Sufficient Conditions For Non-negative Impulse Response of Arbitrary-Order Systems

Y. Liu, Student Member, IEEE
Dept. of Electrical Engineering
University of Notre Dame
yliu5@nd.edu

Peter H. Bauer, Fellow, IEEE
Dept. of Electrical Engineering
University of Notre Dame
pbauer@nd.edu

Abstract—This paper presents a set of sufficient conditions that ensure an arbitrary-order transfer function to exhibit a non-negative impulse response. Such conditions are given for both continuous-time and discrete-time systems. This set of sufficient conditions expose a simple geometric pole-zero pattern that can be easily employed for designing filters with a non-negative impulse response. Most of the existing work can be obtained as a special case of this work. This paper significantly extends the class of pole-zero patterns that are known to exhibit a non-negative impulse response, in both continuous-time and discrete-time domains.

I. INTRODUCTION

NON-NEGATIVITY of the impulse response is a widely required feature in many applications, such as congestion control systems, machine tool axis control, trajectory following in robotics, etc. [1]. In those applications, local extrema of the step response are not acceptable. One recently emerging class of applications comes from the design of so called Evidence Filters [2] and requires the feature of a non-negative impulse response in addition to other constraints.

Unfortunately, necessary and sufficient conditions on zero-pole patterns that guarantee the non-negativity of the impulse response of a general SISO linear system are still not available. Most research efforts produced sufficient conditions only. However, most of the existing results on sufficient conditions are applicable only to first-order and second-order systems. Construction of high-order systems has to rely on the convolution property, i.e., the fact that cascading two non-negative impulse response systems results in a non-negative impulse response system. However, such an approach imposes significant limits on the frequency domain performance.

Instead, this paper directly explores sufficient conditions on pole-zero patterns for arbitrary-order systems, in both continuous-time and discrete-time domains. These conditions significantly extend the class of pole-zero patterns that are known to exhibit a non-negative impulse response.

II. RELATED AND PREVIOUS WORK

The results in [1], [3]–[10] represent some of the major findings in the continuous-time domain: The work in [3] gives necessary and sufficient conditions on pole patterns (poles being negative real) for an all-pole system. The work in [4] provides sufficient conditions for a system with zeros and poles on a vertical line. The work in [5] indicates that if it is possible to pair each complex zero/pole with a negative real pole whose magnitude is no greater than the magnitude of the real part of that complex zero/pole, then the step response is monotonically non-decreasing. The approach in [6] provides sufficient conditions on pole-zero patterns for either a first-order system with one negative real pole and one negative zero, or an all-pole second-order system with one negative real pole. The work in [7] generates sufficient conditions for a second-order transfer function with negative real poles and zeros. The results in [11] provide necessary and sufficient conditions for a third-order system with negative real poles, and sufficient conditions for a third-order system with a complex conjugate pole pair and a real negative pole satisfying certain constraints. However, such conditions only state the relationship between the poles and the numerator coefficients, not the pole-zero patterns. Transferring such conditions to the form of pole-zero patterns is limited to second-order systems, due to the high complexity of the problem. The result in [1] gives the bounding theorem that ensures no step response overshoot occurs. The theorems in [10] discuss the influence of the real open right half plane zeros on the step response.

The work in [12]–[14] constitute the major results in the discrete-time domain: The results in [12] provide the sufficient conditions for systems with zeros and poles located on a given circle or concentric circles, which constitutes the discrete-time counterpart of [4]. The work in [13] presents the discrete-time counterpart of [7], and shows necessary and sufficient conditions for second-order systems with positive real zeros and poles. The findings in [14] provide necessary and sufficient conditions for the step response sequence to exhibit initial undershoot and final overshoot, for strictly proper rational systems.

While existing work only shows sufficient conditions on pole-zero patterns for first-order or second-order systems with a non-negative impulse response, this paper gives sufficient conditions for arbitrary-order systems. We will show that most of the existing results can be treated as special cases of this work.

Section III and section IV describe our results in continuous-time and discrete-time domains, respectively. Section V discusses the specific conditions for third-order systems, as well as the relationship between this work and previous work.
III. SUFFICIENT CONDITIONS FOR NON-NEGATIVE IMPULSE RESPONSE CONTINUOUS-TIME SYSTEMS

Theorem 1: If a \( n^{th} \)-order stable continuous-time rational transfer function denoted by

\[
G(s) = \frac{\prod_{i=1}^{n} (s + z_i)}{\prod_{i=1}^{n} (s + p_i)} \quad \text{(III.1)}
\]

satisfies:

1) \( p_i, z_i > 0 \), \( p'_i \), \( z'_i \) are simple, \( 1 \leq i \leq n \)
2) \( \min\{p_i\} \leq z_i \leq \max\{p_i\}, 1 \leq i \leq n \)
3) \( \sum_{i=1}^{n} z_i \geq \sum_{i=1}^{n} p_i \)

and additionally, the number of the zeros between two consecutive poles satisfies one of the following two conditions:

(i) Only one interval \( I_1 \) has three zeros, only one interval has no zeros, and the remaining \( n - 3 \) intervals each have exactly one zero.

(ii) Only one interval \( I_2 \) has two zeros, and the remaining \( n - 2 \) intervals each have exactly one zero.

then the impulse response satisfies:

\[
g(t) \geq 0, \quad t \geq 0.
\]

To prove Theorem 1, it is convenient to consider Lemma 1 first.

Lemma 1: For the partial fraction expansion of Equation (III.1) and its inverse Laplace transform:

\[
G(s) = 1 + \sum_{i=1}^{n} \frac{C_i}{s + p_i} \quad \text{(III.2)}
\]

\[
g(t) = \delta(t) + \sum_{i=1}^{n} C_i e^{-p_i t} \quad \text{(III.3)}
\]

where

\[
C_i = \frac{\prod_{j=1}^{p_i} (z_j - p_i)}{\prod_{j=1, j \neq i}^{p_i} (p_j - p_i)} \quad \text{(III.4)}
\]

a sign-change occurs only once from \( C_1 \) to \( C_n \) from “+” to “−”, as shown in Fig. 1, is equivalent to that \( z'_i \) and \( p'_i \) fall into one of two cases described by the additional conditions (i) and (ii) in Theorem 1 respectively.

<table>
<thead>
<tr>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>...</th>
<th>( C_{k-1} )</th>
<th>( C_k )</th>
<th>...</th>
<th>( C_{n-1} )</th>
<th>( C_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>...</td>
<td>+</td>
<td>...</td>
<td>-</td>
<td>...</td>
<td>-</td>
</tr>
</tbody>
</table>

Fig. 1. Valid sign pattern of \( C_i \)’s

Proof: Without loss of generality, we assume \( p_i \leq p_{i+1}, z_i \leq z_{i+1}, i = 1, \ldots, n \). Clearly, the denominators of consecutive \( C_i \)’s have alternating signs. Therefore, the sign of a \( C_i \) depends on the sign of its numerator, or equivalently, the distribution of \( z_i \)’s. If the number of \( z_i \)’s that lie in the interval sandwiched by \( p_{i-1} \) and \( p_i \) is odd, \( C_{i-1} \) and \( C_i \) have the same sign. Otherwise, \( C_{i-1} \) and \( C_i \) have opposite signs, and the corresponding interval sandwiched by \( p_{i-1} \) and \( p_i \) is denoted as “boundary interval”. Therefore, if the sign-change occurs only once, it must occur on the boundary interval which has an even number of \( z_i \)’s. Any other intervals on either side of the boundary interval each have an odd number of \( z_i \)’s.

With Lemma 1, we are in the position to prove Theorem 1.

Proof: Suppose the \( k^{th} \) interval, \( 1 \leq k \leq n - 1 \), is the boundary interval when (i) or (ii) is satisfied. Since

\[
C_i > 0, \quad i \leq k \leq n - 1
\]

the impulse response given in (III.3) becomes

\[
g(t) = \delta(t) + \sum_{i=1}^{k} C_i e^{-p_i t} + \sum_{i=k+1}^{n} C_i e^{-p_i t}
\]

\[
\geq \delta(t) + (\sum_{i=1}^{k} C_i) \cdot e^{-p_k t} + (\sum_{i=k+1}^{n} C_i) \cdot e^{-p_{k+1} t}
\]

Since \( e^{-p_k t} \geq e^{-p_{k+1} t} \), the inequality

\[
\sum_{i=1}^{n} C_i \geq \sum_{i=1}^{k} C_i + \sum_{i=k+1}^{n} C_i \geq 0 \quad \text{(III.5)}
\]

is sufficient to ensure the non-negativity of \( g(t) \) for \( t > 0 \).
Simplifying (III.5), we have:
\[
\sum_{i=1}^{n} C_i = \frac{(p_2 - p_1)(p_3 - p_1) \cdots (p_n - p_{n-1})}{(p_2 - p_1)(p_3 - p_1) \cdots (p_n - p_{n-1})} \sum_{i=1}^{n} \frac{z_i - \sum_{i=1}^{n} p_i}{z_i - \sum_{i=1}^{n} p_i} \geq 0
\]

Therefore, it is sufficient that
\[
\sum_{i=1}^{n} z_i - \sum_{i=1}^{n} p_i \geq 0
\]

This is the condition 3) in Theorem 1.

It is very important to note that the additional conditions (i) and (ii) do not necessarily decompose the transfer function into the form shown in [5], [7]. The result in Theorem 1 can be extended to contain a complex conjugate pair. This result is given in Lemma 2 with the proof omitted due to similarity w.r.t. the proof of Theorem 1.

Lemma 2: If (III.1) has a complex conjugate zero pair \( z_k, z_{k+1} = \sigma \pm j\delta \), and
1) \( \sigma, p_i, z_j > 0 \), \( p'_i, s, z'_j \) are simple, \( 1 \leq i, j \leq n, j \neq k, k + 1 \)
2) \( \min \{p_i\} \leq \sigma, z_j \leq \max \{p_i\}, 1 \leq i, j \leq n, j \neq k, k + 1 \)
3) \( \sum_{i=1}^{n} z_i \geq \sum_{i=1}^{n} p_i \)

and, additionally, one of the following two conditions is satisfied:
(i) Only one interval \( I_3 \) has three zeros with \( z_k, z_{k+1} \in I_3 \), only one interval has no zeros, and the remaining \( n - 3 \) intervals each have exactly one zero.
(ii) \( z_k, z_{k+1} \in I_2 \), and the remaining \( n - 2 \) intervals each have exactly one zero.

Then the impulse response satisfies:
\[
g(t) \geq 0, \quad t \geq 0.
\]

Since \( z_i \)'s and \( p_i \)'s represent the positions of zeros and poles, from the proofs of Lemma 1 and Theorem 1, we see that zeros and poles show a simple geometric pattern.

IV. SUFFICIENT CONDITIONS FOR NON-NEGATIVE IMPULSE RESPONSE DISCRETE-TIME SYSTEMS

Theorem 2: For a \( n^{th} \)-order stable discrete-time rational transfer function denoted by
\[
G(z) = \frac{\prod_{i=1}^{n} (z - z_i)}{\prod_{i=1}^{n} (z - p_i)}
\]

the poles \( p'_i \)'s and the zeros \( z'_i \)'s satisfy:
1) \( 0 < p_i, z_i < 1 \), \( p'_i, s, z'_j \) are simple, \( 1 \leq i \leq n \)
2) \( \min \{p_i\} \leq z_i \leq \max \{p_i\}, 1 \leq i \leq n \)
3) \( \sum_{i=1}^{n} z_i \leq \sum_{i=1}^{n} p_i \)

and additionally, the number of the zeros between two consecutive poles satisfies one of the following two conditions:
(i) Only one interval \( I_1 \) has three zeros, only one interval has no zeros, and the remaining \( n - 3 \) intervals each have exactly one zero.
(ii) Only one interval \( I_2 \) has two zeros, and the remaining \( n - 2 \) intervals each have exactly one zero.

Then the impulse response satisfies:
\[
g(n) \geq 0, \quad n \geq 0.
\]

Proof: Via partial fraction expansion, we have:
\[
G(z) = 1 + \sum_{i=1}^{n} \frac{D_i}{z - p_i},
\]
\[
g(n) = \delta(n) + \sum_{i=1}^{n} D_i p_i^{n-1} u(k-1)
\]
\[
D_i = \prod_{j=1, j \neq i}^{n} (p_i - p_j)
\]

Similarly, the one-time occurrence of a sign-change (from “+” to “−”) from \( D_{n-1} \) to \( D_{n} \) (not \( D_{2} \) to \( D_{n-1} \)) is equivalent to the pole-zero pattern falling into one of the two conditions (i) and (ii) in Theorem 2. Suppose the \( k^{th} \) \( (1 \leq k \leq n - 1) \) interval is the boundary interval, we have
\[
D_i > 0, \quad \text{when} \quad (k + 1) \leq i \leq n
\]
\[
D_i < 0, \quad \text{when} \quad 1 \leq i \leq k.
\]

\[
\therefore \quad \sum_{i=1}^{n} D_i \geq 0 \quad \text{(IV.2)}
\]

is sufficient to ensure the non-negativity of the impulse response. Simplifying (IV.2), we have:
\[
\sum_{i=1}^{n} p_i - \sum_{i=1}^{n} z_i \geq 0
\]

which is the condition 3) in Theorem 2.

Likewise, the result in Theorem 2 can be extended to contain a complex zero pair. This result is given in Lemma 3.
**Lemma 3:** If (IV.1) has a complex conjugate zero pair \(z_k, z_{k+1} = \sigma \pm j\mu\), and the poles \(p_i's\) and the zeros \(z_i's\) satisfy:

1) \(0 < p_i, z_j, \sigma < 1, \ p_i's, z_j's\ are\ simple, \ 1 \leq i,j \leq n, \ j \neq k,k+1\)
2) \(\min\{p_i\} \leq \sigma, z_j \leq \max\{p_i\}, \ 1 \leq i,j \leq n, \ j \neq k,k+1\)
3) \(\sum_{i=1}^{n} z_i \leq \sum_{i=1}^{n} p_i\)

and additionally, one of the following two conditions is satisfied:

(i) Only one interval \(\tilde{I}_3\) has three zeros with \(z_k, z_{k+1} \in \tilde{I}_3\), only one interval has no zeros, and the remaining \(n-3\) intervals each have exactly one zero.

(ii) \(z_k, z_{k+1} \in \tilde{I}_2\), and the remaining \(n-2\) intervals each have exactly one zero.

then the impulse response satisfies:

\[g(n) \geq 0, \ n \geq 0.\]

The proof is omitted since it is similar to that of Theorem 2.

**V. DISCUSSION**

**A. Sufficient Conditions for Third-Order Systems**

From the proofs of Theorems 1 and 2, it is obvious that for a third-order system, the additional conditions (i) and (ii) are automatically satisfied. Therefore, conditions 1 – 3 in Theorems 1 and 2 are already sufficient.

**B. Relationship With Existing Work**

The major results reported in the literature can be obtained as a special case of this work.

1) When \(n = 1\), the additional conditions in Theorem 1 and Theorem 2 are trivially satisfied, because \(C_i\) or \(D_i\) do not change sign. The conditions then degenerate to the ones given in [5] [6] and [13].

2) When \(n = 2\), the additional conditions in Theorem 1 and Theorem 2 are automatically satisfied. The conditions then match the ones given in [7] and [13].

**C. A Fourth-Order System Example**

A fourth-order discrete-time example is given to illustrate the achieved results:

\[G(z) = \frac{(z - 0.41)(z - 0.723)}{(z - 0.4)(z - 0.68)} \cdot \frac{[z - (0.738 + 0.5j)](z - (0.738 - 0.5j))}{(z - 0.72)(z - 0.84)}\]

Since all the conditions in Lemma 3 are satisfied, the impulse response must be non-negative. Indeed, this is the case, as shown in Fig. 3. However, whichever way is used to decompose the system into a cascade of first-order or second-order sub-systems, the sufficient conditions reported from previous work cannot ensure each sub-system to exhibit a non-negative impulse response. Thus the non-negativity of the overall system cannot be determined with existing methods.

**VI. CONCLUSION**

This work contributes to the research on pole-zero patterns for non-negative impulse response systems. It provides sufficient conditions that ensure a non-negative impulse response for an arbitrary-order system, in both continuous-time and discrete-time domains. This work significantly extends the class of the pole-zero patterns that are known to exhibit a non-negative impulse response.

**ACKNOWLEDGMENT**

This work was supported by the Defense Threat Reduction Agency grant DTRA N00164-07-C-8570. The authors would like to thank DTRA and CRANE NAVAL for their support.

**REFERENCES**


