Abstract—This paper provides insights into some fundamental properties of non-negative impulse response (NNIR) digital filters. In particular, performance restrictions in the frequency domain due to the non-negativity constraint are investigated. It is shown that the gain drop in the frequency response near zero frequency affects the response over the entire frequency spectrum. The passband and stopband gains are shown to be severely limited. It is further shown that stopband and passband locations can be chosen freely with the exception of a mandatory passband around zero. Extensions of the presented results to the continuous time domain as well as duality results are also briefly presented.

Index Terms—Non-negative impulse response, filter, frequency response, frequency-domain bounds, time-domain bounds.

I. INTRODUCTION

The relationship between the characteristics of the frequency responses of systems and the characteristics of their transient responses has been attracting a lot of research attentions. Some of the early contributions [1]–[4] discussed this relationship in an empirical way. It became one of the most active research topics from the 1940’s through the 1960’s when linear network (electric network) analysis via system functions (transfer/immittance functions) was popular. It was then well recognized that linear networks cannot have arbitrary frequency characteristics. The positive real characteristic of a driving point immittance function is an example.

At that time, a lot of research was directed towards discovering the corresponding restrictions on the time domain responses [5]–[9]. Likewise, due to the requirements on prescribed transient behaviors [5] [10] [11], some studies were also given for the restrictions on the frequency domain responses imposed by time domain constraints. For illustration, the work in [5] [6] presents several bounds on transient responses (specifically, impulse/step/ramp/parabolic responses) due to the non-negativity/monotonicity of the real/imaginary parts of system functions. It also provides a few bounds on the real/imaginary part of a system function when the impulse response is non-negative. The work in [8] gives the bounds on the attenuation of the real part of the transfer or immittance function when the impulse response is restricted to be non-negative. A detailed survey of the work related to the NNIR feature can be found in [12].

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However, the majority of prior work on non-negative impulse response (or monotonic step response) constraints focuses on studying the restriction on the real or imaginary part of a system function, approaching the problem from an electric circuits point of view. Very few results provide performance boundary studies on the frequency responses of filters. The work in [13] and [14] is an exception that made some initial efforts. The work in [13] shows that if an impulse response is non-negative, the attenuation over subsequent octaves on the frequency axis is severely limited when the attenuation over the first octave is given. The work in [14] investigates the relationship between the non-negativity/non-positivity of the step response and the asymptotic roll-off near the origin of a high-pass filter.

This paper aims at filling the gaps in the frequency-domain performance boundary study of NNIR filters. An in-depth discussion on the fundamental properties of an NNIR filter is given in terms of frequency domain performance limitations. This investigation is of particular importance to the design of NNIR systems, as they occur in various industrial processes [10] [11] [15]. For example, operation of a CCD sensor in FIR mode requires the impulse response of the FIR filter to have non-negative coefficients. This limitation is due to physical principle of CCD sensor, because the coefficients correspond to the integration time and the time can only be non-negative. An emerging application called “evidence filters” also requires the feature of NNIR [16].

A closely related but different system is called a positive linear system [17]. Typical applications of a positive linear system include Charge Routing Networks [18], compartmental systems, storage systems, heat exchangers and models used in economy and sociology. A theoretic approach to positive linear systems has been first introduced in [19]. Since then, many issues related to positive linear systems including reachability, controllability, realizability, and stability have been investigated [20]–[23]. The work in [24] gives an essentially complete characterization of any realizable rational transfer function with an NNIR in terms of the location of poles. The work in [25] gives necessary and sufficient conditions for positive realizability using a convex analysis.

This paper is organized as follows: Section II constitutes the main part of this paper. Frequency responses at different frequency points are found to be inherently correlated. The upper-bounds on the maximal attenuation at a frequency point and on the change of attenuation/gain from one end to the other end of a frequency region are provided, respectively. Tightness of the different bounds due to the different choices of frequency granularity is discussed. The discussions in this section are given for systems in discrete-time (D-T) settings.
Section III illustrates the analogous results for continuous-time (C-T) systems. Section IV gives the “dual” results regarding the fundamental limitations imposed on the time domain when the spectral domain response is restricted to be non-negative. Section V offers conclusions.

II. FUNDAMENTAL PROPERTIES IN THE FREQUENCY DOMAIN OF AN NNIR SYSTEM

Before the discussion is started, we define the following symbols that will be used throughout the remainder of this paper.

\( \omega \) : angular frequency in D-T domain

\( \Omega \) : angular frequency in C-T domain

\( H(e^{j\omega}) \) : frequency response of a D-T NNIR filter

\( H(j\Omega) \) : frequency response of a C-T NNIR filter

\( h(n) : \mathbb{Z} \rightarrow \mathbb{R} \): impulse response function of a real-valued D-T NNIR filter

\( h(t) : \mathbb{R} \rightarrow \mathbb{R} \): impulse response function of a real-valued C-T NNIR filter

* : convolution operator

\( \mathbb{Z} \) : set of integers

\( \mathbb{Z}^+ \) : set of positive integers

\( \mathcal{F} \) : Fourier-transform

\( U_l[x] \) : \( l \)th-degree Chebyshev polynomial of the second kind in variable \( x \)

Lemma 1: If the impulse response of a filter is non-negative, i.e.,

\[ h(n) \geq 0, \ n \in \mathbb{Z} \]

then the squared magnitude response denoted by \( |H(e^{j\omega})|^2 \) satisfies:

\[ |H(e^{j\omega})|^2 \leq H^2(1), \tag{1} \]

where \( 0 \leq \omega \leq \pi \).

Proof:

\[
|H(e^{j\omega})|^2 = \left| \sum_{n=-\infty}^{\infty} h(n) e^{-jn\omega} \right|^2 \\
\leq \left( \sum_{n=-\infty}^{\infty} |h(n) e^{-jn\omega}| \right)^2 = \left( \sum_{n=-\infty}^{\infty} h(n) \right)^2 = H^2(1)
\]

Theorem 1: If the impulse response of a filter is non-negative, i.e.,

\[ h(n) \geq 0, \ n \in \mathbb{Z} \]

and the attenuation at the frequency point \( \omega_o \) is bounded by:

\[ H^2(1) - \delta \leq |H(e^{j\omega_o})|^2 \leq H^2(1) \quad \tag{2} \]

where \( \delta > 0, \ \omega_o \in \left(0, \frac{\pi}{mk}\right], \ m, k \in \mathbb{Z}^+ \), then the attenuation at the frequency point \( m^k \omega_o \) is bounded by:

\[ H^2(1) - m^{2k} \delta \leq |H(e^{jm^k\omega_o})|^2 \leq H^2(1) \quad \tag{3} \]

Proof: Using mathematical induction, it can be shown that:

\[
\frac{1 - \cos(m^k \omega)}{1 - \cos(\omega)} \leq m^{2k} \tag{3}
\]

(Readers may refer to the addendum in [26] for a detailed proof.)

\[ \therefore h(n) > 0 : \tilde{h}(n) = h(n) * h(-n) \geq 0 \]

Since \( h(n) \) is real, we have:

\[ h(n) * h^*(-n) = h(n) * h(-n) = \tilde{h}(n) \]

Therefore

\[ |H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = \tilde{h}(n) * h(-n) = \tilde{h}(n) \]

Since \( |H(e^{j\omega})|^2 \) is real and symmetric about \( \omega = 0 \), we have:

\[ |H(e^{j\omega})|^2 = \sum_{n=-\infty}^{\infty} \tilde{h}(n) e^{-jn\omega} = 2 \sum_{n=0}^{\infty} \tilde{h}(n) \cos(n\omega) \]

Therefore, we have:

\[
H^2(1) - |H(e^{j\omega})|^2 = 2 \sum_{n=0}^{\infty} \tilde{h}(n) (1 - \cos(n\omega)) \\
\leq m^{2k} \cdot 2 \sum_{n=0}^{\infty} \tilde{h}(n) (1 - \cos(n\omega)) \\
= m^{2k} \left( H^2(1) - |H(e^{j\omega})|^2 \right) \quad \tag{4}
\]

Since

\[ H^2(1) - |H(e^{j\omega})|^2 \leq \delta \]

inequality (4) becomes:

\[ H^2(1) - |H(e^{j\omega})|^2 \leq m^{2k} \delta \]

Together with Lemma 1, we have:

\[ H^2(1) - m^{2k} \delta \leq |H(e^{jm^k\omega_o})|^2 \leq H^2(1) \]

Theorem 1 is demonstrated in Fig. 1.

Obviously, the results in [13] are a special case (\( m = 2 \)) of Theorem 1.

Corollary 1: If the impulse response of a filter is non-negative, i.e.,

\[ h(n) \geq 0, \ n \in \mathbb{Z} \]

and it satisfies:

\[ |H(e^{j\omega})|^2 = H^2(1) \]
Substituting $m$ as well, i.e.:

$IIR$ multi-band evidence filter $H$

Least squares $\omega$

Component polyphase structure $H$

have:

$\left|H \left(e^{jm\omega_o}\right)\right|^2 = H^2 (1)$

Proof: From (4) in the proof of Theorem 1, we know that:

$H^2 (1) - \left|H \left(e^{jm\omega_o}\right)\right|^2 \leq m^2 k \left(H^2 (1) - \left|H \left(e^{j\omega_o}\right)\right|^2\right)$

$= 0$

Therefore, multiple upper-bounds can be obtained depending on the choice of $\omega_o$. Theorem 3 sheds some light on the relationships between these upper-bounds. It is convenient to present Theorem 2 first, which Theorem 3 is based on.

Theorem 2: If the impulse response is non-negative, i.e.,

$h \left(n\right) \geq 0, \ n \in \mathbb{Z}$

and the attenuation at the frequency point $m\omega_o$ is bounded by:

$H^2 (1) - \delta \leq \left|H \left(e^{jm\omega_o}\right)\right|^2 \leq H^2 (1)$

where $\delta > 0, \omega_o \in \left(0, \frac{\pi}{m+\delta}\right], m, k \in \mathbb{Z}^+$, then the change of attenuation from frequency $m\omega_o$ to frequency $(m+k)\omega_o$ is bounded by:

$\left|H \left(e^{jm\omega_o}\right)\right|^2 - \left|H \left(e^{j(m+k)\omega_o}\right)\right|^2 \leq \frac{k^2 + 2mk}{m^2} \delta$

Proof: To find the relationship between the attenuations in the two frequency regions, consider the function $f(\omega)$:

$f \left(\omega\right) \triangleq \frac{\cos \left(m\omega\right) - \cos \left((m+k)\omega\right)}{1 - \cos \left(m\omega\right)}$

Then

$p \triangleq \frac{d}{d\omega} f \left(\omega\right) \bigg|_{\omega=\omega_o} = \frac{A}{\left(1 - \cos \left(m\omega_o\right)\right)^2}$
Inequality (7) indicates that \( f(\omega_o) \) is monotonically decreasing in \( (0, \frac{\pi}{k+m}] \). Therefore, a maximum is obtained when \( \omega_o \to 0 \).

\[
\max_{\omega_o \in (0, \frac{\pi}{k+m}]} f(\omega_o) = \lim_{\omega_o \to 0} f(\omega_o) = \frac{k^2 + 2mk}{m^2} \quad (8)
\]

From (8) we have:

\[
|H(e^{j\omega_o})|^2 - |H(e^{j(m+k)\omega_o})|^2 = 2 \sum_{n=0}^{\infty} \hat{h}(n) \cos(nm\omega_o - (n+m+k)\omega_o)) \leq \frac{k^2 + 2mk}{m^2} \cdot 2 \sum_{n=0}^{\infty} \hat{h}(n) (1 - \cos(nm\omega_o)) \leq \frac{k^2 + 2mk}{m^2} \left(H^2(1) - |H(e^{j\omega_o})|^2 \right) \leq \frac{k^2 + 2mk}{m^2} \delta \quad (9)
\]

Now we are in position to present Theorem 3.

Theorem 3 shows that the largest such \( \omega_i \) results in the tightest bound for the attenuation at \( \omega_x \).

**Theorem 3:** Assume \( \omega_x \) is given and

\[ m_i^k \omega_i = \omega_x \quad m_i, k_i \in \mathbb{Z}^+, \quad i = 0, 1, 2, \ldots \]

and

\[ \omega_o > \omega_1 > \omega_2 > \ldots > \omega_i > \ldots \]

then

\[ \Delta_o = \min_i \Delta_i \]

where

\[ \Delta_i = m_i^{2k_i} \left(H^2(1) - |H(e^{j\omega_i})|^2 \right) \]

**Proof:** See appendix.

**Comment:**

Theorem 3 reveals the relationship among the magnitude response at frequencies \( m_i^k \omega_i \) \((\forall m_i, k_i \in \mathbb{Z}^+)\). It shows that among all \( \omega_i \)'s satisfying \( \omega_x = m_i^{k_i} \omega_i \), the amount of attenuation at \( \omega_x \) is most tightly bounded by the \( \omega_i \) closest to the frequency of interest \( \omega_x \). The smallest possible attenuation ratio is given by \( \frac{\omega_o}{\omega_x} = 2 \).

**Example:** Fig. 3(a) shows the magnitude response of a \( 30^{th} \)-order half-band evidence filter, designed using the method in [16]. It is noticed that the magnitude response is below the 0dB line, as described by the result in Lemma 1.

The results in Theorem 1 and Theorem 3 are best demonstrated by the magnitude response at frequencies near the origin. Consider \( \omega_0 = 0.03\pi, \omega_o = 0.015\pi, \omega_1 = 0.01\pi \) now. As shown by the magnitude response at frequencies near the origin in Fig.3(b), the attenuations at \( \omega_x, \omega_o \) and \( \omega_1 \) are observed to be:

\[
\begin{align*}
\Delta_x &= 1 - |H(e^{j\omega_x})|^2 = 0.2488 \\
\Delta_o &= 1 - |H(e^{j\omega_o})|^2 = 0.0697 \\
\Delta_1 &= 1 - |H(e^{j\omega_1})|^2 = 0.0317,
\end{align*}
\]

respectively. Since

\[ \omega_x = 3\omega_1 = 2\omega_o, \]

From Theorem 1, we obtain two upper-bounds \( B_{x0} \) and \( B_{x1} \) for the attenuation at \( \omega_x \) using \( \omega_o \) and \( \omega_1 \) as the “reference frequency” respectively:

\[ B_{x0} = 2^2 \cdot \Delta_o = 0.2788 \quad B_{x1} = 3^2 \cdot \Delta_1 = 0.2853 \]

Observing \( \Delta_x, B_{x0}, \) and \( B_{x1} \), we find:

\[ \Delta_x < B_{x0}, \Delta_x < B_{x1}. \]

Therefore, the result in Theorem 1 is verified. Then comparing \( B_{x0} \) and \( B_{x1} \), we have:

\[ B_{x0} < B_{x1}, \]
which indicates that $B_{x_0}$ is tighter, as is described by the result in Theorem 3.

In contrast, the magnitude response of a conventional filter is not constrained by the above results at any frequency [28]. To illustrate this, magnitude responses of half-band Butterworth filters of different orders are shown in Fig. 3(c). Observe that a drop in the gain near frequency zero from 0dB is not required in order to produce a steep gain drop at higher frequencies.

**Theorem 4:** If the impulse response is non-negative, i.e.,

$$h(n) \geq 0, \quad n \in \mathbb{Z}$$

and the attenuation at the frequency point $\omega_o$ is bounded by:

$$H^2(1) - \delta \leq |H(e^{j\omega_o})|^2 \leq H^2(1)$$

where $\delta > 0$, $\omega_o \in (0, \pi]$, then the change of attenuation or gain from frequency $m \omega_o$ to $(m + k) \omega_o$ is bounded by:

$$|H(e^{j(m \omega_o)})|^2 - |H(e^{j((m+k) \omega_o)})|^2 \leq (k^2 + 2mk) \delta$$

where $m, k \in \mathbb{Z}^+, m \geq 2$.

**Proof:** See appendix.

If one intends to apply Theorem 4 to estimate the change of attenuation/gain from frequency $\omega_x$ to frequency $\omega_y$ ($\omega_x, \omega_y \in (0, \pi]$), one has infinitely many choices. By Theorem 4, these choices ($\omega_i$) are:

$$l_i \omega_i = \omega_x, \quad (l_i + r_i) \omega_i = \omega_y$$

where $l_i, r_i \in \mathbb{Z}^+, i = 0, 1, 2, \ldots$. Corollary 2 shows that the largest such $\omega_i$ results in the tightest bound for the change of attenuation/gain from $\omega_x$ to $\omega_y$.

**Corollary 2:** Assume $[\omega_x, \omega_y]$ is given and

$$l_i \omega_i = \omega_x, \quad (l_i + r_i) \omega_i = \omega_y$$

$$l_i, r_i \in \mathbb{Z}^+, i = 0, 1, 2, \ldots$$

and

$$\omega_0 > \omega_1 > \omega_2 \cdots > \omega_i > \ldots$$

Let $\Delta_i$ denote the upper-bound on the change of attenuation/gain from frequency $\omega_x$ to frequency $\omega_y$. Then

$$\Delta_o = \min_i \Delta_i$$

where

$$\Delta_i = (r_i^2 + 2l_i r_i) \left( H^2(1) - |H(e^{j\omega_i})|^2 \right)$$

**Proof:** See appendix.

**Comment:**

Corollary 2 reveals the relationship between the change of attenuation within the frequency region of interest $[\omega_x, \omega_y]$. It shows that among all the $\omega_i$’s satisfying $\omega_x = l_i \omega_i$, $\omega_y = (l_i + r_i) \omega_i$ ($\forall m_i, l_i, r_i \in \mathbb{Z}^+$), the amount of the change of attenuation from $\omega_x$ to $\omega_y$ is most tightly bounded by the $\omega_i$ closest to $\omega_x$. The smallest possible attenuation ratio is given by $\omega_o = \gcd(\omega_x, \omega_y)$.

**Example:** The results in Theorem 2, Theorem 4 and Corollary 2 can be illustrated using the same half-band evidence filter given in Fig. 3(a).
Consider \( \omega_x = 0.04\pi, \omega_y = 0.05\pi, \omega_o = 0.01\pi, \) and \( \omega_1 = 0.005\pi. \) As shown in Fig. 4, the attenuations at \( \omega_o \) and \( \omega_1 \) are observed to be:

\[
\Delta_o = 1 - |H(e^{j\omega_o})|^2 = 0.0317 \\
\Delta_1 = 1 - |H(e^{j\omega_1})|^2 = 0.008 \\
\Delta_x = 1 - |H(e^{j\omega_x})|^2 = 0.3945 \\
\Delta_y = 1 - |H(e^{j\omega_y})|^2 = 0.5361.
\]

The attenuation within the transition band \( [\omega_x, \omega_y] \) is observed to be:

\[
\Delta_{xy} = |H(e^{j\omega_x})|^2 - |H(e^{j\omega_y})|^2 = 0.1416.
\]

Since 
\[
\omega_x = 4\omega_o, \quad \omega_y = (4 + 1) \omega_o \\
\omega_x = 8\omega_1, \quad \omega_y = (8 + 2) \omega_1,
\]

according to Corollary 2, we obtain two upperbounds \( B_o \) and \( B_1 \) that describe the change of attenuation from \( \omega_x \) to \( \omega_y \), using \( \omega_o \) and \( \omega_1 \) as reference frequency respectively:

\[
B_o = (1^2 + 2 \times 4 \times 1) \cdot \Delta_o = 0.2853 \\
B_1 = (2^2 + 2 \times 8 \times 2) \cdot \Delta_1 = 0.288.
\]

The result in Theorem 4 is illustrated by the fact that:

\[
\Delta_{xy} < B_o, \quad \Delta_{xy} < B_1,
\]

The result in Corollary 2 is illustrated by comparing \( B_o \) and \( B_1 \):

\[
B_o < B_1,
\]

which indicates that bound \( B_o \) is tighter.

According to Theorem 2, we obtain the upperbound \( B_{xy} \) on the change of attenuation from \( \omega_x \) to \( \omega_y \), using \( \omega_o \) as the reference frequency:

\[
B_{xy} = \frac{1^2 + 2 \times 4 \times 1}{4^2} \Delta_x = 0.2219
\]

Observing \( B_{xy} \) and \( \Delta_{xy} \), we find:

\[
\Delta_{xy} < B_{xy},
\]

which illustrates the result in Theorem 2.

In contrast, the frequency response of a conventional NNIR filter is not constrained in this way [28].

### III. RESULTS FOR CONTINUOUS-TIME SYSTEMS

In section II, the developed results are reported for a discrete-time system. In this section, we show that all the proceeding results also apply to a continuous-time system.

**Theorem 5:** If the impulse response of a filter is non-negative, i.e.,

\[
h(t) \geq 0
\]

and the attenuation at the frequency point \( \Omega_o \) is bounded by:

\[
H^2(0) - \delta \leq |H(j\Omega_o)|^2 \leq H^2(0)
\]

where \( \delta > 0, \Omega_o > 0 \), then the attenuation at the frequency point \( m^k\Omega_o \) is bounded by:

\[
H^2(0) - m^{2k}\delta \leq |H(jm^k\Omega_o)|^2 \leq H^2(0)
\]

where \( m, k \in \mathbb{Z}^+ \)

**Proof:** Similar to inequality (3) in Theorem 1, it can be proven that:

\[
\frac{1 - \cos(m^k\Omega)}{1 - \cos(\Omega)} \leq m^{2k}
\]

Then:

\[
\therefore h(t) \geq 0 \quad \therefore \tilde{h}(t) \triangleq h(t) * h(-t) \geq 0
\]

Since \( h(t) \) is real, we have:

\[
H^*(j\Omega) = H(-j\Omega)
\]

Therefore

\[
|H(j\Omega)|^2 = H(j\Omega) H^*(j\Omega) = H(j\Omega) H(-j\Omega)
\]

\[
\therefore h(t) * h(-t) = \tilde{h}(t)
\]

Since \( |H(j\Omega)|^2 \) is real and symmetric about \( \Omega = 0 \), we have:

\[
|H(j\Omega)|^2 = \int_{-\infty}^{\infty} \tilde{h}(\tau) e^{-j\Omega \tau} d\tau = 2 \int_{0}^{\infty} \tilde{h}(\tau) \cos(\Omega \tau) d\tau
\]

Therefore, we have:

\[
H^2(0) - |H(jm^k\Omega_o)|^2 = 2 \int_{0}^{\infty} \tilde{h}(\tau) (1 - \cos(m^k\Omega_o \tau)) d\tau \\
\leq m^{2k} \cdot 2 \int_{0}^{\infty} \tilde{h}(\tau) (1 - \cos(\Omega_o \tau)) d\tau \\
= m^{2k} \left( H^2(0) - |H(j\Omega_o)|^2 \right)
\]

(10)

Since

\[
H^2(0) - |H(j\Omega_o)|^2 \leq \delta
\]

inequality (10) becomes:

\[
H^2(0) - |H(jm^k\Omega_o)|^2 \leq m^{2k}\delta
\]

(11)
Similar to Lemma 1, it can be readily proven that:

$$|H(j\Omega)|^2 \leq H^2(0)$$

(12)

Combining (11) and (12), we have:

$$H^2(0) - m^{2k\delta} \leq |H(jm^k\Omega_o)|^2 \leq H^2(0)$$

IV. DUALITY

Since both the auto-correlation sequence \( \hat{h}(n) \) of impulse response \( h(n) \) and the power spectrum samples \( |H(2\pi k/N)|^2 \), \( k = 0, 1, \ldots, N - 1, N \in \mathbb{Z}^+ \), are non-negative real, we have the following Discrete Fourier Transform (DFT) pair:

$$|H(k)|^2 = \left| H\left(\frac{2\pi k}{N}\right) \right|^2 = \sum_{k=0}^{N-1} \hat{h}(n) \cos(2\pi kn/N),$$

\( k = 0, 1, \ldots, N - 1 \)

(13)

and

$$\hat{h}(n) = \frac{1}{N} \sum_{k=0}^{N-1} H\left(\frac{2\pi k}{N}\right)^2 \cos(2\pi kn/N)$$

\( k = 0, 1, \ldots, N - 1 \)

(14)

Similarly, in the continuous-time domain, we have the Fourier Transform pair:

$$|H(j\Omega)|^2 = \int_{-\infty}^{\infty} \hat{h}(t) \cos(\Omega t) \, dt$$

(15)

and

$$\hat{h}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\Omega)|^2 \cos(\Omega t) \, d\Omega$$

(16)

where \( \hat{h}(t) \) is the auto-correlation function of the impulse response \( h(t) \).

Due to the dualities between (13) and (14), (15) and (16), respectively, dual bounds on the attenuation/gain of the auto-correlation function exist.

V. CONCLUSION

In this paper we analyze the limitations in shaping the frequency response of linear filters, if a non-negativity condition is imposed on the impulse response. The results show that in contrast to classical linear filter and system design [29], non-negative impulse response filters are severely constrained in the system frequency response. Among other relationships it is shown that the gain drop near zero frequency has implications throughout the entire frequency spectrum. Unlike in classical filter design, increasing the filter order does not necessarily allow for sharper passbands, higher attenuation in the stopband, or a narrower transition bandwidth. While passbands and stopbands can be placed arbitrarily in the frequency spectrum, the achievable gains in both are severely limited. Some of these fundamental limiting relationships have been derived for discrete time systems and a brief extension to the continuous time case is offered. Results on duality between time and frequency response are also offered.

PROOF OF THEOREM 3:

Proof: Choose any two frequency points \( \omega_p, \omega_q \), such that

$$\omega_x = m^k \omega_p = m^k \omega_q = m^k \omega_o$$

(17)

where \( k_p, m_p, k_q, m_q \in \mathbb{Z}^+ \).

Let \( \delta_p, \delta_q \) be the amount of attenuation at \( \omega_p \) and \( \omega_q \) respectively, i.e.:

$$\delta_p = H^2(1) - |H(e^{j\omega_p})|^2 \geq 0$$

$$\delta_q = H^2(1) - |H(e^{j\omega_q})|^2 \geq 0$$

Let \( \Delta_p, \Delta_q \) be the upper-bound on the attenuation at \( \omega_x \) when \( \omega_p \) and \( \omega_q \) are chosen to compute the bound at \( \omega_x \) respectively. From Theorem 1, we have:

$$\Delta_p = m_p^{2k_p} \delta_p, \quad \Delta_q = m_q^{2k_q} \delta_q$$

Now we will prove that:

If

$$\omega_p < \omega_q$$

then

$$\Delta_p > \Delta_q.$$
And since
\[
\frac{|H (e^{j\omega_p})|^2 - |H (e^{j\omega_q})|^2}{H^2 (1) - |H (e^{j\omega_q})|^2} = \left[ H^2 (1) - |H (e^{j\omega_q})|^2 \right] \frac{\omega_p^2}{\omega_q^2} = \frac{\omega_p^2}{\omega_q^2} - 1
\]
we have:
\[
\frac{\omega_p^2}{\omega_q^2} - 1 \leq \frac{r^2 + 2lr}{l^2} + 1 = \frac{(l + r)^2}{l^2}
\]
\[
\Rightarrow \frac{\omega_p^2}{\omega_q^2} \leq \frac{r^2 + 2lr}{l^2} + 1 = \frac{(l + r)^2}{l^2}
\]

That is \( \frac{\omega_p^2}{\omega_q^2} \leq \frac{r^2 + 2lr}{l^2} \) has to be satisfied. However, this contradicts (18). Therefore, it has to be true that \( \Delta_q \leq \Delta_p \).

Since \( \omega_p, \omega_q \) are arbitrarily-chosen frequency points that satisfy (17), it can be concluded that the frequency point \( \omega_o \) delivers the tightest upper-bound \( \Delta_o \) on the attenuation at \( \omega_x \).

Proof of Theorem 1:

\[ \Delta_o = m_2^2 \omega_o \left( H^2 (1) - |H (e^{j\omega_o})|^2 \right) \]

Proof of Theorem 4:

Proof: Here only the proof for the attenuation case is given. The gain case can be proven similarly.

Choose any two frequency points \( \omega_p, \omega_q, p, q \in \mathbb{Z}^+ \), \( \omega_p, \omega_q \in [0, \omega_o], \omega_p < \omega_q \), such that
\[
\begin{align*}
l_p \omega_p &= l_q \omega_q = \omega_x, \\
(l_p + r_p) \omega_p &= (l_q + r_q) \omega_q = \omega_y
\end{align*}
\]
where \( l_p, r_p, l_q, r_q \in \mathbb{Z}^+ \).

Let \( \delta_p, \delta_q \) be the amount of attenuation at \( \omega_p \) and \( \omega_q \) respectively, i.e.:
\[
\begin{align*}
\delta_p &= H^2 (1) - |H (e^{j\omega_p})|^2 \\
\delta_q &= H^2 (1) - |H (e^{j\omega_q})|^2
\end{align*}
\]

Let \( \Delta_p, \Delta_q \) be the upper-bound on the attenuation from \( \omega_x \) to \( \omega_y \) when \( \omega_p \) and \( \omega_q \) are chosen to compute the bound respectively.

From Theorem 4, we have:
\[
\begin{align*}
\Delta_p &= (r_p^2 + 2l_pr_p) \delta_p \\
\Delta_q &= (r_q^2 + 2l_qr_q) \delta_q
\end{align*}
\]

Now we will prove that:
If
\[ \omega_p < \omega_q \]
then
\[ \Delta_p > \Delta_q \]

Again we choose a proof by contradiction.
Suppose $\Delta_q > \Delta_p$, then we have:

$$
\frac{\omega_q^2}{\omega_p^2} > \frac{(\omega_q^2 + 2l_q r_q) \delta_q}{(\omega_p^2 + 2l_p r_p) \delta_p} \Rightarrow \frac{\delta_q}{\delta_p} > \frac{(\omega_p^2 + 2l_p r_p)}{(\omega_q^2 + 2l_q r_q)} = \frac{(l_p + r_p)^2 - l_p^2}{(l_q + r_q)^2 - l_q^2} = \frac{\omega_p^2}{\omega_q^2} \cdot \frac{\omega_q^2}{\omega_p^2}
$$

From (20), we have:

$$
l_p \omega_p = l_q \omega_q, \quad (l_p + r_p) \omega_p = (l_q + r_q) \omega_q
$$

Therefore, (21) becomes:

$$
\frac{\omega_q}{\omega_p} > \frac{\delta_q}{\delta_p}
$$

(22)

Since $\frac{\omega_q}{\omega_p}$ is rational, it is always possible to choose $\tilde{\omega}$ and $s, t$ such that:

$$
\omega_p = s \tilde{\omega}, \quad \omega_q = (s + t) \tilde{\omega}, \quad s, t \in \mathbb{Z}^+, \quad \tilde{\omega} \in [0, \omega_p]
$$

Since the frequency response for $(0, \omega_p)$ is known or specified, it has to satisfy the necessary condition provided in Theorem 2. From (9) in the proof of Theorem 2, we have:

$$
\frac{|H(e^{j\omega_p})|^2 - |H(e^{j\omega_q})|^2}{H^2(1) - |H(e^{j\omega_p})|^2} = \frac{H^2(1) - |H(e^{j\omega_q})|^2}{H^2(1) - |H(e^{j\omega_p})|^2}
$$

And since

$$
\frac{|H(e^{j\omega_p})|^2 - |H(e^{j\omega_q})|^2}{H^2(1) - |H(e^{j\omega_q})|^2} = \frac{H^2(1) - |H(e^{j\omega_q})|^2}{H^2(1) - |H(e^{j\omega_p})|^2} = \frac{\delta_q}{\delta_p} = \frac{\delta_q}{\delta_p} - 1
$$

we have:

$$
\frac{\delta_q}{\delta_p} - 1 \leq \frac{t^2 + 2st}{s^2}
$$

That is, $\frac{\delta_q}{\delta_p} \leq \frac{\omega_q}{\omega_p}$ has to be satisfied. However, this contradicts (22). Therefore, it has to be true that $\Delta_q \leq \Delta_p$.

Since $\omega_p, \omega_q$ are arbitrarily-chosen frequency points that satisfy (20), it can be concluded that when the boundary point $\omega_x$ is chosen, the obtained upper-bound on the attenuation from $\omega_x$ to $\omega_y$ is tightest.

It should be noted that when $[\omega_x, \omega_y]$ is a geometrically spaced frequency region, Corollary 2 also holds.


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