}_e$.

The key is the the only time we act for some $H_{e,i}$ by enumeration is to get a *global win* for H_e. Thus all positive action is finitary.

The way this coheres with the $\mathcal{G}_i$ and $\mathcal{N}_i$ is now described. It is easiest to construct this as a tree argument. We already know the strategy for the $\mathcal{N}_i$ and $\mathcal{G}_e$. We will have some top node τ devoted to measuring if $W_e \cap V_e = \emptyset$, and measuring if $\ell(e, s) \rightarrow \infty$. It has three outcomes. $s <_L \infty < f$ where s denotes the "stop" *global win* outcome.

The $H_{e,i}$ are spread out in the tree below the outcome ∞.

Suppose that τ is on the true path and we have $\mathcal{K}_{\sigma_0}$, $\mathcal{N}_{\sigma_1}$ and $\mathcal{G}_{\sigma_2}$ above τ (here trees grow downwards). Assuming both are infinitary (the most difficult case). $\mathcal{K}_{\sigma_0}$, $\mathcal{G}_{\sigma_1}$ insist that certain elements don't enter A (so that A is built from the "other half") and $\mathcal{N}_{\sigma_2}$ processes elements putting some into Q_{σ_2} and the rest will be a well behaved stream all in the high σ_2-state.

H_e and hence $H_{e,i}$ will choose followers from this stream and hence *cannot* injure $\mathcal{N}_{\sigma_1}$ or $\mathcal{G}_{\sigma_2}$. We ask that all versions of $H_{e,i}$ occur at some fixed level below $|\tau|$, and furthermore along any path there is a fixed collection of $H_{f,j}$, $\mathcal{K}_n$ and $\mathcal{G}_k$ of higher priority. The point of this is that this convention means that *enumeration* can cause initialization of $H_{e,i}$ at most finitely often.

All other activity is negative. The apparent problem is really the following. We have a version of $H_{e,i}$ sitting at some level ρ below τ. At some stage it might well be the case that we assign to $H_{e,i}$ some follower x which is good from the point of view of τ and hence acceptable to $\mathcal{K}_{\sigma_0}$, $\mathcal{G}_{\sigma_1}$ and $\mathcal{N}_{\sigma_2}$.

However, x might well be bad from the point of view of, say, $\mathcal{N}_\gamma$ for some γ below τ. Perhaps x is not in the correct γ-state, and at some stage γ acts, putting x into Q_{σ_2} at stage s. This could appear to be bad from $H_{e,i}$'s point of view.

However, our solution is to *not* initialize this $H_{e,i}$ setup, but to regard it as assigned to $\hat{\rho}$ where $\hat{\rho} \supset \gamma$ is the string of the same length as ρ (such a $\hat{\rho}$ is assigned to $H_{e,i}$ via the convention above).

The idea is that x will be reassigned to $\hat{\rho}$ and so its restraint will assume priority $\hat{\rho}$ in place of ρ. *The point to notice is that x won't be put into A anyway, unless we put a win at* τ (not at $\hat{\rho}$). If we get a win at τ we can injure γ anyway!

It is important to note that, while x might be assigned to $\hat{\rho}$, it can be *reassigned* to some ρ_1, for $\gamma \supset \rho_1 \supset \hat{\rho}$, which is associated with some $H_{f,j}$. This is okay since it corresponds to finite injury. $\square$

8 Degrees of $S(A)$

As we mentioned in the last section, the degrees associated with $S(A)$ are related to certain dynamical considerations. To facilitate our discussion, we briefly review the methodology used to construct a set satisfying $S(A)$.

To build a set satisfying $S(A)$ we must build c.e. sets A, C and B_X that satisfy the requirements below:

$R_X : A \cup X \supseteq C \to (\exists B_X)(A \cup B_X \supseteq X \wedge (\forall Y)(A \cup Y = \omega \to |(X \cap Y) - B| = \infty))$.

We break these into

$$\begin{aligned} R_X :& A \cup X \supseteq C \to (\exists B_X)(A \cup B_X \supseteq X \wedge (\forall Y, i)(R_{X,Y,i})), \text{ where} \\ R_{X,Y,i} :& A \cup Y = \omega \to |(X \cap Y) - B| \geq i. \end{aligned}$$

The strategy for satisfying R_X is fairly simple, and for a single A (rather than a degree), the strategies work independently. Any c targeted C not associated with R_X (but for $R_{\hat{X}}$ for some $\hat{X} \neq X$) will be put into B_X. So we can concentrate upon a single R_X.

To meet a single $R_{X,Y,i}$ (assuming we have met $R_{X,Y,i-1}$), we pick a witness x.

We keep x out of $A \cup C$ until x enters Y. If this does not happen, $A \cup Y \neq \omega$ with witness x. If Y responds, eating x, then at some stage we can put x into

C, and wait for X to eat x. If X does not respond, then we get a global X-win since then $A \cup X \not\supseteq C$. If x enters X at stage s, we can *immediately* put x into A increasing the count of $(X \cap Y) - B_X$.

It would seem reasonable to believe that the construction above involves the notion of promptness since the witness x "promptly" enters A. The reader should recall that a set B is called *promptly simple* if there is a computable function g such that

$$(\forall e)(|W_e| = \infty \Rightarrow (\exists x, s)(x \in W_{e \text{ at } s} \wedge x \in A_{g(s)})).$$

A set is called *prompt* if it has promptly simple degree. Alternatively, C is prompt iff there is a computable h such that

$$(\forall e)(|W_e| = \infty \Rightarrow (\exists x, s)(x \in W_e \text{ at } s \upharpoonright A_{h(s)} \uparrow x \neq A_x \upharpoonright x)).$$

Clearly the construction above seems to make A prompt. One is naturally lead to investigate how promptness and $S(A)$ relate.

We remark that promptness considerations are central to recent investigations about orbits and complete sets (e.g. Cholak, Downey, Stob [4], Cholak [1, 2], Harrington-Soare [12, 13, 14], Wald [27]). We need the following definition.

Definition 8.1 We say a set A is *effectively* S if A satisfies $S(A)$ and moreover, this is an effective procedure to compute an index for B_X from one for X.

The construction outlined above clearly makes a set that is effectively S. Our intuition concerning promptness is realized via:

Theorem 8.2 *Suppose A is* effectively S. *Then A is prompt.*

Proof. We assume that A is effectively S. We build a computable g to meet the requirements

$$R_e : |W_e| = \infty \rightarrow (\exists x)(\exists s)(x \in W_{e,s} \rightarrow A_{g(s)} \upharpoonright x \neq A_s \upharpoonright x).$$

To facilitate this, for a fixed e, we build a set Y_e.

We have our set C and given enumeration of A. $C - A$ is infinite. As well we have a computable function f such that for all n, if W_n is an index for a set with

$W_n \subset C \cup A$ and $W_n \cup A \supseteq C$, then $W_{f(n)}$ has the role of B_{W_n}. Of course, we can assume that at each stage s, $B_{W_n,s} \subseteq W_{n,s}$, and run the enumerations so that $W_{n,s} \cap \overline{A}_s = B_{W_n,s} \cap \overline{A}_s$.

The argument is no injury argument. We build $X = X_e$ and $Y = Y_X$ for the sake of $\mathcal{R}_e$. For the sake of $\mathcal{R}_e$, for the $n + 1$-st try, let us assume that we have previously used the numbers $y_1, ..., y_n$. Also we have defined Y so that at the current stage s we have $A_s \cup Y_s \supseteq \{y : y \leq y_n\}$. Now $\mathcal{R}_e$ requires attention if it is not currently met, and we see some element $y_{n+1} > y_n$ enter $W_{e,s}$. Now $\mathcal{R}_e$ will assert control of $X_{e,s} \upharpoonright y_{n+1}$.

We will immediately put into Y all of $\{z : y_n < z \leq y_{n+1}\}$ not yet in A_s. (Of course here, and below, we pretend that we can put numbers directly into a set whereas, we will have some computably overheads given by the s-m-n or recursion theorem. For simplicity we pretend that there are no such overheads.) Now put into $X_{e,s}$ all numbers c with $c \leq y_{n+1}$ and $c \notin A_s$. Wait till the least stage $t \geq s$ such that

$$A_s \cup X_{e,t} \upharpoonright y_{n+1} = A_s \cup B_{X_{e,t}} \upharpoonright y_{n+1}.$$

We force such a stage to occur by additionally making $(X_{e,u} \cap \overline{A}_u)(z) = (C_u \cap \overline{A}_u)(z)$ for $u \geq s$, and $z \leq u$, while we are waiting for t. Notice that we cannot wait forever, since this action will force $X_e \cap \overline{A} = C \cap \overline{A}$. Hence since A is effectively S, $B_{X_e} \cap \overline{A} = X_e \cap \overline{A}$.

- If $A_t \upharpoonright y_{n+1} \neq A_s \upharpoonright y_{n+1}$ declare $\mathcal{R}_e$ as met.
- Otherwise, enumerate no further numbers into Y, or X until $\mathcal{R}_e$ again gets a candidate.

We claim first that $\mathcal{R}_e$ is met. If we suppose otherwise then since y_n are monotone increasing, and computable, $Y \cup A = \omega$. Since we never meet $\mathcal{R}_e$, all numbers entering X_e from Y must enter B before they enter A. This contradicts property S.

Finally we need to argue that A is prompt. This is achieved by dovetailing the constructions above for many e. For instance, we only allow an attack to begin for $\mathcal{R}_e$ at a stage $s \geq e$. Then define $h(s)$ to be the maximum of s and all of the t's corresponding to attacks begun at stage s. $\square$

Notice that by Harrington and Soare [12], we have the corollary:

Corollary 8.3 Suppose that A is effectively S. Then A is automorphic to a complete set.

Proof. Harrington and Soare proved that all prompt sets are automorphic to complete sets. $\square$

We'd like to improve the theorem above to sets satisfying S, rather than *effectively* S. However, this is not possible, meaning that the relationship between promptness and S is quite obscure. Harrington and Soare introduced the notion of tardiness to explain various phenomena from the automorphism machinery.

We need some definitions.

Definition 8.4 (Harrington and Soare [13]) Let $e = \langle e_1, ..., e_n \rangle$. Then define a standard enumeration of n-c.e. sets via

$$X^n_{e,s} = (W_{e_1,s} - W_{e_2,s}) \cup (W_{e_3,s} - W_{e_4,s}) \cup ...,$$

where the last part of the union is either $W_{e_n,s}$ if n is odd, and $W_{e_{n-1},s} - W_{e_n}$ if n is even. The we say a c.e. set A is

(i) *almost prompt* if there is a nondecreasing computable function p such that for all n and e,

$$X^e_n = \overline{A} \to (\exists x, s)[x \in X^n_{e,s} \wedge x \in A_{p(s)}].$$

(ii) *very tardy* if A is not almost prompt.

(iii) *n-tardy* if there for every nondecreasing computable function p, there exists an e such that

$$X^e_n = \overline{A} \wedge (\forall y)(\forall s)[y \in X^n_{e,s} \to y \notin A_{p(s)}].$$

We remark that the difference between very tardy and n-tardy is that the fixed n kills all potential p. As Harrington and Soare pointed out, A is 0-tardy iff $A = \omega$ and A is 1-tardy iff A is computable. So the first place A can be nontrivially very tardy is 2-tardy. Harrington and Soare demonstrated that a certain $\mathcal{E}$-definable property $Q(A)$ implies that A is 2-tardy. The point of this result is the following.

Theorem 8.5 (Harrington and Soare [12, 13]) *Suppose that A is 2-tardy. Then A is not Turing complete.*

Theorem 8.6 *There is a set A with $S(A)$ holding and such that A is 2-tardy.*

Proof sketch. This is a straightforward combination of the tree method and the $S(A)$ construction. In view of this result as having only technical interest, we only sketch the proof. We need to meet the $\mathcal{R}_X$ as before, and also the 2-tardy requirements below.

$$\mathcal{T}_e : \varphi_e \text{ nondecreasing and total} \rightarrow (\exists W, V)[Z =_{\text{def}} W - V = \overline{A} \wedge$$

$$(\forall y)(\forall s)[y \in Z_s \rightarrow y \notin A_{\varphi_e(s)}]$$

To meet the requirement $\mathcal{T}_e$, we process numbers through a node γ devoted to slowing down their enumeration into A. We will be currently maintaining $Z_s = \overline{A}_s \restriction r(e, s)$. Some requirement $\mathcal{R}$ will desire to put some number x into A. The action is simple. We take x out of Z at stage s (by putting it into V). Now we do not put x into A at all unless a stage t occurs where $\varphi_{e,t}(s') \downarrow$ for all $s' \leq s$, and nondecreasing on all such s', and $t > \varphi_e(s)$. The x is free to enter A at any stage $t' \geq t$. At stage t we increase $r(e, t)$ to t, and make $Z_t = \overline{A}_t \restriction r(e, t)$. Thus if φ_e is increasing and total, γ has the infinite outcome, then $r(e, t) \rightarrow \infty$ and hence $Z = \overline{A}$. Otherwise γ has the finite outcome.

We concentrate the $\mathcal{R}_X$ at a single node σ on the tree. We will use σ to encode whether $\overline{A} \cap C = \overline{A} \cap X$. If σ believes that $\overline{A} \cap C = \overline{A} \cap X$, we will build a version of B_X at σ. The σ nodes devoted to $\mathcal{R}_{X,Y,i}$ for various Y, i will be spread out in the tree below the infinitary outcome of τ, in the usual $\mathbf{0}'''$ way. (Although this is a finite injury $\mathbf{0}''$ argument like a minimal pair.) Thus a single node σ devoted to meeting $\mathcal{R}_{X,Y,i}$ runs through the cycle of

(i) picking an element x

(ii) waiting till x enters Y (realization),

(iii) (eventually) putting x into C,

(iv) when x enters X, putting x "immediately" into A.

The thing to notice here is that nothing is committed until we get beyond step (iii). Once x is realized at σ it can begin its journey up the tree being processed by the needed γ nodes one at a time, in increasing order of priority. Remember, these γ nodes are guessing that τ has the infinitary outcome, and hence guess that this x will eventually get into A. If x gets stuck then we win γ and no harm is done to $\mathcal{R}_X$, since x is not yet in C. So suppose then we work our way back up the tree and get to τ where we are building X. It is here that we will put x into C. x will reside here unless it enters X. Now (iv) must happen. However, what we do is release x to continue its journey up the tree. If τ really is on the true path then x will enter A. $\square$

Corollary 8.7 There are sets satisfying $S(A)$ but not effectively $S(A)$.

We remark that one can also have a set A satisfying $\neg S(A)$ that is 2-tardy, since one can have a 2-tardy maximal set. (Harrington and Soare [13], Theorem 3.11.) It would be interesting to know if one can have sets A, B satisfying Harrington and Soare's $Q(-)$ as well as $S(A)$ and $\neg S(B)$.

We remark that one can use promptness to solve a question implicit in Downey-Harrington [8]. As we have seen, there it is proven that each high$_n$-low$_n$ class contains degrees whose members are either purely $S(A)$ or purely $\neg S(A)$. Theorem 8.10, below, says that the strongest possible extension of this fact would be that for any degree $\mathbf{c} \neq \mathbf{0}''$ computable enumerable in and above $\mathbf{0}'$ there are degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}' = \mathbf{b}' = \mathbf{c}$, and for all $A \in \mathbf{a}$ and all $B \in \mathbf{b}$, $S(A)$ and $\neg S(B)$. Unfortunately, this attractive conjecture fails.

Theorem 8.8 *(i) Suppose that $\neg S(A)$ holds for all A of degree $\mathbf{a}$. Then $\mathbf{a}$ is tardy.*

(ii) There is a degree $\mathbf{c} \neq \mathbf{0}''$ computable enumerable in and above $\mathbf{0}'$, such that if $\mathbf{a}' = \mathbf{c}$, then there is a set A in $\mathbf{a}$ with $S(A)$.

Proof. (i) It is a routine finite injury argument to construct a promptly simple set N with a semilow complement satisfying $S(A)$. By Maass [19], such sets are all automorphic, and hence, all promptly simple sets with semilow complements satisfy $S(A)$. Of course all prompt degrees contain promptly simple sets with semilow complements.

(ii) Cooper [6], and independently, Shore [23] constructed a degree $\mathbf{c} \neq \mathbf{0}''$ computable enumerable in, and above $\mathbf{0}'$, such that if $\mathbf{a}' = \mathbf{c}$, then $\mathbf{a}$ is prompt. The result follows by (i). □

Another result relating promptness to $S(A)$ is the following.

Theorem 8.9 *(i) Suppose that* $\mathbf{a}$ *is low and prompt. Then* $S(A)$ *holds for all* $A \in \mathbf{a}$.

(ii) Suppose that A is hemimaximal and has low degree. The A is not prompt.

Proof. We begin by constructing a low prompt $\mathbf{a}$ such that for all $A \in \mathbf{a}$, $S(A)$ holds. The theorem will then follow by a result of Wald [27] who proved that if $\mathbf{a}$ and $\mathbf{b}$ are low and prompt, then every set in $\mathbf{a}$ can be sent to one in $\mathbf{b}$.

We turn to the proof that there are low prompt $S(A)$ degrees.

To prove Theorem 8.9, we build sets D, C_i and $B_{e,i}$ (the sets A_e and X_i are given to us by the requirements) with D promptly simple, in stages to satisfy the requirements $\mathcal{R}_{e,j}$ below.

$$\mathcal{R}_{e,i} \;:\; \Gamma_e^D = A_e \wedge \Delta_e^{A_e} = D \to (\forall j)\mathcal{R}_{e,i,j}, \text{ where}$$
$$\mathcal{R}_{e,i,j} \;:\; X_i \subseteq C_e \wedge X_i \cup A_e \supseteq C_e \to$$
$$(B_{e,i} \subseteq X_i \wedge A_e \cup B_{e,i} \supseteq X_i \wedge X_i \cup Y_j = \omega \to |(X_i \cap Y_j) - B_{e,i}| = \infty).$$

To make D promptly simple, we will ensure the requirements below are met.

$$\mathcal{P}_e \;:\; |W_e| = \infty \to \exists s, x(x \in W_{e,at\ s} \wedge x \in D_{s+1}).$$

This is done in the standard fashion. When we see some unrestrained x enter W_e at stage s, we simply put it into D at the very next stage. The construction below is easily seen to be amenable to lowness, so it remains to say how we meet the $\mathcal{R}_{e,i}$ and then how this coheres with the promptly simple requirements.

First we recall the strategy from Downey-Harrington [8]. Let $\ell(e, s)$ denote the D-controllable length of agreement between D and A_e. That is

$$\ell(e, s) = \max\{x : \forall y \leq x[\Delta_{e,s}^{A_{e,s}} = D_s(y) \wedge (\forall z \leq \delta_{e,s}(y)[\Gamma_{e,s}^{D_s}(z) = A_{e,s}(z)]]\}$$

Naturally we regard D as indirectly controlling A_e and hence once $x \leq \ell(e, s)$ then $A_{e,s} \restriction \delta_{e,s}(x)$ is fixed unless we change D_s on some argument $\leq \gamma_{e,s}(\delta_{e,s}(x))$. We break the $\mathcal{R}_{e,i,j}$ into infinitely many subrequirements of the form $\mathcal{R}_{e,i,j,k}$ which are the same as the $\mathcal{R}_{e,i,j}$ except that they assert that $|(X_i \cap Y_j) - B_{e,i}| \geq k$ instead of $|(X_i \cap Y_j) - B_{e,i}| = \infty$. Clearly if we meet all the $\mathcal{R}_{e,i,j,k}$ for all k then $\mathcal{R}_{e,i,j}$ will be met too.

We meet the requirements $\mathcal{R}_{e,i,j,k}$ using the finite injury method. It will essentially suffice to describe the strategy for a single requirement. Thus we will drop the subscript "e" from the sets and functionals. Clearly if $\limsup \ell(e, s) \not\to \infty$ we are done and since we are using a finitary methods we can thus without loss of generality suppose that $\ell(e, s) \to \infty$. Furthermore there is an easy win on $\mathcal{R}_{e,i,j}$ if we ever see $X_i \backslash C \neq \emptyset$. This is because we control C and we can thus win $\mathcal{R}_{e,i,j}$ by simply restraining any element of $X_{i,s} - C_s$ from entry to C thereby negating one of the hypotheses of $\mathcal{R}_{e,i,j}$. (For instance, we will do this if we see a win for $\mathcal{R}_{e,i,j}$ with priority $\langle e, i\rangle$.) Therefore, without loss of generality we will suppose that $|X_i \backslash C| = 0$. The cycle for a single $\mathcal{R}_{e,i,j,k}$ requirement is a follows.

8.1. Pick a follower d targeted for D which is large. Do this at an e-expansionary stage. (Namely when $\ell(e, s)$ exceeds $m = m\ell(e, s)$ which denotes the maximum of $\ell(e, t)$ for $t < s$.)

8.2. Freeze $D \restriction d$ and C until a stage s_1 where $\ell(e, s_1) > d$. The region $J = [\delta_s(m) + 1, \delta_{s_1}(d)]$ now becomes $\mathcal{R}_{e,i,j,k}$'s *critical region.* At stage s_1, $\mathcal{R}_{e,i,j,k}$ continues to restrain $C \restriction \delta_{s_1}(d)$ as well as $D \restriction s_1$ but otherwise imposes no further restraint. (This it why the other requirements can be met.)

8.3. Wait for a stage s_2 such that $A_{s_2} \cup Y_{j,s_2} \supseteq \{0, ..., \delta_{s_1}(d)\}$. *If such a stage s_2 does not occur then $A \cup Y_j \neq \omega$ and hence we win $\mathcal{R}_{e,i,j}$.* At this stage s_2 we declare that $\mathcal{R}_{e,i,j,k}$ is *active*.

8.4. Put all of the interval $[\delta_s(m) + 1, \delta_{s_1}(d)]$ into C but still continue to freeze $D \upharpoonright s_1$.

8.5. Wait for a stage s_3 where $X_{i,s_3} \supseteq (C_{s_3} \cup A_{s_3}) \upharpoonright s_1$. *If such a stage s_3 does not occur then we win $\mathcal{R}_{e,i,j}$.*

8.6. When s_3 occurs then put d into D_{s_3} but continue to restrain $D \upharpoonright d - 1$.

8.7. Wait for the next e-expansionary stage s_3. Notice that some number n, say, from $(C_{s_1} - A_{s_1}) \cap [\delta(m)+1, \delta s_1(d)]$ must have entered A_{s_3}. Furthermore notice that $n \in Y_{j,s_3}$ since we knew in Step 8.3 that $A \cup Y_{j,s_2} \supseteq \{0, ..., \delta_{s_1}(d)\}$ and since $A_{s_1} \upharpoonright \delta_{s_1}(d) = A_{s_2} \upharpoonright \delta_{s_1}(d)$ by D-restraint, we know that, in particular, n must have entered Y_j since it was not in A. Our action is to put into B_{i,s_3} the least collection of elements to cause $A_{s_3} \cup B_{i,s_3} \supset C_{s_3}$ for arguments less than or equal to $\ell(e, s_3)$.

The key point is that n will not enter B_i and hence we have increased the value of $|(X_{i,s_3} \cap Y_{j,s_3}) - B_{i,s_3}|$ by one since stage s, the beginning of the cycle. In this way we force $|X_i - B_i| \to \infty$. Notice that we only put elements into C in response to Y_j gaining new elements. Moreover, whenever we put elements into C provided that X_i responds by making $X_i \cup A \supseteq C$ (locally), we will later ensure that $B_i \cup A \supseteq X_i$. Of course, for any $i' \neq i$, while we are attacking some $\mathcal{R}_{e,i,j,k}$ we simply make $B_{i'} = X_{i'}$ locally with no effect on $\mathcal{R}_{e,i,j,k}$. Thus the requirements cohere exactly by a standard application of the bounded injury priority method.

Finally, observe that adding prompt simplicity has no effect since all the actions are totally finitary, and any injuries from below are simply dumped into B. $\square$

We close this paper with one final demonstration that there is no simple relationship between $S(A)$ and previously considered promptness considerations. The theorem below was stated, without proof, in Downey-Harrington [8].

Theorem 8.10 (Downey and Harrington [8]) *Let* **a** *be any high degree. There exist sets A and E in* **a** *such that $S(A)$ and $\neg S(E)$.*

Proof. Since Herrmann (and Hemimaximal) sets obey $\neg S(E)$ and exist in all high degrees, we see **a** contains a set E satisfying $\neg S(E)$. For A we will use permitting and high coding. Actually, we first observe that high coding is enough.

Lemma 8.11 *Suppose that* $\mathbf{a} < \mathbf{b}$ *and* $\mathbf{a}$ *contains a set satisfying* $S(A)$*. Then so does* $\mathbf{b}$.

Proof. Suppose that $S(A)$ holds with witness C. Then for any set B, $S(A \oplus B)$ holds with witness $C \oplus \emptyset$. $\square$

Recall that to meet the $\mathcal{R}_X$ via the $\mathcal{R}_{X,Y,i}$ we pick x outside of $C \cup A$, wait till x enters Y, put x into C, and wait for x to enter X. At that stage, we would put x into A (and not B) increasing the count on $(X \cap Y) - B$. If we fail to put x into B, because we are awaiting some permission, then we run the risk of having $x \notin A \cup B_X$. However it is clear that $S(A)$ holds iff A only meets the definition of $S(A)$ but with $A \cup B \supseteq^* X$ in place of $A \cup B \supseteq X$. Thus we are okay provided that we can guarantee that for almost all x following some $R_{X,Y,i}$, if x enters C and then X we will put x into A. This is the familiar high permission scenario.

Let D be c.e. with $\overline{D}$'s computation function dominant, as in Soare [25], Ch XI, say Exercise 2.15 (or Theorem 2.1). (That is, Cooper's proof that each high c.e. degree bounds a minimal pair.) One can then think of the $\mathcal{R}_{X,Y,i}$ separately for each Y, as trying to eliminate followers from a hole. When x some such follower x is realized into Y we wait for a normal permission from D to allow us to put it into C. While we wait we will pick a new follower x', etc. Now of course, C is acting like a gate in a pinball construction. Namely we will wait for x inside of C to enter X as before. Let $x_1, x_2, \ldots$ be the elements arriving into C for the sake of $\mathcal{R}_X$. When x_n arrives in C at stage s_n we are waiting for $t_n > s_n$ for x_n to enter X. When this happens we define $f_X(s_n) = t_n$, and we, as usual, wait for $d_{s_n}^{v+1} \neq d_{s_n}^{v}$, precisely as in Soare [25], Ch XI, say Exercise 2.15. If such a stage occurs then we can put x_n into A. We then complete f_X to a computable function by defining $f_X(z) = 0$ if $z \notin \{s_n : n \in \omega\}$. Then D can always compute the final position of a follower, since D permission is needed to move a follower. Finally, the dominance of D makes sure that almost all x_n enter A, since f_X is computable and $\overline{D}$'s computation function is dominant. $\square$

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