Singular Learning Theory Problems: Day 4

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1 Problems

1. Find RLCT of $f(x, y) = y^2 - x^3$ at the origin by using a resolution of singularities.

2. In this exercise, we will compute the leading coefficient C in the asymptotic formula

$$Z(N) \;=\; \int_{[0,1]^2} (1 - x^2y^2)^{N/2}\, dxdy \;\approx\; CN^{-1/2}(\log N).$$

   (a) Write down the corresponding zeta function $\zeta(z)$ for this integral.

   (b) Given $K(x, y) = -\frac{1}{2}\log(1 - x^2y^2)$, show that $K(x, y)$ has the Taylor expansion

$$K(x, y) = \frac{x^2y^2}{2}\left(1 + \frac{x^2y^2}{2} + \frac{x^4y^4}{3} + \cdots\right)$$

   (c) For now, suppose that z is a negative real number. Using the generalized binomial theorem, we get the power series expansion

$$K(x, y)^{-z} = \sum_{i=0}^{\infty} 2^z \cdot h_i(z) \cdot (xy)^{-2z+2i}.$$

   Prove that the coefficients $h_i(z)$ are polynomials in z, and find $h_0(z)$.

   (d) Integrate $K(x, y)^{-z}$ term-by-term over $\{(x, y) \in [0, 1]^2\}$ to get a Laurent series expansion for $\zeta(z)$. For each term in the series, find its contribution

$$\frac{a}{(z - \frac{1}{2})^2}, \qquad \text{for some } a \in \mathbb{R},$$

   to the pole at $z = \frac{1}{2}$ of multiplicity 2.

   (e) Asymptotic theory tells us that while $\zeta(z)$ is well-defined for $z < 0$, the zeta function has an analytic continuation to the whole complex plane $z \in \mathbb{C}$. Deduce that the coefficient of $(z - \frac{1}{2})^{-2}$ in the Laurent expansion of $\zeta(z)$ is $d = 1/\sqrt{8}$.

   (f) Finally, use Theorem 3.4 below and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ to compute the leading coefficient C.

2 What is a Blow-up?

2.1 Blow-up of the origin

Consider $\mathbb{R}^n \times \mathbb{P}^{n-1}$ with coords

$$((x_1, x_2, \dots, x_n), (\xi_1 : \xi_2 : \dots : \xi_n)).$$

Let V be subset of points

$$((x_1, x_2, \dots, x_n), (x_1 : x_2 : \dots : x_n)), \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Then, $X = \overline{V} \subset \mathbb{R}^d \times \mathbb{P}^{n-1}$ is the blow-up of the origin in $\mathbb{R}^n$. The projection $\pi : X \to \mathbb{R}^n, (x, \xi) \mapsto x$, is the blow-up map.

1. $X = V \cup E$ where $E = \{0\} \times \mathbb{P}^{n-1}$ is the exceptional divisor. The map π is an isomorphism from V to $\mathbb{R}^n \setminus \{0\}$, while $\pi^{-1}(0) = E$.

2. X is a toric variety (defined by binomials).

$$X = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{P}^{n-1} \mid x_i \xi_j = x_j \xi_i \text{ for all } i, j\}$$

3. X covered by affine charts $U_i = \{(x, \xi) \in X \mid \xi_i \neq 0\} \simeq \mathbb{R}^n$ with coords

$$\{(y_1, \dots, y_n) = (\frac{\xi_1}{\xi_i}, \dots, x_i, \dots, \frac{\xi_n}{\xi_i}\}$$

so π is given by affine maps $\pi_i : U_i \to \mathbb{R}^n$

$$(y_1, \dots, y_n) \mapsto (y_1 y_i, \dots, y_i, \dots, y_n y_i)$$

2.2 Blow-up of a linear subspace

$\pi \times id : X \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ is the blowing-up of $\{0\} \times \mathbb{R}^m$ in $\mathbb{R}^{n+m}$.

2.3 Blow-up of a smooth center

Let $Z \subset \mathbb{R}^d$ be a smooth variety whose ideal is $\langle f_1, \dots, f_r \rangle$. For instance, when Z is the origin, its ideal is $\langle x_1, \dots, x_d \rangle$. Let us blow up $\mathbb{R}^d$ with *center* Z.

For each point $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ not in Z, let us tag it with the point

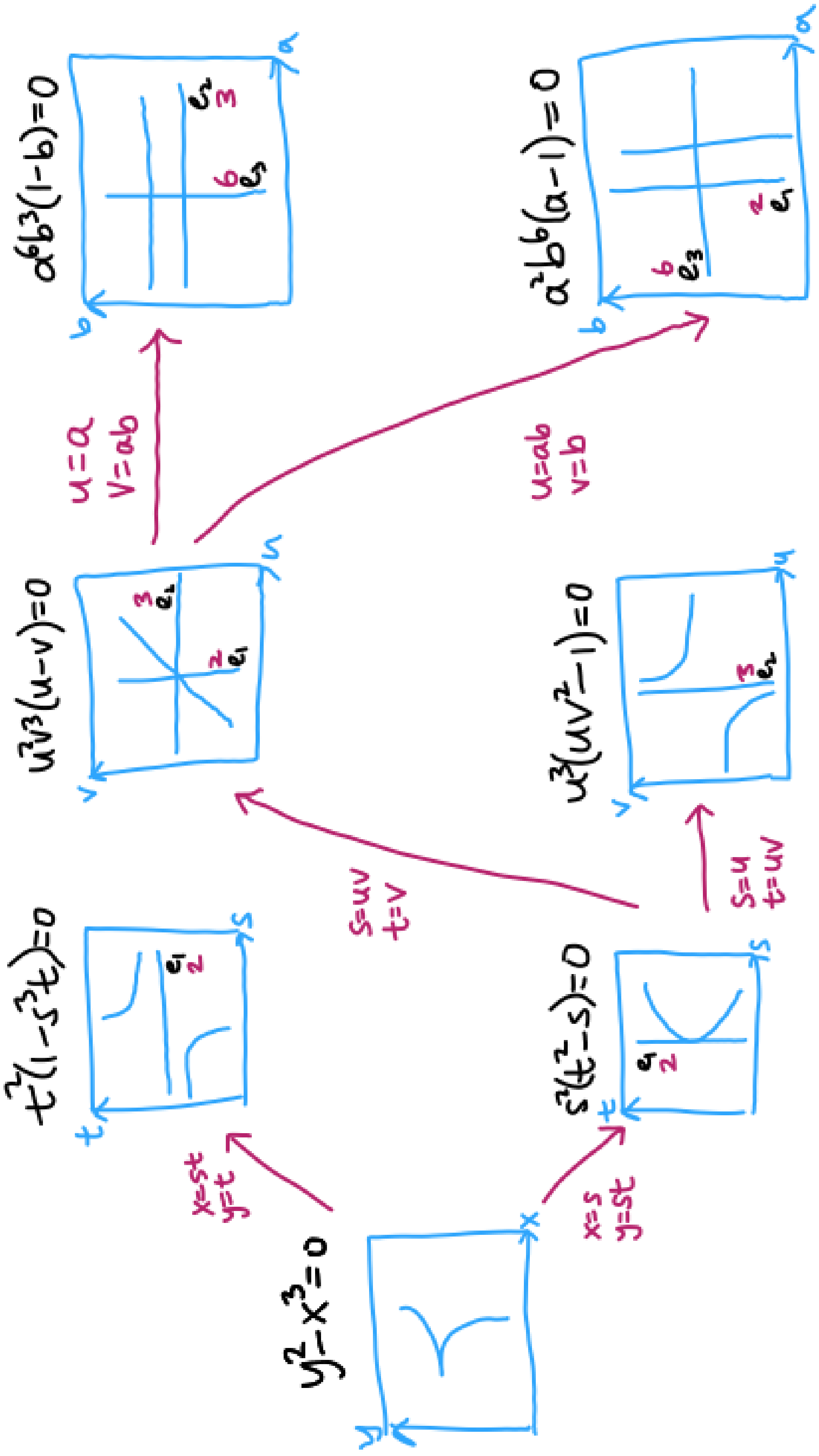
$$f^{\mathbb{P}}(x) = (f_1(x) : \dots : f_r(x)) \in \mathbb{P}^{r-1}.$$

The set X of points

$$(x, f^{\mathbb{P}}(x)) \in \mathbb{R}^d \times \mathbb{P}^{r-1}, \quad x \in \mathbb{R}^d \backslash Z$$

has a Zariski closure $\widetilde{X}$ called the *blowup* of $\mathbb{R}^d$ with center Z.

The projection $\pi : \widetilde{X} \subset \mathbb{R}^d \times \mathbb{P}^{r-1} \to \mathbb{R}^d$ is the *blowup map*. This map restricts to an isomorphism $X \to \mathbb{R}^d \backslash Z$, while the preimage $E = \pi^{-1} Z \simeq Z \times \mathbb{P}^{\text{codim(Z)}-1}$ is the *exceptional divisor*.



y²−x³=0
x=st
y=t
x=s
y=st
t²(1−s³t)=0
s²(t²−s)=0
s=uv
t=v
s=u
t=uv
u²v³(u−v)=0
u³(uv²−1)=0
u=a
v=ab
u=ab
v=b
a⁶b³(1−b)=0
a²b⁶(a−1)=0

3 Zeta Functions

Laplace integrals occur frequently in physics, statistics and other applications. At first glance, the relationship between their asymptotic expansions and the Laurent expansion of the zeta function seems strange. The key is to write these integrals as

$$Z(n) = \int_\Omega e^{-n|f(\omega)|}|\varphi(\omega)|\,d\omega = \int_0^\infty e^{-nt}v(t)\,dt$$
$$\zeta(z) = \int_\Omega \big|f(\omega)\big|^{-z}|\varphi(\omega)|\,d\omega = \int_0^\infty t^{-z}v(t)\,dt$$

where $v(t)$ is the state density function or Gelfand-Leray function

$$v(t) = \frac{d}{dt}\int_{0<|f(\omega)|<t}|\varphi(\omega)|\,d\omega.$$

Formally, $Z(n)$ is the *Laplace transform* of $v(t)$ while $\zeta(z)$ is its *Mellin transform.* Note that contrary to its name, $v(t)$ is generally not a function but a Schwartz distribution. We study the series expansions

$$Z(n) \approx \sum_\alpha \sum_{i=1}^d c_{\alpha,i} n^{-a}(\log n)^{i-1} \tag{1}$$
$$v(t)dt \approx \sum_\alpha \sum_{i=1}^d b_{\alpha,i}\, t^\alpha(\log t)^{i-1}dt \tag{2}$$
$$\zeta(z) \sim \sum_\alpha \sum_{i=1}^d d_{\alpha,i}(z-\alpha)^{-i} \tag{3}$$

where the series (1) and (2) are asymptotic expansions while (3) is the principal part of the Laurent series expansion. Formulas relating their coefficients are then deduced from the Laplace and Mellin transforms of $t^\alpha(\log t)^i$ which appears in $v(t)$. Detailed expositions on this subject have been written by Arnol'd–Guseĭn-Zade–Varchenko [1], Watanabe [5] and Greenblatt [2].

Proposition 3.1. *The asymptotic expansion of the Laplace transform of* $t^{\alpha-1}(\log t)^i$ *is*

$$\int_0^\infty e^{-nt}\, t^{\alpha-1}(\log t)^i\,dt \approx \sum_{j=0}^i \binom{i}{j}(-1)^j\Gamma^{(i-j)}(\alpha)\,n^{-\alpha}(\log n)^j$$

while the Mellin transform of $t^{\alpha-1}(\log t)^i$ *is*

$$\int_0^1 t^{-z}\,t^{\alpha-1}(\log t)^i\,dt = -\,i!\,(z-\alpha)^{-(i+1)}.$$

Proof. See [1, Thm 7.4] and [5, Ex 4.7] respectively. □

In this section, we employ standard techniques to derive the asymptotic expansion of the Laplace integral from the Laurent expansion of the zeta function. Recall that Γ is the Gamma function and that $\Gamma^{(i)}$ is its i-th derivative.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^d$ be a compact semianalytic subset and $\varphi : \Omega \to \mathbb{R}$ be nearly analytic. If $f \in \mathcal{A}_\Omega$ with $f(x) = 0$ for some $x \in \Omega$, then the Laplace integral*

$$Z(n) = \int_\Omega e^{-n|f(\omega)|} |\varphi(\omega)| \, d\omega$$

has the asymptotic expansion

$$\sum_\alpha \sum_{i=1}^{d} c_{\alpha,i} \, n^{-\alpha} (\log n)^{i-1}. \tag{4}$$

The α in this expansion range over positive rational numbers which are poles of

$$\zeta(z) = \int_{\Omega_\delta} \big|f(\omega)\big|^{-z} |\varphi(\omega)| \, d\omega \tag{5}$$

where $\delta \in \mathbb{R}$ is any $\delta > 0$ and $\Omega_\delta = \{\omega \in \Omega : |f(\omega)| < \delta\}$. The coefficients $c_{\alpha,i}$ satisfy

$$c_{\alpha,i} = \frac{(-1)^i}{(i-1)!} \sum_{j=i}^{d} \frac{\Gamma^{(j-i)}(\alpha)}{(j-i)!} \, d_{\alpha,j} \tag{6}$$

where $d_{\alpha,j}$ is the coefficient of $(z-\alpha)^{-j}$ in the Laurent expansion of $\zeta(z)$.

Proof. See [4, Thm 3.16]. □

Definition 3.3. The *leading coefficient* $\mathrm{coef}_\Omega(f;\varphi)$ is the coefficient $c_{\lambda,\theta}$ of the leading term in the asymptotic expansion of $Z(n)$. Note that (λ, θ) is the real log canonical threshold $\mathrm{RLCT}_\Omega(f;\varphi)$.

Proposition 3.4. *The leading coefficient* $\mathrm{coef}_\Omega(f;\varphi)$ *is given by*

$$c_{\lambda,\theta} = \frac{(-1)^\theta \, \Gamma(\lambda)}{(\theta-1)!} \, d_{\lambda,\theta}$$

where $d_{\lambda,\theta}$ is the coefficient of $(z-\lambda)^{-\theta}$ in the Laurent expansion of $\zeta(z)$.

References

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[3] H. Hironaka: Resolution of singularities of an algebraic variety over a field of characteristic zero I, II, *Ann. of Math.* (2) **79** (1964) 109–203.

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