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Exceptional Stewart–Gough Platforms, Segre Embeddings, and the Special Euclidean Group*

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Abstract. Stewart–Gough platforms are mechanisms which consist of two rigid objects, a base and a platform, connected by six legs via spherical joints. For fixed leg lengths, a generic Stewart–Gough platform is rigid with 40 assembly configurations (over the complex numbers), while exceptional Stewart–Gough platforms have infinitely many assembly configurations and thus have self-motion. We define a family of exceptional Stewart–Gough platforms called Segre-dependent Stewart–Gough platforms which arise from a linear dependency of point-pairs under the Segre embedding and compute an irreducible decomposition of this family. We also consider Stewart–Gough platforms which move with two degrees of freedom. Since the Segre embedding arises from a representation of the special Euclidean group in three dimensions which has degree 40, we consider the special Euclidean group in other dimensions and compute spatial Stewart–Gough platforms that move in 4-dimensional space.

Key words. Stewart–Gough platform, architecturally singular, Segre embedding, special Euclidean group, numerical algebraic geometry

AMS subject classifications. Primary, 65H10; Secondary, 13P05, 14Q99, 68W30

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1. Introduction.

A Stewart–Gough platform consists of rigid base and platform objects connected by six legs via spherical (i.e., ball-and-socket) joints. Gough utilized one in the 1950's [20] to mechanically test tires, while Stewart [50] devised a related kinematical arrangement in the 1960's for use as a flight simulator. There is extensive literature on Stewart–Gough platforms and other parallel-link robots: we refer the interested reader to the book [34] and websites [10, 35].

Mathematically, we will model a Stewart–Gough platform by fixing a coordinate system for the base $\mathcal{B}$ and a coordinate system for the platform $\mathcal{P}$. Then we select six connection points $b_1, \ldots, b_6$ on the base object with respect to $\mathcal{B}$ and $p_1, \ldots, p_6$ on the platform object with respect to $\mathcal{P}$. The points b_i and p_i are connected by the ith leg, which has length $d_i \geq 0$, with spherical joints, as shown in Figure 1.1. In normal operation, the six leg lengths are adjusted under computer control to produce coordinated motion of the platform with respect to the base. In most situations, the mapping between the six leg lengths and the 6-dimensional

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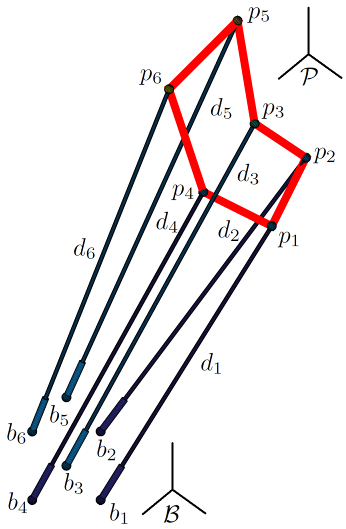


Figure 1.1. *Stewart–Gough platform.*

space of rigid-body motion is nonsingular, and hence locally invertible. In particular, when the leg lengths are held constant, the mechanism becomes rigid.

We shall refer to the position and orientation of the platform with respect to the base as its *pose*. Given a pose, the relative positions of points $b_1, \ldots, b_6$ and $p_1, \ldots, p_6$ are known, so the leg lengths $d_1, \ldots, d_6$ are also known. However, given the leg lengths, the pose may not be unique. For generic choices of the parameters b_i, p_i, and d_i, the polynomial system relating the pose to the leg lengths has 40 solutions, allowing complex numbers [29, 33, 43, 44, 51]. Each real solution among these represents a physically achievable way to assemble the Stewart–Gough platform. Dietmaier [15] showed that parameters exist such that all 40 assembly configurations are real.

Since a generic Stewart–Gough platform is rigid, the location of the platform object with respect to the base can only continuously change by adjusting the leg lengths. Exceptional Stewart–Gough platforms are those which have infinitely many different assembly configurations and thus have self-motion, i.e., move even with fixed leg lengths. Some known exceptional platforms are special cases of Griffis–Duffy platforms [22, 30] and Geiss–Scheyer platforms [19], a special case of Borel's Fb1 family of icosapods [12], as shown in [18]. A complete classification of all exceptional Stewart–Gough platforms remains an open problem.

A subset of the exceptional Stewart–Gough platforms is called *architecturally singular*. These have the property that their exceptional motion is determined by just the geometry of the base and platform. If one chooses the leg lengths corresponding to any pose of the platform, the architecturally singular platform still moves with at least one degree of freedom. This may be contrasted with more general families of exceptional platforms for which only special poses of the platform result in motion. The classification of all architecturally singular Stewart–Gough platforms has been carried out by Karger [31, 32] (see also [38]).

In this paper, we define a subfamily of exceptional Stewart–Gough platforms, called *Segre-dependent Stewart–Gough platforms*, which arise from a linear dependence among point-pairs under the Segre embedding related to a representation of the special Euclidean group in three dimensions. More specifically, the Segre-dependent platforms are a subfamily of the architecturally singular platforms. One motivation for studying Segre-dependent Stewart–Gough platforms is to highlight the algebraic geometric relationship between the Segre embedding and Stewart–Gough platforms. Another motivation is this provides for new computational approaches to compute exceptional mechanisms. A third motivation is to understand the relationship between the generic number of assembly configurations and the degree of the corresponding group, as discussed in section 5.

The remainder of the paper is organized as follows. In section 2, we formulate the assembly of Stewart–Gough platforms as a polynomial system and observe the presence of point-pairs under the Segre embedding and the special Euclidean group in three dimensions. Section 3 defines Segre-dependent mechanisms and describes using numerical algebraic geometry (e.g., see [8, 49]) to compute them. Section 4 presents our computational results for Segre-dependent mechanisms, including platforms with two degrees of freedom and spatial platforms that move in 4-dimensional space. The special Euclidean group in other dimensions is considered in section 5 with Theorem 5.1 relating the degree of the special Euclidean group with the degree of the special orthogonal group. A conclusion is provided in section 6. Information on the computations can be downloaded from the repository at https://doi.org/10.7274/R0R20Z94.

2. Polynomial system formulation.

Following the formulations of [17, 46] with a slight modification, we present a polynomial system for the assembly of Stewart–Gough platforms in which each leg imposes a linear condition. Each assembly configuration can be specified as the relative position and orientation of the platform coordinate system $\mathcal{P}$ with respect to the base coordinate system $\mathcal{B}$. That is, each assembly configuration of the Stewart–Gough platform corresponds to an element of the special Euclidean group SE(3) consisting of rotations and translations in $\mathbb{R}^3$.

Each element in SE(3) will be represented by a matrix $M \in \mathbb{R}^{3\times 3}$, two vectors $x, y \in \mathbb{R}^3$, and a scalar $r \in \mathbb{R}$. The matrix M represents the relative rotation from $\mathcal{B}$ to $\mathcal{P}$ so that $M \in \mathrm{SO}(3)$, i.e., $M^T = M^{-1}$ and $\det M = 1$. The vector y represents translations from $\mathcal{B}$ to $\mathcal{P}$. Thus, for $b \in \mathcal{B}$, the corresponding point in $\mathcal{P}$ is $p = Mb + y$. This representation also accounts for the map from $\mathcal{P}$ to $\mathcal{B}$ which is given by M^T and $x = -M^T y$ so that $y = -Mx$. That is, the corresponding point to $p \in \mathcal{P}$ is $b = M^T p + x \in \mathcal{B}$ so that $M(M^T p + x) + y = p$ and $M^T(Mb + y) + x = b$. In particular, we must have $x^T x = y^T y$. We depart from the formulation provided in [17, 46] by defining $r = -x^T x/2 = -y^T y/2$ so that $x^T x = y^T y = -2r$, which is selected to simplify computations later.

Putting everything together, SE(3) is represented by the real solution set of the polynomial system

$$f(r, x, y, M) = \begin{bmatrix} M^T M - I \\ M M^T - I \\ y + Mx \\ x + M^T y \\ 2r + x^T x \\ 2r + y^T y \\ \det M - 1 \end{bmatrix}, \tag{2.1}$$

where I is the 3×3 identity matrix. By treating the variables in $\mathbb{C}^{16}$, the solution set of $f=0$, denoted by $\mathcal{SE}_3 \subset \mathbb{C}^{16}$, is irreducible with $\dim \mathcal{SE}_3 = 6$ and $\deg \mathcal{SE}_3 = 40$. In particular, using this representation of SE(3), the algebraic set $\mathcal{SE}_3$ is the Zariski closure in $\mathbb{C}^{16}$ of SE(3), and SE(3) is the real subset of $\mathcal{SE}_3$, i.e., $\mathrm{SE}(3) = \mathcal{SE}_3 \cap \mathbb{R}^{16}$.

The distance constraint imposed by each leg corresponds to the intersection of $\mathcal{SE}_3$ with a hyperplane as follows. Since the ith leg connects b_i with p_i, we can measure distance by placing them in a common coordinate system. Since $b_i \in \mathcal{B}$ corresponds with $Mb_i + y \in \mathcal{P}$, the leg length condition

$$(Mb_i + y - p_i)^T (Mb_i + y - p_i) = d_i^2$$

on $\mathcal{SE}_3$ is equivalent to

$$\begin{aligned} d_i^2 &= b_i^T b_i + p_i^T p_i + 2b_i^T M^T y - 2p_i^T y - 2p_i^T M b_i + y^T y \\ &= b_i^T b_i + p_i^T p_i - 2(b_i^T x + p_i^T y + p_i^T M b_i + r). \end{aligned} \tag{2.2}$$

Let $\ell_i = (b_i^T b_i + p_i^T p_i - d_i^2)/2$. Hence, if points b_i and p_i are given, then knowing the leg length $d_i \geq 0$ is equivalent to knowing ℓ_i. With this, (2.2) becomes

$$r + b_i^T x + p_i^T y + p_i^T M b_i = \ell_i, \tag{2.3}$$

which is a linear equation in $(r, x, y, M) \in \mathcal{SE}_3$. In fact, the set of coefficients on the left-hand side of (2.3) is the Segre embedding of the point-pairs b_i and p_i, where the Segre embedding $\sigma : \mathbb{C}^3 \times \mathbb{C}^3 \hookrightarrow \mathbb{C}^{16}$ is

(2.4)

$$\sigma(\alpha,\beta) = [\, 1 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta_1 \quad \beta_2 \quad \beta_3 \quad \alpha_1\beta_1 \quad \alpha_2\beta_1 \quad \alpha_3\beta_1 \quad \alpha_1\beta_2 \quad \alpha_2\beta_2 \quad \alpha_3\beta_2 \quad \alpha_1\beta_3 \quad \alpha_2\beta_3 \quad \alpha_3\beta_3 \,].$$

Suppose that we are given parameters b_i, p_i, and ℓ_i for a Stewart–Gough platform. Let $\mathcal{L}_6 \subset \mathbb{C}^{16}$ denote the linear space arising from the intersection of the six hyperplanes defined by (2.3). Then assembling the corresponding Stewart–Gough platform is equivalent to computing $\mathcal{SE}_3 \cap \mathcal{L}_6$, which generically consists of $\deg \mathcal{SE}_3 = 40$ points.

As an illustration of this formulation, the following examples show that one is able to design mechanisms with certain properties or show that none exists.

Example 2.1. To show that there does not exist an exceptional Stewart–Gough platform such that the motion is a linear translation, we can, without loss of generality, consider all mechanisms which move along the curve $\{(-z^2/2, -z\mathbf{e}_1, z\mathbf{e}_1, I) \mid z \in \mathbb{C}\} \subset \mathcal{SE}_3$, where $\mathbf{e}_1 = [1\ 0\ 0]^T$. Thus, to reach a contradiction, we assume that there exist constants b_i, p_i, and ℓ_i such that

$$-z^2/2 + (p_{i1} - b_{i1})z + p_i^T b_i = \ell_i$$

for all $z \in \mathbb{C}$. This is impossible since the coefficient of z^2 is nonzero.

Example 2.2. There exists exceptional Stewart–Gough platforms such that the platform moves along a circle keeping the same relative rotation with respect to the base [40, Thm. 2]. For example, to move along a unit circle in the (y_1, y_2)-plane, we consider the curve $C = \{(-1/2, -y, y, I) \mid y_1^2 + y_2^2 = 1, y_3 = 0\} \subset \mathcal{SE}_3$. We aim to find parameters b_i, p_i, and ℓ_i such that

$$-1/2 + (p_{i1} - b_{i1})y_1 + (p_{i2} - b_{i2})y_2 + p_i^T b_i = \ell_i$$

whenever $y_1^2 + y_2^2 = 1$. Thus, we need $p_{i1} = b_{i1}$, $p_{i2} = b_{i2}$, and $\ell_i = p_i^T b_i - 1/2$. These parameters are physically meaningful with

$$d_i = \sqrt{1 + (b_{i3} - p_{i3})^2} > 0.$$

Thus, the family of exceptional Stewart–Gough platforms which can move along the curve C is

$$\left\{(b_1, p_1, \ell_1, \ldots, b_6, p_6, \ell_6) \mid b_{ij} = p_{ij} \text{ and } \ell_i = p_i^T b_i - 1/2 \text{ for } i = 1, \ldots, 6, j = 1, 2\right\},$$

one of which is shown in Figure 2.1.

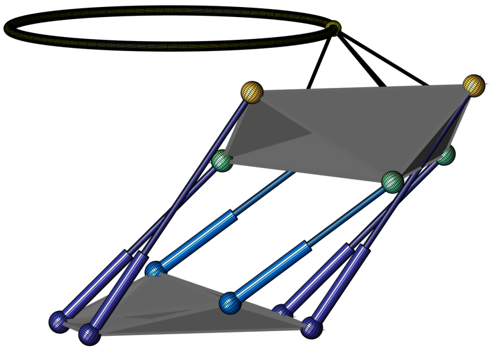


Figure 2.1. *A Stewart–Gough platform with motion along a unit circle.*

3. Segre-dependent Stewart–Gough platforms.

3.1. Definition of Segre-dependent. Given parameters b_i, p_i, and ℓ_i, the six hyperplanes defined by (2.3) generically define a codimension 6 linear space $\mathcal{L}_6 \subset \mathbb{C}^{16}$. If the parameters are selected so that $\operatorname{codim} \mathcal{L}_6 < 6$ and $\mathcal{SE}_3 \cap \mathcal{L}_6 \neq \emptyset$, then we trivially have that the corresponding Stewart–Gough platform has a rank deficient Jacobian matrix. Therefore, we say that b_i and p_i form a *Segre-dependent Stewart–Gough platform* if, for σ as in (2.4),

$$\operatorname{rank} \begin{bmatrix} \sigma(b_1, p_1) \\ \vdots \\ \sigma(b_6, p_6) \end{bmatrix} < 6. \tag{3.1}$$

For any such $(b_1, p_1, \ldots, b_6, p_6)$ that is real, we may pick any point in SE(3) and generate a compatible real set of leg lengths d_i using (2.2). Since the resulting Stewart–Gough platform has a rank deficient Jacobian, every Segre-dependent Stewart–Gough platform is architecturally singular. For architecturally singular Stewart–Gough platforms that are not Segre-dependent, see Remark 4.11, [38, Cor. 1], and [45, Thm. 4.1].

Remark 3.1. Generically, every 6×6 submatrix of the the 6×16 matrix arising from the Segre embedding σ in (2.4) has rank 6. In order to be Segre-dependent, every 6×6 submatrix must have rank at most 5. For a general mechanism from the Geiss–Schreyer family of exceptional mechanisms [19], precisely $\binom{8}{6} = 28$ of the 6×6 submatrices have rank 5 arising from the following 6×8 submatrix having rank 5:

$$\begin{bmatrix} b_{12} & b_{13} & b_{12}p_{11} & b_{13}p_{11} & b_{12}p_{12} & b_{13}p_{12} & b_{12}p_{13} & b_{13}p_{13} \\ b_{22} & b_{23} & b_{22}p_{21} & b_{23}p_{21} & b_{22}p_{22} & b_{23}p_{22} & b_{22}p_{23} & b_{23}p_{23} \\ b_{32} & b_{33} & b_{32}p_{31} & b_{33}p_{41} & b_{32}p_{32} & b_{33}p_{32} & b_{32}p_{33} & b_{33}p_{33} \\ b_{42} & b_{43} & b_{42}p_{41} & b_{43}p_{41} & b_{42}p_{42} & b_{43}p_{42} & b_{42}p_{43} & b_{43}p_{43} \\ b_{52} & b_{53} & b_{52}p_{51} & b_{53}p_{51} & b_{52}p_{52} & b_{53}p_{52} & b_{52}p_{53} & b_{53}p_{53} \\ b_{62} & b_{63} & b_{62}p_{61} & b_{63}p_{61} & b_{62}p_{62} & b_{63}p_{62} & b_{62}p_{63} & b_{63}p_{63} \end{bmatrix}.$$

Since this mechanism is not architecturally singular and hence not Segre-dependent, the existence of a 6×6 submatrix of rank 6 was assured. Nonetheless, this shows that the location of the points is not in general position with respect to the Segre embedding. Since the Geiss–Schreyer family is a special case of Borel's Fb1 family of icosapods [12], as shown in [18], we note that every 6×6 submatrix generically has rank 6 on Fb1.

We generalize condition (3.1) in the following problem.

Problem 3.2. *For $1 \leq m \leq n$, let $\sigma_{m,n} : \mathbb{C}^m \times \mathbb{C}^n \hookrightarrow \mathbb{C}^{(m+1)(n+1)}$ be the Segre embedding. For all $N \geq 3$ and $2 \leq R < \min\{N, (m+1)(n+1)\}$, compute an irreducible decomposition of the set of points $(b_1, p_1, \ldots, b_N, p_N) \in (\mathbb{C}^m \times \mathbb{C}^n)^N$ such that*

$$\text{rank} \begin{bmatrix} \sigma_{m,n}(b_1, p_1) \\ \vdots \\ \sigma_{m,n}(b_N, p_N) \end{bmatrix} \leq R. \tag{3.2}$$

As a physical interpretation of this problem, the values of m and n correspond to the dimension of the spaces for the base and platform points. The number N is the number of legs so that the matrix in (3.2) has size $N \times (m+1)(n+1)$. The number R is the requested upper bound on the rank which is at most the minimum of the number of rows and columns. For example, $m = 2$, $n = 3$, $N = 6$, and $R = 5$ correspond to Segre-dependent mechanisms with at least one degree of freedom with six legs having a planar base and a spatial platform (or, equivalently, a spatial base and a planar platform).

If $R = 1$, then all points (b_i, p_i) are equal, which physically corresponds with all legs coinciding. In the general case, a matrix has rank at most R if and only if every $(R+1)\times(R+1)$ minor vanishes so that constructing the ideal for this problem is trivial. Computing the geometric irreducible decomposition corresponds with computing the prime decomposition of this ideal generated by such minors. In the following, we describe how to compute such a decomposition for specific instances using numerical algebraic geometry.

3.2. Computing using numerical algebraic geometry. Since the total number of minors, namely $\binom{N}{R+1} \cdot \binom{(m+1)(n+1)}{R+1}$, can be large, we prefer to employ a null space approach [4] to solve relevant cases of Problem 3.2 in section 4. Using the notation of Problem 3.2, we let $A := A(b_1, p_1, \ldots, b_N, p_N)$ be the $N \times (m+1)(n+1)$ matrix in (3.2), and we let $D_\ell := N - R > 0$ and $D_r := (m+1)(n+1) - R > 0$. Thus, rank $A \leq R$ if and only if the left null space of A has dimension at least D_ℓ, which happens if and only if the right null space of A has dimension at least D_r. Let $B_\ell \in \mathbb{C}^{N\times N}$ and $B_r \in \mathbb{C}^{(m+1)(n+1)\times(m+1)(n+1)}$ be general, and let $\Lambda_\ell \in \mathbb{C}^{D_\ell \times R}$ and $\Lambda_r \in \mathbb{C}^{R\times D_r}$ be matrices of indeterminants. With this setup, we have

$$\begin{aligned} &\{(b_1,p_1,\ldots,b_N,p_N) \mid \operatorname{rank} A(b_1,p_1,\ldots,b_N,p_N) \le R\} \\ &= \overline{\left\{(b_1,p_1,\ldots,b_N,p_N) \mid \begin{bmatrix} I & \Lambda_\ell \end{bmatrix} \cdot B_\ell \cdot A(b_1,p_1,\ldots,b_N,p_N) = 0 \text{ for some } \Lambda_\ell\right\}} \\ &= \overline{\left\{(b_1,p_1,\ldots,b_N,p_N) \;\middle|\; A(b_1,p_1,\ldots,b_N,p_N)\cdot B_r \cdot \begin{bmatrix} I \\ \Lambda_r \end{bmatrix} = 0 \text{ for some } \Lambda_r\right\}}. \end{aligned}$$

The choice between the left and right null spaces is based on the relative sizes of N and $(m+1)(n+1)$.

For concreteness, we will formulate the remaining based on using the left null space, say

$$F(b_1,p_1,\ldots,b_N,p_N,\Lambda_\ell) = \begin{bmatrix} I & \Lambda_\ell \end{bmatrix} \cdot B_\ell \cdot A(b_1,p_1,\ldots,b_N,p_N). \tag{3.3}$$

Treating F as a vector and the variables in two groups, $(b_1,p_1,\ldots,b_N,p_N) \in (\mathbb{C}^m \times \mathbb{C}^n)^N$ and $\Lambda_\ell \in \mathbb{C}^{D_\ell\times R}$, the system F consists of $(m+1)(n+1)D_\ell$ polynomials:

- D_ℓ polynomials of multidegree $(0,1)$,
- $(m+n)D_\ell$ polynomials of multidegree $(1,1)$, and
- mnD_ℓ polynomials of multidegree $(2,1)$.

Let $\mathcal{V}(F)$ be the set of solutions of $F=0$. For the projection

$$\pi(b_1,p_1,\ldots,b_N,p_N,\Lambda_\ell) = (b_1,p_1,\ldots,b_N,p_N),$$

we compute an irreducible decomposition of $X = \overline{\pi(\mathcal{V}(F))}$ using numerical algebraic geometry.

In numerical algebraic geometry (e.g., see [8, 49]), an irreducible decomposition of an algebraic set $Y \subset \mathbb{C}^K$ is computed via a union of *witness sets* forming a *numerical irreducible decomposition*. A witness set for an irreducible algebraic set $Z \subset \mathbb{C}^K$ of dimension k and degree d is a triple $\mathcal{Z} = \{f, L, W\}$, where

- (witness system) f is a polynomial system such that Z is an irreducible component of $\mathcal{V}(f) \subset \mathbb{C}^K$,
- (witness slice) L is a system of k general affine linear polynomials in K variables, and
- (witness point set) $W = Z \cap \mathcal{V}(L) \subset \mathbb{C}^K$ which consists of d points.

Hence, if $Y = Y_1 \cup \cdots \cup Y_u$ is an irreducible decomposition with corresponding witness set $\mathcal{Y}_1,\ldots,\mathcal{Y}_u$, then the formal union $\mathcal{Y}_1 \cup\cdots\cup \mathcal{Y}_u$ is a numerical irreducible decomposition for Y.

One option for computing a numerical irreducible decomposition for X, as first described in [23], is to first compute a numerical irreducible decomposition for $\mathcal{V}(F)$ treated as an algebraic set in the affine space $(\mathbb{C}^m \times \mathbb{C}^n)^N \times \mathbb{C}^{D_\ell\times R}$. To do so, we may begin with the regenerative cascade [27], which uses homotopy path tracking to compute a finite superset, say $\widehat{W_k}$, of the set of isolated points of $\mathcal{V}(F)\cap\mathcal{V}(\ell_1,\ldots,\ell_k)$, say W_k, for all possible values of k, where each ℓ_i is a general affine linear polynomial. Then, for each k, the local dimension test

[3] applied to each point in $\widehat{W_k}$ identifies which ones are isolated points, thereby extracting W_k from $\widehat{W_k}$. Finally, monodromy [48] and a trace test [47] are used to partition W_k into a union of witness point sets for the irreducible components of $\mathcal{V}(F)$ of codimension k.

From this computed numerical irreducible decomposition for $\mathcal{V}(F)$, a numerical irreducible decomposition for X is computed as follows. For each irreducible component $Z \subset \mathcal{V}(F)$ with witness set $\mathcal{Z}$, a (pseudo)witness set for $\overline{\pi(Z)}$ is computed via homotopy continuation, as described in [26]. The projection membership test of [25] yields the irreducible components of X since they are the inclusion maximal elements of

$$\left\{ \overline{\pi(Z)} \;\middle|\; Z \text{ is an irreducible component of } \mathcal{V}(F) \right\}.$$

An alternative option for computing a numerical irreducible decomposition for X is to utilize the multihomogeneous regeneration presented in [24]. Since only the image under the projection map π is of interest, one can simplify this computation using [1] by not having to consider all possible slices in the auxiliary variables Λ_ℓ. This regeneration procedure produces unions of witness point sets which can be decomposed into witness points sets for the irreducible components using monodromy and a trace test, as above.

These techniques depend upon the selection of elements that are general in a family, i.e., outside of a proper algebraic subset of the family, and also depend on path tracking in homotopy continuation. In finite precision arithmetic, both processes may fail to achieve the probability-one success rate predicted by theory based on selecting elements from a continuum and exact path tracking. To enhance reliability in our work, we employ adaptive step-size and adaptive precision path tracking methods [5, 6, 9], and we select general elements over $\mathbb{C}$ by using a random number generator with at least as many digits as the adaptive precision encounters. Moreover, the trace test [47] provides an a posteriori check that complete witness point sets have been computed, so it flags missing witness points. Finally, by keeping all of the points (b_i, p_i) fully general, we maintain symmetry in the solution set, as shown in the computations below.

4. Computational results.

The following were obtained using the numerical algebraic geometric methods described in section 3.2 using the software package `Bertini` [7]. The computations were performed in parallel using a total of 64 cores in four AMD Opteron 6378 2.4 GHz processors. To illustrate computing times, Computations 4.9 and 4.12 each took about 2.5 hours, while Computation 4.7 took about three days.

From the witness sets (see section 3.2) computed in these computations, exact defining equations can be recovered [2], which allows each component computed to be certified, e.g., using `Macaulay2` [21].

4.1. Planar-planar with six legs in SE(3).

A planar Stewart–Gough platform has both $b_1, \ldots, b_6$ and $p_1, \ldots, p_6$ lying in a plane. Since we are free to independently select the coordinate frame of references $\mathcal{B}$ and $\mathcal{P}$, we can, without loss of generality, assume that the third coordinate of each point is 0, i.e., $b_i, p_i \in \mathbb{C}^2$. Thus, planar Stewart–Gough platforms which are Segre-dependent correspond to solving Problem 3.2 with $N = 6$, $R = 5$, and $m = n = 2$.

In particular, for the 6×9 matrix

$$A = \begin{bmatrix} 1 & b_{11} & b_{12} & p_{11} & p_{12} & b_{11}p_{11} & b_{12}p_{11} & b_{11}p_{12} & b_{12}p_{12} \\ 1 & b_{21} & b_{22} & p_{21} & p_{22} & b_{21}p_{21} & b_{22}p_{21} & b_{21}p_{22} & b_{22}p_{22} \\ 1 & b_{31} & b_{32} & p_{31} & p_{32} & b_{31}p_{31} & b_{32}p_{31} & b_{31}p_{32} & b_{32}p_{32} \\ 1 & b_{41} & b_{42} & p_{41} & p_{42} & b_{41}p_{41} & b_{42}p_{41} & b_{41}p_{42} & b_{42}p_{42} \\ 1 & b_{51} & b_{52} & p_{51} & p_{52} & b_{51}p_{51} & b_{52}p_{51} & b_{51}p_{52} & b_{52}p_{52} \\ 1 & b_{61} & b_{62} & p_{61} & p_{62} & b_{61}p_{61} & b_{62}p_{61} & b_{61}p_{62} & b_{62}p_{62} \end{bmatrix}, \tag{4.1}$$

the problem is to describe the irreducible components of the set of points $(b_1, p_1, \ldots, b_6, p_6) \in (\mathbb{C}^2 \times \mathbb{C}^2)^6$ such that $\operatorname{rank} A \leq 5$. Following section 3.2, we obtain the following decomposition.

Computation 4.1. *The solution to Problem 3.2 when $N = 6$, $R = 5$, and $m = n = 2$ is provided by the following list of irreducible components:*

1. *(Two legs coincide) $\binom{6}{2} = 15$ irreducible components of dimension 20 and degree 1 such that, for distinct $i, j \in \{1, \ldots, 6\}$,*
   - *$b_i = b_j$ and $p_i = p_j$.*
2. *(General planar case) One irreducible component of dimension 20 and degree 306. In particular, the projection of this component onto $(b_1, \ldots, b_6, p_1, \ldots, p_4) \in \mathbb{C}^{20}$ is generically one-to-one. Thus, on a Zariski dense subset $(b_1, \ldots, b_6, p_1, \ldots, p_4) \in \mathbb{R}^{20}$, there is a unique $(p_5, p_6) \in \mathbb{R}^4$ yielding a Segre-dependent planar Stewart–Gough platform.*

The general planar component in item 2 has been geometrically described [16, 36, 39, 45]. Moreover, with a left null vector $\lambda = [\lambda_1, \ldots, \lambda_6] \in \mathbb{P}^5$, the system $\lambda \cdot A = 0$ consists of nine linear equations in the unknowns

$$\lambda_1, \ldots, \lambda_6, p_{51}\lambda_5, p_{52}\lambda_5, p_{61}\lambda_6, p_{62}\lambda_6.$$

As long as $\lambda_5, \lambda_6 \neq 0$, which happens generically, this determines unique values of p_5, p_6.

If we treat $(b_1, \ldots, b_6, p_1, \ldots, p_6) \in (\mathbb{C}^2)^{12}$, this yields a term in the multidegree of this general planar component. Since two linear constraints are placed on each $b_1, \ldots, b_6, p_1, \ldots, p_4$ and none of p_5, p_6, following the notation of [24], we can write this term as

$$1\omega^{(2,2,2,2,2,2,2,2,2,2,0,0)}.$$

In fact, due to symmetry, this actually yields a family consisting of $2 \cdot \binom{6}{2} = 30$ terms of the multidegree where the 2 corresponds with selecting either the base or the platform and the binomial coefficient corresponds to selecting two of the six locations which have no constraints.

Based on symmetry, this is just one of the 15 different families of codimension 20 linear slices in $(\mathbb{C}^2)^{12}$. The following table summarizes all terms of the multidegree of this general planar component.

Term in multidegree	Total number of elements in family		
$1\omega^{(2,2,2,2,2,2,2,2,2,2,0,0)}$	$2 \cdot \binom{6}{2}$	$=$	30
$1\omega^{(2,2,2,2,2,2,2,2,2,1,1,0)}$	$2 \cdot \binom{6}{1} \cdot \binom{5}{2}$	$=$	120
$1\omega^{(2,2,2,2,2,2,2,2,1,1,1,1)}$	$2 \cdot \binom{6}{4}$	$=$	30
$3\omega^{(2,2,2,2,2,1,2,2,2,2,1,0)}$	$2 \cdot \binom{6}{1} \cdot \binom{5}{1}$	$=$	60
$2\omega^{(2,2,2,2,1,2,2,2,2,2,1,0)}$	$2 \cdot \binom{6}{1} \cdot \binom{5}{1}$	$=$	60
$1\omega^{(2,2,2,1,2,2,2,2,2,2,1,0)}$	$2 \cdot \binom{6}{1} \cdot \binom{5}{1} \cdot \binom{4}{1}$	$=$	240
$2\omega^{(2,2,2,2,2,1,2,2,2,1,1,1)}$	$2 \cdot \binom{6}{3} \cdot \binom{3}{1}$	$=$	120
$1\omega^{(2,2,1,2,2,2,2,2,2,1,1,1)}$	$2 \cdot \binom{6}{3} \cdot \binom{3}{1}$	$=$	120
$1\omega^{(2,2,2,2,2,0,2,2,2,2,2,0)}$	$\binom{6}{1}$	$=$	6
$0\omega^{(2,2,2,2,0,2,2,2,2,2,2,0)}$	$\binom{6}{1} \cdot \binom{5}{1}$	$=$	30
$2\omega^{(2,2,2,2,2,0,2,2,2,2,1,1)}$	$2 \cdot \binom{6}{2} \cdot \binom{2}{1}$	$=$	60
$1\omega^{(2,2,2,0,2,2,2,2,2,2,1,1)}$	$2 \cdot \binom{6}{2} \cdot \binom{4}{1}$	$=$	120
$3\omega^{(2,2,2,2,1,1,2,2,2,2,1,1)}$	$\cdot\binom{6}{2}$	$=$	15
$2\omega^{(2,2,2,1,2,1,2,2,2,2,1,1)}$	$\binom{6}{2} \cdot \binom{2}{1} \cdot \binom{4}{1}$	$=$	120
$1\omega^{(2,2,1,1,2,2,2,2,2,2,1,1)}$	$\binom{6}{2} \cdot \binom{4}{2}$	$=$	90

Example 4.2. To demonstrate item 2, we consider the following sufficiently general collection of points:

$$b_1 = (0,0),\ b_2 = (1,0),\ b_3 = (2,1),\ b_4 = (3,-1),\ b_5 = (-2,3),\ b_6 = (-1,-2),$$
$$p_1 = (0,0),\ p_2 = (2,0),\ p_3 = (1,1),\ p_4 = (-2,-3).$$

Using the "linearity" of $\lambda \cdot A$, as described above, it is easy to check that the unique planar Segre-dependent Stewart–Gough platform has

$$p_5 = (-4/9, -2/3) \text{ and } p_6 = (10/37, 8/37) \text{ with left null vector } \lambda = [84, 7, -28, -8, -18, -37],$$

as shown in Figure 4.1.

Remark 4.3. One family of planar Stewart–Gough platforms that has been extensively studied is the family of Griffis–Duffy platforms [22]. In this family, the base and platform are triangles with connection points at each vertex and along each edge where the legs connect a vertex to a point on an edge in a cyclical fashion. We take the following collections to be collinear:

$$\{b_1, b_2, b_3\},\ \{b_3, b_4, b_5\},\ \{b_5, b_6, b_1\},\ \{p_2, p_3, p_4\},\ \{p_4, p_5, p_6\},\ \{p_6, p_1, p_2\}.$$

A generic element in this family is rigid with 16 assembly configurations. The following derives the family of Segre-dependent Griffis–Duffy platforms, which is equivalent to the family of exceptional Griffis–Duffy platforms derived using the results of [13, 30, 36, 45].

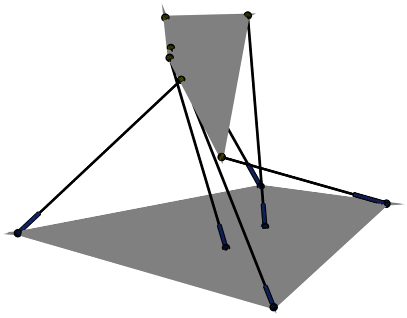


Figure 4.1. *A planar Segre-dependent Stewart–Gough platform.*

Suppose that we construct a Griffis–Duffy platform where

$$\begin{aligned} p_1 &= p_6\mu_1 + (1-\mu_1)p_2, \quad p_3 = p_2\mu_3 + (1-\mu_3)p_4, \quad p_5 = p_4\mu_5 + (1-\mu_5)p_6, \\ b_2 &= b_1\mu_2 + (1-\mu_2)b_3, \quad b_4 = b_3\mu_4 + (1-\mu_4)b_5, \quad b_6 = b_5\mu_6 + (1-\mu_6)b_1. \end{aligned} \tag{4.2}$$

Then the platform is Segre-dependent if and only if one of the following holds:

1. $b_i = b_j$ for $(i,j) = (1,3)$, $(3,5)$, or $(1,5)$;
2. $p_i = p_j$ for $(i,j) = (2,4)$, $(4,6)$, or $(2,6)$;
3. b_1, b_3, b_5 and p_2, p_4, p_6 are both collinear;
4. $$\begin{aligned} R(\mu) := \mu_1\mu_2\mu_3 \det\begin{bmatrix} \mu_4 & \mu_6 - 1 \\ 1-\mu_5 & \mu_5+\mu_6-1 \end{bmatrix} + \mu_4\mu_5\mu_6 \det\begin{bmatrix} \mu_1 & \mu_3 - 1 \\ 1-\mu_2 & \mu_2+\mu_3-1 \end{bmatrix} \\ - \det\begin{bmatrix} \mu_1 & \mu_3 - 1 \\ 1-\mu_2 & \mu_2+\mu_3-1 \end{bmatrix} \det\begin{bmatrix} \mu_4 & \mu_6 - 1 \\ 1-\mu_5 & \mu_5+\mu_6-1 \end{bmatrix} = 0. \end{aligned}$$

The quintic polynomial $R(\mu)$ in item 4 is linear in each μ_i. In particular, for general $\mu_1, \ldots, \mu_5 \in \mathbb{R}$, there exists a unique $\mu_6 \in \mathbb{R}$ such that every $b_1, p_2, b_3, p_4, b_5, p_6 \in \mathbb{R}^2$ with (4.2) yields a Griffis–Duffy platform that is Segre-dependent. We illustrate by considering the sufficiently general values

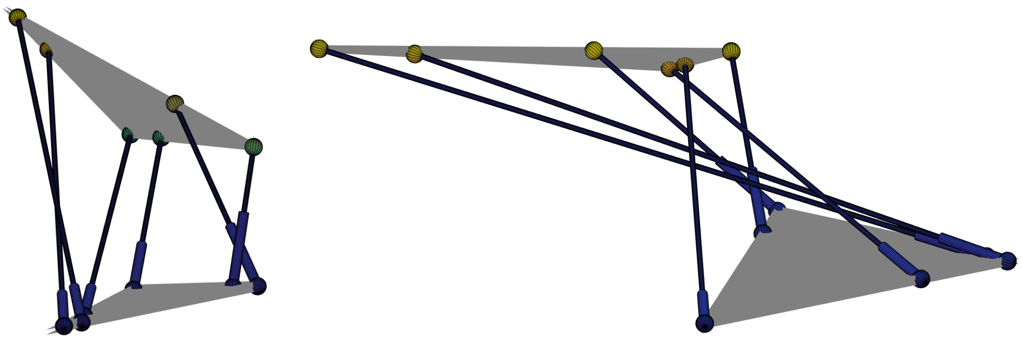
$$\mu_1 = 1/3,\ \mu_2 = 4/5,\ \mu_3 = 1/4,\ \mu_4 = 2/7, \text{ and } \mu_5 = 3/11.$$

The unique value of μ_6 satisfying item 4 is $\mu_6 = 10/11$. Hence, every Griffis–Duffy mechanism built from these values of μ_i will be Segre-dependent. We illustrate this with two examples presented in Figure 4.2:

(a) $b_1 = (0,0)$, $b_3 = (0,1)$, $b_5 = (-2,2)$, $p_2 = (0,0)$, $p_4 = (2,0)$, and $p_6 = (3,1)$;
(b) $b_1 = (0,0)$, $b_3 = (0,5)$, $b_5 = (-2,2)$, $p_2 = (0,0)$, $p_4 = (1/2,0)$, and $p_6 = (3,1)$.

Example 4.4. Griffis–Duffy Type I platforms [30] are Griffis–Duffy platforms (see Remark 4.3) whose base and platform consist of equilateral triangles with connection points at the midpoint of each edge, i.e., $\mu_i = 1/2$. Figure 4.3 shows an example, as does one of the seven pictures featured on the main cover of the *SIAM Journal on Applied Algebraic Geometry* (SIAGA). Every such mechanism is Segre-dependent by item 4. In fact, taking the midpoints of each edge for *any* two triangles will yield a Segre-dependent mechanism.

4.2. Planar-planar with six legs in SE(3) and two degrees of freedom. In section 4.1, the exceptional mechanisms described generically had self-motion with one degree of freedom. We



(a) (b)

Figure 4.2. *Two mobile Griffis–Duffy platforms.*

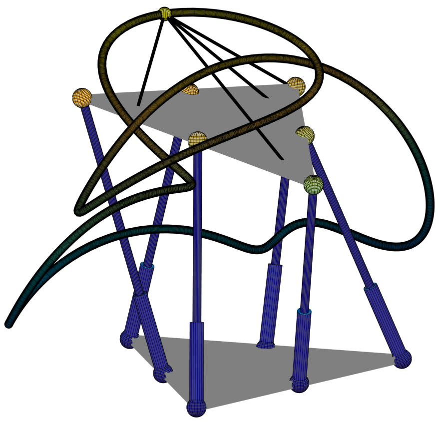


Figure 4.3. *Griffis–Duffy Type* I *platform with its coupler curve.*

now consider planar Segre-dependent Stewart–Gough mechanisms with at least two degrees of freedoms. With the setup from section 4.1, this means that the 6×9 matrix A in (4.1) has rank at most 4, i.e., the dimension of the left null space of A is at least 2. We note that all nonarchitecturally singular Stewart–Gough platforms with two degrees of freedom are given in [42, Thm. 5].

Computation 4.5. *The solution to Problem 3.2 when $N = 6$, $R = 4$, and $m = n = 2$ is provided by the following list of irreducible components:*

1. *(Three legs coincide)* $\binom{6}{3} = 20$ *irreducible components of dimension* 16 *and degree* 1 *such that, for distinct $i, j, k \in \{1, \ldots, 6\}$,*
   - $b_i = b_j = b_k$ *and* $p_i = p_j = p_k$.
2. *(Two pairs of legs coincide)* $\frac{\binom{6}{2}\cdot\binom{4}{2}}{2!} = 45$ *irreducible components of dimension* 16 *and degree* 1 *such that, for distinct $i, j, k, \ell \in \{1, \ldots, 6\}$,*
   - $b_i = b_j$, $b_k = b_\ell$, $p_i = p_j$, *and* $p_k = p_\ell$.
3. *(Five points coincident)* $2 \cdot \binom{6}{5} = 12$ *irreducible components of dimension* 16 *and*

degree 1 *such that, for distinct* $h, i, j, k, \ell \in \{1, \dots, 6\}$,

- $b_h = b_i = b_j = b_k = b_\ell$ *or*
- $p_h = p_i = p_j = p_k = p_\ell$.

4. *(Four point-pairs: one set coincident and one set collinear)* $2 \cdot \binom{6}{4} = 30$ *irreducible components of dimension* 16 *and degree* 3 *such that, for distinct* $i, j, k, \ell \in \{1, \dots, 6\}$,
   - $b_i = b_j = b_k = b_\ell$ *and* $\operatorname{rank} \begin{bmatrix} p_j - p_i \\ p_k - p_i \\ p_\ell - p_i \end{bmatrix} \leq 1$ *or*
   - $\operatorname{rank} \begin{bmatrix} b_j - b_i \\ b_k - b_i \\ b_\ell - b_i \end{bmatrix} \leq 1$ *and* $p_i = p_j = p_k = p_\ell$.
5. *(Six point-pairs: both collinear) One irreducible component of dimension* 16 *and degree* 25 *such that*
   - $\operatorname{rank} \begin{bmatrix} b_2 - b_1 \\ b_3 - b_1 \\ b_4 - b_1 \\ b_5 - b_1 \\ b_6 - b_1 \end{bmatrix} \leq 1$ *and* $\operatorname{rank} \begin{bmatrix} p_2 - p_1 \\ p_3 - p_1 \\ p_4 - p_1 \\ p_5 - p_1 \\ p_6 - p_1 \end{bmatrix} \leq 1.$
6. *(Five point-pairs: both collinear)* $\binom{6}{5} = 6$ *irreducible components of dimension* 16 *and degree* 93 *such that, for distinct* $h, i, j, k, \ell \in \{1, \dots, 6\}$,
   - $\operatorname{rank} \begin{bmatrix} b_i - b_h \\ b_j - b_h \\ b_k - b_h \\ b_\ell - b_h \end{bmatrix} \leq 1$, $\operatorname{rank} \begin{bmatrix} p_i - p_h \\ p_j - p_h \\ p_k - p_h \\ p_\ell - p_h \end{bmatrix} \leq 1$, *and, for* $\alpha, \beta \in \{1, 2\}$,

$$\operatorname{rank} \begin{bmatrix} 1 & b_{h\alpha} & p_{h\beta} & b_{h1}p_{h1} & b_{h1}p_{h2} & b_{h2}p_{h1} & b_{h2}p_{h2} \\ 1 & b_{i\alpha} & p_{i\beta} & b_{i1}p_{i1} & b_{i1}p_{i2} & b_{i2}p_{i1} & b_{i2}p_{i2} \\ 1 & b_{j\alpha} & p_{j\beta} & b_{j1}p_{j1} & b_{j1}p_{j2} & b_{j2}p_{j1} & b_{j2}p_{j2} \\ 1 & b_{k\alpha} & p_{k\beta} & b_{k1}p_{k1} & b_{k1}p_{k2} & b_{k2}p_{k1} & b_{k2}p_{k2} \\ 1 & b_{\ell\alpha} & p_{\ell\beta} & b_{\ell 1}p_{\ell 1} & b_{\ell 1}p_{\ell 2} & b_{\ell 2}p_{\ell 1} & b_{\ell 2}p_{\ell 2} \end{bmatrix} \leq 3.$$

7. *(Six points collinear) Two irreducible components of dimension* 16 *and degree* 583 *such that*
   - $\operatorname{rank} \begin{bmatrix} b_2 - b_1 \\ b_3 - b_1 \\ b_4 - b_1 \\ b_5 - b_1 \\ b_6 - b_1 \end{bmatrix} \leq 1$ *and, for* $\alpha \in \{1, 2\}$,

$$\operatorname{rank} \begin{bmatrix} \mathbf{1} & b_{:\alpha} & p_{:1} & p_{:2} & b_{:1} \cdot p_{:1} & b_{:1} \cdot p_{:2} & b_{:2} \cdot p_{:1} & b_{:2} \cdot p_{:2} \end{bmatrix} \leq 4$$

or

- $\operatorname{rank}\begin{bmatrix} p_2-p_1 \\ p_3-p_1 \\ p_4-p_1 \\ p_5-p_1 \\ p_6-p_1 \end{bmatrix} \le 1$ *and, for* $\alpha \in \{1,2\}$,

$$\operatorname{rank}\begin{bmatrix} \mathbf{1} & b_{:1} & b_{:2} & p_{:\alpha} & b_{:1}\cdot p_{:1} & b_{:1}\cdot p_{:2} & b_{:2}\cdot p_{:1} & b_{:2}\cdot p_{:2} \end{bmatrix} \le 4,$$

where $\mathbf{1}$ *is the vector of* 1*'s and "*:*" corresponds to all values* 1 *to* 6*.*

8. *(Two legs coincide and three point-pairs: one set coincident and one set collinear)* $2\cdot\binom{6}{2}\cdot\binom{4}{3} = 120$ *irreducible components of dimension* 15 *and degree* 2 *such that, for distinct* $h,i,j,k,\ell \in \{1,\dots,6\}$,
   - $b_h = b_i$, $b_j = b_k = b_\ell$, $p_h = p_i$, *and* $\operatorname{rank}\begin{bmatrix} p_k-p_j \\ p_\ell-p_j \end{bmatrix} \le 1$ *or*
   - $b_h = b_i$, $\operatorname{rank}\begin{bmatrix} b_k-b_j \\ b_\ell-b_j \end{bmatrix} \le 1$, $p_h = p_i$, *and* $p_j = p_k = p_\ell$.
9. *(Two legs coincide and four point-pairs: both collinear)* $\binom{6}{4} = 15$ *irreducible components of dimension* 15 *and degree* 24 *such that, for distinct* $g,h,i,j,k,\ell \in \{1,\dots,6\}$,
   - $b_g = b_h$, $\operatorname{rank}\begin{bmatrix} b_j-b_i \\ b_k-b_i \\ b_\ell-b_i \end{bmatrix} \le 1$, $p_g = p_h$, $\operatorname{rank}\begin{bmatrix} p_j-p_i \\ p_k-p_i \\ p_\ell-p_i \end{bmatrix} \le 1$, *and, for* $\alpha,\beta \in \{1,2\}$,

$$\operatorname{rank}\begin{bmatrix} 1 & b_{i\alpha} & p_{i\beta} & b_{i1}p_{i1} & b_{i1}p_{i2} & b_{i2}p_{i1} & b_{i2}p_{i2} \\ 1 & b_{j\alpha} & p_{j\beta} & b_{j1}p_{j1} & b_{j1}p_{j2} & b_{j2}p_{j1} & b_{j2}p_{j2} \\ 1 & b_{k\alpha} & p_{k\beta} & b_{k1}p_{k1} & b_{k1}p_{k2} & b_{k2}p_{k1} & b_{k2}p_{k2} \\ 1 & b_{\ell\alpha} & p_{\ell\beta} & b_{\ell1}p_{\ell1} & b_{\ell1}p_{\ell2} & b_{\ell2}p_{\ell1} & b_{\ell2}p_{\ell2} \end{bmatrix} \le 3.$$

10. *(Two collections of three point-pairs: one coincident and one collinear)* $\binom{6}{3} = 20$ *irreducible components of dimension* 14 *and degree* 4 *such that, for distinct* $g,h,i,j,k,\ell \in \{1,\dots,6\}$,
    - $b_g = b_h = b_i$, $\operatorname{rank}\begin{bmatrix} b_k-b_j \\ b_\ell-b_j \end{bmatrix} \le 1$, $\operatorname{rank}\begin{bmatrix} p_h-p_g \\ p_i-p_g \end{bmatrix} \le 1$, *and* $p_j = p_k = p_\ell$.

4.3. Planar-spatial case with six legs in SE(3). For a Stewart–Gough platform where one body is planar and one is spatial, we can, without loss of generality, assume that the points in the base $b_1,\dots,b_6$ lie in $\mathbb{C}^2$, i.e., the third coordinate of each b_i is zero. Thus, solving Problem 3.2 with $N=6$, $R=5$, $m=2$, and $n=3$ requires computing where the 6×12 matrix

$$A = \begin{bmatrix} 1 & b_{11} & b_{12} & p_{11} & p_{12} & p_{13} & b_{11}p_{11} & b_{12}p_{11} & b_{11}p_{12} & b_{12}p_{12} & b_{11}p_{13} & b_{12}p_{13} \\ 1 & b_{21} & b_{22} & p_{21} & p_{22} & p_{23} & b_{21}p_{21} & b_{22}p_{21} & b_{21}p_{22} & b_{22}p_{22} & b_{21}p_{23} & b_{22}p_{23} \\ 1 & b_{31} & b_{32} & p_{31} & p_{32} & p_{33} & b_{31}p_{31} & b_{32}p_{31} & b_{31}p_{32} & b_{32}p_{32} & b_{31}p_{33} & b_{32}p_{33} \\ 1 & b_{41} & b_{42} & p_{41} & p_{42} & p_{43} & b_{41}p_{41} & b_{42}p_{41} & b_{41}p_{42} & b_{42}p_{42} & b_{41}p_{43} & b_{42}p_{43} \\ 1 & b_{51} & b_{52} & p_{51} & p_{52} & p_{53} & b_{51}p_{51} & b_{52}p_{51} & b_{51}p_{52} & b_{52}p_{52} & b_{51}p_{53} & b_{52}p_{53} \\ 1 & b_{61} & b_{62} & p_{61} & p_{62} & p_{63} & b_{61}p_{61} & b_{62}p_{61} & b_{61}p_{62} & b_{62}p_{62} & b_{61}p_{63} & b_{62}p_{63} \end{bmatrix} \tag{4.3}$$

has rank at most 5.

Computation 4.6. *The solution to Problem 3.2 when $N = 6$, $R = 5$, $m = 2$, and $n = 3$ is provided by the following list of irreducible components:*

1. *(Two legs coincide)* $\binom{6}{2} = 15$ *irreducible components of dimension* 25 *and degree* 1 *as in item 1 of Computation 4.1.*
2. *(Three point-pairs: base coincident, platform collinear)* $\binom{6}{3} = 20$ *irreducible components of dimension* 24 *and degree* 3 *such that, for distinct* $i, j, k \in \{1, \ldots, 6\}$,
   - $b_i = b_j = b_k$ *and* $\operatorname{rank} \begin{bmatrix} p_j - p_i \\ p_k - p_i \end{bmatrix} \leq 1$.
3. *(Three point-pairs: base collinear, platform coincident)* $\binom{6}{3} = 20$ *irreducible components of dimension* 23 *and degree* 2 *such that, for distinct* $i, j, k \in \{1, \ldots, 6\}$,
   - $p_i = p_j = p_k$ *and* $\operatorname{rank} \begin{bmatrix} b_j - b_i \\ b_k - b_i \end{bmatrix} \leq 1$.
4. *(Four point-pairs: base coincident, platform coplanar)* $\binom{6}{4} = 15$ *irreducible components of dimension* 23 *and degree* 3 *such that, for distinct* $i, j, k, \ell \in \{1, \ldots, 6\}$,
   - $b_i = b_j = b_k = b_\ell$ *and* $\operatorname{rank} \begin{bmatrix} p_j - p_i \\ p_k - p_i \\ p_\ell - p_i \end{bmatrix} \leq 2$.
5. *(Four point-pairs: both collinear)* $\binom{6}{4} = 15$ *irreducible components of dimension* 23 *and degree* 42 *such that, for distinct* $i, j, k, \ell \in \{1, \ldots, 6\}$,
   - $\operatorname{rank} \begin{bmatrix} b_j - b_i \\ b_k - b_i \\ b_\ell - b_i \end{bmatrix} \leq 1$, $\operatorname{rank} \begin{bmatrix} p_j - p_i \\ p_k - p_i \\ p_\ell - p_i \end{bmatrix} \leq 1$, *and, for* $\alpha \in \{1, 2\}$ *and* $\beta \in \{1, 2, 3\}$,

$$\operatorname{rank} \begin{bmatrix} 1 & b_{i\alpha} & p_{i\beta} & b_{i1}p_{i1} & b_{i1}p_{i2} & b_{i1}p_{i3} & b_{i2}p_{i1} & b_{i2}p_{i2} & b_{i2}p_{i3} \\ 1 & b_{j\alpha} & p_{j\beta} & b_{j1}p_{j1} & b_{j1}p_{j2} & b_{j1}p_{j3} & b_{j2}p_{j1} & b_{j2}p_{j2} & b_{j2}p_{j3} \\ 1 & b_{k\alpha} & p_{k\beta} & b_{k1}p_{k1} & b_{k1}p_{k2} & b_{k1}p_{k3} & b_{k2}p_{k1} & b_{k2}p_{k2} & b_{k2}p_{k3} \\ 1 & b_{\ell\alpha} & p_{\ell\beta} & b_{\ell1}p_{\ell1} & b_{\ell1}p_{\ell2} & b_{\ell1}p_{\ell3} & b_{\ell2}p_{\ell1} & b_{\ell2}p_{\ell2} & b_{\ell2}p_{\ell3} \end{bmatrix} \leq 3.$$

6. *(Five point-pairs: base collinear, platform coplanar)* $\binom{6}{5} = 6$ *irreducible components of dimension* 23 *and degree* 216 *such that, for distinct* $h, i, j, k, \ell \in \{1, \ldots, 6\}$,
   - $\operatorname{rank} \begin{bmatrix} b_i - b_h \\ b_j - b_h \\ b_k - b_h \\ b_\ell - b_h \end{bmatrix} \leq 1$, $\operatorname{rank} \begin{bmatrix} p_i - p_h \\ p_j - p_h \\ p_k - p_h \\ p_\ell - p_h \end{bmatrix} \leq 2$, *and, for* $\alpha \in \{1, 2\}$ *and distinct* $\beta, \gamma \in \{1, 2, 3\}$,

$$\operatorname{rank} \begin{bmatrix} 1 & b_{h\alpha} & p_{h\beta} & p_{h\gamma} & b_{h1}p_{h1} & b_{h1}p_{h2} & b_{h1}p_{h3} & b_{h2}p_{h1} & b_{h2}p_{h2} & b_{h2}p_{h3} \\ 1 & b_{i\alpha} & p_{i\beta} & p_{i\gamma} & b_{i1}p_{i1} & b_{i1}p_{i2} & b_{i1}p_{i3} & b_{i2}p_{i1} & b_{i2}p_{i2} & b_{i2}p_{i3} \\ 1 & b_{j\alpha} & p_{j\beta} & p_{j\gamma} & b_{j1}p_{j1} & b_{j1}p_{j2} & b_{j1}p_{j3} & b_{j2}p_{j1} & b_{j2}p_{j2} & b_{j2}p_{j3} \\ 1 & b_{k\alpha} & p_{k\beta} & p_{k\gamma} & b_{k1}p_{k1} & b_{k1}p_{k2} & b_{k1}p_{k3} & b_{k2}p_{k1} & b_{k2}p_{k2} & b_{k2}p_{k3} \\ 1 & b_{\ell\alpha} & p_{\ell\beta} & p_{\ell\gamma} & b_{\ell1}p_{\ell1} & b_{\ell1}p_{\ell2} & b_{\ell1}p_{\ell3} & b_{\ell2}p_{\ell1} & b_{\ell2}p_{\ell2} & b_{\ell2}p_{\ell3} \end{bmatrix} \leq 4.$$

7. *(Six base points collinear) One irreducible component of dimension* 23 *and degree* 369 *such that*

- rank $\begin{bmatrix} b_2 - b_1 \\ b_3 - b_1 \\ b_4 - b_1 \\ b_5 - b_1 \\ b_6 - b_1 \end{bmatrix} \leq 1$ *and, for* $\alpha \in \{1, 2\}$,

$$\text{rank} \begin{bmatrix} \mathbf{1} & b_{:\alpha} & p_{:1} & p_{:2} & p_{:3} & b_{:1} \cdot p_{:1} & b_{:1} \cdot p_{:2} & b_{:1} \cdot p_{:3} & b_{:2} \cdot p_{:1} & b_{:2} \cdot p_{:2} & b_{:2} \cdot p_{:3} \end{bmatrix} \leq 5,$$

where $\mathbf{1}$ *is the vector of* 1*'s and "*$:$*" corresponds to all values* 1 *to* 6.

8. *(Planar-planar) One irreducible component of dimension* 23 *and degree* 1700 *as in item* 2 *of Computation* 4.1.

4.4. Planar-spatial with six legs in SE(3) and two degrees of freedom. Similarly to section 4.2, we next consider where the 6×12 matrix A in (4.3) has rank at most 4.

Computation 4.7. *The solution to Problem* 3.2 *when* $N = 6$, $R = 4$, $m = 2$, *and* $n = 3$ *is provided by the following list of irreducible components:*

1. *(Three legs coincide)* $\binom{6}{3} = 20$ *irreducible components of dimension* 20 *and degree* 1 *as in item* 1 *of Computation* 4.5.
2. *(Two pairs of legs coincide)* $\frac{\binom{6}{2}\cdot\binom{4}{2}}{2!} = 45$ *irreducible components of dimension* 20 *and degree* 1 *as in item* 2 *of Computation* 4.5.
3. *(Six base points coincident) One irreducible component of dimension* 20 *and degree* 1 *such that*
   - $b_1 = b_2 = b_3 = b_4 = b_5 = b_6$.
4. *(Five point-pairs: base coincident, platform coplanar)* $\binom{6}{5} = 6$ *irreducible components of dimension* 20 *and degree* 6 *as in item* 3 *of Computation* 4.5.
5. *(Four point-pairs: base coincident, platform collinear)* $\binom{6}{4} = 15$ *irreducible components of dimension* 20 *and degree* 6 *as in item* 4 *of Computation* 4.5.
6. *(Four point-pairs: base collinear, platform coincident)* $\binom{6}{4} = 15$ *irreducible components of dimension* 19 *and degree* 3 *as in item* 4 *of Computation* 4.5.
7. *(Two legs coincide and three point-pairs: base coincident, platform collinear)* $\binom{6}{2}\cdot\binom{4}{3} = 60$ *irreducible components of dimension* 19 *and degree* 3 *as in item* 8 *of Computation* 4.5.
8. *(Five point-pairs: both collinear)* $\binom{6}{5} = 6$ *irreducible components of dimension* 19 *and degree* 186 *as in item* 6 *of Computation* 4.5.
9. *(Six point-pairs: base collinear, platform coplanar) One irreducible component of dimension* 19 *and degree* 2547 *as in item* 7 *of Computation* 4.5.
10. *(Five platform points coincident)* $\binom{6}{5} = 6$ *irreducible components of dimension* 18 *and degree* 1 *as in item* 3 *of Computation* 4.5.
11. *(Two legs coincide and four point-pairs: base coincident, platform coplanar)* $\binom{6}{4} = 15$ *irreducible components of dimension* 18 *and degree* 3 *such that, for distinct* $g, h, i, j, k, \ell \in \{1, \ldots, 6\}$,
    - $b_g = b_h$, $b_i = b_j = b_k = b_\ell$, $p_g = p_h$, *and* rank $\begin{bmatrix} p_j - p_i \\ p_k - p_i \\ p_\ell - p_i \end{bmatrix} \leq 2$.
12. *(Two collections of three point-pairs: base coincident, platform collinear)* $\frac{\binom{6}{3}}{2!} = 10$ *irre-*

ducible components of dimension 18 *and degree* 9 *such that, for distinct* $g, h, i, j, k, \ell \in \{1, \ldots, 6\}$,

- $b_g = b_h = b_i$, $b_j = b_k = b_\ell$, rank $\begin{bmatrix} p_h - p_g \\ p_i - p_g \end{bmatrix} \le 1$, *and* rank $\begin{bmatrix} p_k - p_j \\ p_\ell - p_j \end{bmatrix} \le 1$.

13. *(Platform coplanar, two legs coincide, and four point-pairs: both collinear)* $\binom{6}{4} = 15$ *irreducible components of dimension* 18 *and degree* 50 *as in item* 9 *of Computation* 4.5.
14. *(Six point-pairs: both collinear) One irreducible component of dimension* 18 *and degree* 75 *as in item* 5 *of Computation* 4.5.
15. *(Six platform points collinear) One irreducible component of dimension* 18 *and degree* 1329 *as in item* 7 *of Computation* 4.5.
16. *(Two collections of three point-pairs: one set coincident and one set collinear)* $\binom{6}{3} = 20$ *irreducible components of dimension* 17 *and degree* 6 *as in item* 10 *of Computation* 4.5.

Remark 4.8. Items 6, 8, 9, 10, 13, 14, 15, and 16 of Computation 4.7 are all planar-planar mechanisms. Moreover, for item 3, even though the generic rank of the 6×12 matrix A in (4.3) is 4 corresponding with a two-dimensional left null space, a generic element of this family actually has three degrees of freedom. This does not occur in any other family in the computations presented throughout this section; i.e., for the other families, the generic dimension of the left null space of the matrix in (3.2) is equal to the mobility of the corresponding Stewart–Gough platform.

4.5. Spatial-spatial case with six legs in SE(3). For spatial Stewart–Gough platforms, we consider where the 6×16 matrix A, namely

(4.4)
$$\begin{bmatrix} 1 & b_{11} & b_{12} & b_{13} & p_{11} & p_{12} & p_{13} & b_{11}p_{11} & b_{12}p_{11} & b_{13}p_{11} & b_{11}p_{12} & b_{12}p_{12} & b_{13}p_{12} & b_{11}p_{13} & b_{12}p_{13} & b_{13}p_{13} \\ 1 & b_{21} & b_{22} & b_{23} & p_{21} & p_{22} & p_{23} & b_{21}p_{21} & b_{22}p_{21} & b_{23}p_{21} & b_{21}p_{22} & b_{22}p_{22} & b_{23}p_{22} & b_{21}p_{23} & b_{22}p_{23} & b_{23}p_{23} \\ 1 & b_{31} & b_{32} & b_{33} & p_{31} & p_{32} & p_{33} & b_{31}p_{31} & b_{32}p_{31} & b_{33}p_{31} & b_{31}p_{32} & b_{32}p_{32} & b_{33}p_{32} & b_{31}p_{33} & b_{32}p_{33} & b_{33}p_{33} \\ 1 & b_{41} & b_{42} & b_{43} & p_{41} & p_{42} & p_{43} & b_{41}p_{41} & b_{42}p_{41} & b_{43}p_{41} & b_{41}p_{42} & b_{42}p_{42} & b_{43}p_{42} & b_{41}p_{43} & b_{42}p_{43} & b_{43}p_{43} \\ 1 & b_{51} & b_{52} & b_{53} & p_{51} & p_{52} & p_{53} & b_{51}p_{51} & b_{52}p_{51} & b_{53}p_{51} & b_{51}p_{52} & b_{52}p_{52} & b_{53}p_{52} & b_{51}p_{53} & b_{52}p_{53} & b_{53}p_{53} \\ 1 & b_{61} & b_{62} & b_{63} & p_{61} & p_{62} & p_{63} & b_{61}p_{61} & b_{62}p_{61} & b_{63}p_{61} & b_{61}p_{62} & b_{62}p_{62} & b_{63}p_{62} & b_{61}p_{63} & b_{62}p_{63} & b_{63}p_{63} \end{bmatrix},$$

has rank at most 5.

Computation 4.9. *The solution to Problem* 3.2 *when* $N = 6$, $R = 5$, *and* $m = n = 3$ *is provided by the following list of irreducible components:*

1. *(Two legs coincide)* $\binom{6}{2} = 15$ *irreducible components of dimension* 30 *and degree* 1 *as in item* 1 *of Computation* 4.1.
2. *(Three point-pairs: one set coincident and one set collinear)* $2 \cdot \binom{6}{3} = 40$ *irreducible components of dimension* 28 *and degree* 3 *corresponding to items* 2 *and* 3 *of Computation* 4.6.
3. *(Four point-pairs: both collinear)* $\binom{6}{4} = 15$ *irreducible components of dimension* 27 *and degree* 72 *as in item* 5 *of Computation* 4.6.
4. *(Four point-pairs: one set coincident and one set coplanar)* $2 \cdot \binom{6}{4} = 30$ *irreducible components of dimension* 26 *and degree* 3 *as in item* 4 *of Computation* 4.6.
5. *(Five point-pairs: one set collinear and one set coplanar)* $2 \cdot \binom{6}{5} = 12$ *irreducible components of dimension* 26 *and degree* 444 *as in item* 6 *of Computation* 4.6.
6. *(Planar-planar) One irreducible component of dimension* 26 *and degree* 8445 *as in*

item 2 of Computation 4.1.

7. *(Six points collinear) Two irreducible components of dimension* 25 *and degree* 924 *as in item 7 of Computation 4.6.*

We note that the "2" in several of these families arises due to symmetry of swapping the base and platform.

Remark 4.10. Theorem 2 of [31] states that every architecturally singular Stewart–Gough platform with six distinct legs satisfies at least one of the following conditions:

- $b_1, \ldots, b_6$ are coplanar;
- $p_1, \ldots, p_6$ are coplanar;
- four points of $b_1, \ldots, b_6$ are collinear;
- four points of $p_1, \ldots, p_6$ are collinear.

This theorem must trivially hold for Segre-dependent Stewart–Gough platforms which we can observe by reviewing the last six families provided in Computation 4.9.

Remark 4.11. In Karger's classification of architecturally singular Stewart–Gough platforms [31] (see also [37, 38]), there was a chosen system of Cartesian coordinates. This choice simplified the computation but resulted in additional components as compared with leaving a fully general coordinate system, e.g., as in Computation 4.9. With respect to this general coordinate system, each irreducible component of the families described in items 1, 2, 3, 5, and 6 of Computation 4.9 is an irreducible component of the set of architecturally singular mechanisms.

Each irreducible component of item 4 of Computation 4.9 is a codimension 1 subset of the architecturally singular set where four points are coincident, corresponding to Family 5 in [31]. In particular, coplanarity is needed for Segre-dependency but not for architectural singularity.

Each irreducible component of item 7 of Computation 4.9 is a codimension 3 subset of the architecturally singular set where six points are collinear, corresponding to Family 1 in [31]. In particular, the additional rank constraints are needed for Segre-dependency but not architectural singularity.

4.6. Spatial-spatial case with 10 legs in SE(4). Since SE(3) is 6-dimensional, spatial Stewart–Gough platforms utilize six legs. One can consider a Stewart–Gough platform moving in four dimensions using SE(4) (e.g., see [41]) with $\dim \mathrm{SE}(4) = 10$. Thus, Problem 3.2 with $N = 10$, $R = 9$, and $m = n = 3$ corresponds with Segre-dependent Stewart–Gough platforms having a spatial base and platform that move in 4-dimensional space.

Computation 4.12. *The solution to Problem 3.2 when $N = 10$, $R = 9$, and $m = n = 3$ is provided by the following list of irreducible components:*

1. *(Two legs coincide)* $\binom{10}{2} = 45$ *irreducible components of dimension* 54 *and degree* 1.
2. *(General spatial case) One irreducible component of dimension* 53 *and degree* 147,816. *In particular, the projection onto* $(b_1, \ldots, b_{10}, p_1, \ldots, p_7, p_{81}, p_{82}) \in \mathbb{C}^{53}$ *is generically one-to-one. Thus, on a Zariski dense subset* $(b_1, \ldots, b_{10}, p_1, \ldots, p_7, p_{81}, p_{82}) \in \mathbb{R}^{53}$, *there is a unique* $(p_{83}, p_9, p_{10}) \in \mathbb{R}^7$ *yielding a Segre-dependent spatial Stewart–Gough platform that moves in* 4*-dimensional space.*

Conceptually, spatial Stewart–Gough platforms which move in 4-dimensional space is analogous to the case considered in section 4.1, namely planar Stewart–Gough platforms which move in 3-dimensional space. Computationally, the Segre-dependent mechanisms have a similar decomposition. In particular, we can follow a similar approach to compute the remaining seven coordinates in the general spatial component of item 2. That is, for a left null vector $\lambda = [\lambda_1, \ldots, \lambda_{10}] \in \mathbb{P}^9$, the system $\lambda \cdot A = 0$, where A is the 10×16 matrix where each row has the form of each row in (4.4), consists of 16 linear equations in the unknowns

$$\lambda_1, \ldots, \lambda_{10}, p_{83}\lambda_8, p_{91}\lambda_9, p_{92}\lambda_9, p_{93}\lambda_9, p_{101}\lambda_{10}, p_{102}\lambda_{10}, p_{103}\lambda_{10}.$$

Example 4.13. To demonstrate item 2, we consider the following sufficiently general collection of points:

$$\begin{array}{lllll} b_1 = (0,0,0), & b_2 = (1,0,0), & b_3 = (-1,4,0), & b_4 = (3,4,2), & b_5 = (2,2,-1), \\ b_6 = (1,-3,2), & b_7 = (-4,-2,-4), & b_8 = (-4,3,2), & b_9 = (-2,4,-4), & b_{10} = (-1,-1,2), \\ p_1 = (0,0,0), & p_2 = (2,0,0), & p_3 = (1,1,0), & p_4 = (1,2,2), & p_5 = (-2,2,1), \\ p_6 = (-3,-3,4), & p_7 = (-1,1,-2), & p_8 = (-3,-2,p_{83}). & & \end{array}$$

Using the "linearity" of $\lambda \cdot A$ as described above, it is easy to check that the unique spatial Segre-dependent Stewart–Gough platform that moves in 4-dimensional space has

$$\begin{aligned} &p_{83} = -3794462/313611, \\ &p_9 = (-7809731, -17364829, -1001946)/4163483, \quad \text{and} \\ &p_{10} = (-16495001, 6880823, -29032984)/22069647 \end{aligned}$$

with left null vector

$$\lambda = [-76731972, 30802104, 6959736, -5925332, 12436712, 4871861, 2608205, -1254444, 4163483, 22069647].$$

5. Special Euclidean and special orthogonal groups. The planar pentad consists of two triangles, say $\triangle ABC$ and $\triangle A'B'C'$, in the plane with three binary links constraining the distances $|\overline{AA'}|$, $|\overline{BB'}|$, and $|\overline{CC'}|$. The spherical pentad is a similar mechanism, except the two triangles each lie on a common sphere. The Stewart–Gough platform may be viewed as the natural generalization of these arrangements from plane to sphere to three-space, as summarized in Table 5.1. In this section, we show how the number of assembly configurations of these three mechanical arrangements all fall naturally into a sequence that generalizes to higher dimensions. In particular, as discussed in [17, 46] and summarized in section 2, the reason that a Stewart–Gough platform generically has 40 assembly configurations is due to the fact that $\deg \mathcal{SE}_3 = 40$, where $\mathcal{SE}_3$ is the Zariski closure of a representation of SE(3) where each leg imposes a linear condition. The generic number of assembly configurations for the planar and spherical pentads follows from exactly the same kind of reasoning applied to the spaces SE(2) and SO(3). This idea can be generalized abstractly to mechanisms related to the special orthogonal group SO(N) consisting of $N \times N$ orthogonal matrices with determinant equal to 1 and the special Euclidean group SE(N) in $\mathbb{R}^N$. In each case, the number of "legs," i.e., the number of point-pair distance constraints, is equal to the dimensionality of the space in question, e.g., 10 for 4-dimensional Stewart–Gough platforms as in section 4.6 (see also [11, 41]).

Table 5.1
Mechanisms related to the special Euclidean and special orthogonal groups.

Mechanism	Generic number of assembly configurations	Corresponding group
planar pentad	6	SE(2)
spherical pentad	8	SO(3)
Stewart–Gough platform	40	SE(3)

To understand the generic number of assembly configurations, we consider the following three varieties:

$$\mathcal{SO}_N = \{M \in \mathbb{C}^{N\times N} \mid M^T M = MM^T = I, \det M = 1\}, \tag{5.1}$$

$$\mathcal{SRT}_N = \{(x,y,M) \in \mathbb{C}^N \times \mathbb{C}^N \times \mathbb{C}^{N\times N} \mid M \in \mathcal{SO}_N, Mx + y = 0\}, \tag{5.2}$$

$$\mathcal{SE}_N = \{(r,x,y,M) \in \mathbb{C} \times \mathbb{C}^N \times \mathbb{C}^N \times \mathbb{C}^{N\times N} \mid (x,y,M) \in \mathcal{SRT}_N, 2r + x^T x = 0\}. \tag{5.3}$$

In particular, $\mathcal{SO}_N$ is the Zariski closure of SO(N), $\mathcal{SRT}_N$ is the Zariski closure of a representation of rotations and translations in $\mathbb{R}^N$, and $\mathcal{SE}_N$ is the Zariski closure of a representation of SE(N). The data in Table 5.1 corresponds with

$$\deg \mathcal{SE}_2 = 6, \quad \deg \mathcal{SO}_3 = 8, \quad \text{and} \quad \deg \mathcal{SE}_3 = 40.$$

Thus, we aim to understand the degrees of these varieties in general. We start with $\deg \mathcal{SO}_N$ from [14]:

$$\deg \mathcal{SO}_N = 2^{N-1} \cdot \det \left[\begin{pmatrix} 2(N-i-j) \\ N-2i \end{pmatrix} \right]_{1\le i,j\le \lfloor \frac{N}{2} \rfloor}. \tag{5.4}$$

In particular, the generic number of assembly configurations for spherical pentads is

$$\deg \mathcal{SO}_3 = 2^2 \cdot \binom{2}{1} = 8.$$

The following theorem was motivated by simply observing that

$$\deg \mathcal{SO}_4 = 2^3 \cdot \det \begin{bmatrix} 6 & 1 \\ 1 & 1 \end{bmatrix} = 40 = \deg \mathcal{SE}_3$$

with further computational evidence generated by `Bertini` [7] using [28] presented in Table 5.2.

Theorem 5.1.
1. *For* $N \ge 1$, $\dim \mathcal{SRT}_N = \dim \mathcal{SO}_{N+1}$ *and* $\deg \mathcal{SRT}_N = \frac{1}{2} \cdot \deg \mathcal{SO}_{N+1}$.
2. *For* $k \ge 0$ *and* $N = 2k+1$, $\dim \mathcal{SE}_N = \dim \mathcal{SO}_{N+1}$ *and* $\deg \mathcal{SE}_N = \deg \mathcal{SO}_{N+1}$.
3. *For* $k \ge 1$ *and* $N = 2k$, $\dim \mathcal{SE}_N = \dim \mathcal{SO}_{N+1}$ *and* $\deg \mathcal{SE}_N < \deg \mathcal{SO}_{N+1}$.

One consequence of the proof provided in section 5.1 is that, for $N = 2k \ge 2$,

$$\deg \mathcal{SE}_N \le \deg \mathcal{SO}_{N+1} - 2^{k-1} \cdot \deg \mathcal{SO}_N.$$

Since this inequality is sharp for $N = 2, 4$ but is not for $N = 6$, we have the following open problem.

Table 5.2
Dimensions and degrees for various varieties.

	$\mathcal{SO}_{N+1}$		$\mathcal{SRT}_N$		$\mathcal{SE}_N$	
N	dim	deg	dim	deg	dim	deg
1	1	2	1	1	1	2
2	3	8	3	4	3	6
3	6	40	6	20	6	40
4	10	384	10	192	10	304
5	15	4768	15	2384	15	4768
6	21	111,616	21	55,808	21	90,496

Problem 5.2. *For even $N \geq 8$, compute* $\deg \mathcal{SE}_N$.

Since the degree describes the generic number of assembly configurations over the complex numbers, a natural question to ask is whether all assembly configurations can be real. For Stewart–Gough platforms, this was answered affirmatively by Dietmaier [15], who produced a Stewart–Gough mechanism where all 40 assembly configurations are real. Since each leg corresponds with intersecting $\mathcal{SE}_3$ with a hyperplane, one can view Dietmaier's result as producing a linear space of codimension 6 which intersects $\mathcal{SE}_3$ in 40 real points. That is, there exists a witness set (see section 3.2) for $\mathcal{SE}_3$ such that all 40 witness points are real. This motivates the following, another open problem.

Problem 5.3. *For $N \geq 4$, determine the maximum number of real witness points for $\mathcal{SE}_N$.*

To help understand the odd and even cases, we explicitly consider Theorem 5.1 for $N = 1$ and $N = 2$.

Example 5.4. For $N = 1$, we have

$$\mathcal{SO}_2 = \left\{ \begin{bmatrix} s & -c \\ c & s \end{bmatrix} \;\middle|\; s^2 + c^2 = 1 \right\},$$
$$\mathcal{SRT}_1 = \{(x, y, 1) \mid x + y = 0\}, \text{ and } \mathcal{SE}_1 = \{(r, x, y, 1) \mid x + y = 2r + x^2 = 0\}.$$

Clearly, $\dim \mathcal{SO}_2 = \dim \mathcal{SRT}_1 = \dim \mathcal{SE}_1 = 1$ with $\deg \mathcal{SO}_2 = \deg \mathcal{SE}_1 = 2 \cdot \deg \mathcal{SRT}_1 = 2$. For the embedding $\mathbb{C}^4 \hookrightarrow \mathbb{P}^4$ where $\alpha \mapsto [1, \alpha]$, the Zariski closure of $\mathbb{C} \times \mathcal{SRT}_1$ is

$$\{[h, r, x, y, h] \mid x + y = 0\} \subset \mathbb{P}^4.$$

Hence, the set of points at infinity (corresponding with $h = 0$) is the projective line

$$\{[0, r, x, y, 0] \mid x + y = 0\} \subset \mathbb{P}^4.$$

In particular, $x^T x$ is generically nonzero at infinity in the Zariski closure of $\mathbb{C} \times \mathcal{SRT}_1$ in $\mathbb{P}^4$ so that the hypersurface $2r + x^T x = 0$ intersects $\mathbb{C} \times \mathcal{SRT}_1$ transversely.

Example 5.5. For $N = 2$, we have

$$\mathcal{SO}_3 = \{M \in \mathbb{C}^{3\times 3} \mid M^T M = I, \det M = 1\},$$
$$\mathcal{SRT}_2 = \left\{ \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} m_{11} & m_{12} \\ -m_{12} & m_{11} \end{bmatrix} \right) \;\middle|\; \begin{array}{c} m_{11}^2 + m_{12}^2 - 1 = m_{11}x_1 + m_{12}x_2 + y_1 \\ = m_{12}x_1 - m_{11}x_2 - y_2 = 0 \end{array} \right\},$$
$$\text{and } \mathcal{SE}_2 = \{(r, x, y, M) \in \mathbb{C} \times \mathcal{SRT}_2 \mid 2r + x_1^2 + x_2^2 = 0\}.$$

One can verify that $\dim \mathcal{SO}_3 = \dim \mathcal{SRT}_2 = \dim \mathcal{SE}_2 = 3$ with $\deg \mathcal{SO}_3 = 2 \cdot \deg \mathcal{SRT}_2 = 8 > 6 = \deg \mathcal{SE}_2$. To understand the reason for this drop in degree, we consider the embedding $\mathbb{C}^9 \hookrightarrow \mathbb{P}^9$ where $\alpha \mapsto [1, \alpha]$. The Zariski closure of $\mathbb{C} \times \mathcal{SRT}_2$ is

$$\left\{ \left[h, r, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} m_{11} & m_{12} \\ -m_{12} & m_{11} \end{bmatrix}\right] \;\middle|\; \begin{array}{l} m_{11}^2 + m_{12}^2 - h^2 = x_1^2 + x_2^2 - y_1^2 - y_2^2 \\ = m_{11}x_1 + m_{12}x_2 + hy_1 = m_{12}x_1 - m_{11}x_2 - hy_2 \\ = m_{11}y_1 - m_{12}y_2 + hx_1 = m_{12}y_1 + m_{11}y_2 + hx_2 = 0 \end{array} \right\} \subset \mathbb{P}^9.$$

The set of points at infinity (corresponding with $h = 0$) has dimension 3 with three irreducible components:

$$\left\{ \left[0, r, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, 0\right] \mid x_1^2 + x_2^2 = y_1^2 + y_2^2 \right\}, \left\{ \left[0, r, x_1 \cdot \begin{bmatrix} 1 \\ \pm i \end{bmatrix}, y_1 \cdot \begin{bmatrix} 1 \\ \mp i \end{bmatrix}, m_{11} \cdot \begin{bmatrix} 1 & \pm i \\ \mp i & 1 \end{bmatrix}\right] \right\},$$

where $i = \sqrt{-1}$. Since the two linear irreducible components at infinity have $x^T x = 0$, these two will yield two lines contained in the hypersurface $h = 0$ when intersecting the Zariski closure of $\mathbb{C} \times \mathcal{SRT}_2$ in $\mathbb{P}^9$ with the hypersurface $2rh + x^T x = 0$. Hence, we have $\deg \mathcal{SE}_2 = 2 \cdot \deg \mathcal{SRT}_2 - 2 = 6$.

There are two vectors and two matrices of interest in Example 5.5, namely

$$u = \begin{bmatrix} 1 \\ i \end{bmatrix}, \; \overline{u} = \begin{bmatrix} 1 \\ -i \end{bmatrix}, \; B = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}, \text{ and } \overline{B} = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}. \tag{5.5}$$

These, together with the following lemma, are used in the proof of Theorem 5.1 presented in section 5.1.

Lemma 5.6. *For B and $\overline{B}$ as in (5.5), the following hold:*

1. *there does not exist $M \in \mathcal{SO}_2$ such that $B \cdot M = M \cdot \overline{B}$;*
2. *there exists $M \in \mathbb{C}^{2\times 2}$ with $M^T M = I$ and $\det(M) = -1$ such that $B \cdot M = M \cdot \overline{B}$; and*
3. *there exists $M \in \mathcal{SO}_3$ such that*

$$\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \cdot M = M \cdot \begin{bmatrix} \overline{B} & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. For $M \in \mathcal{SO}_2$, we can write

$$M = \begin{bmatrix} m_{11} & m_{12} \\ -m_{12} & m_{11} \end{bmatrix},$$

where $m_{11}^2 + m_{12}^2 = 1$. If we ignore this quadratic condition, $B \cdot M = M \cdot \overline{B}$ is equivalent to $m_{11} = m_{12} = 0$. Hence, no such $M \in \mathcal{SO}_2$ can exist.

For the second and third statements, it is easy to verify that

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

satisfy the requirements, respectively. ■

5.1. Proof of Theorem 5.1. Let $N \geq 1$. We first address the equality of dimensions. By comparing (5.2) and (5.3), it is clear that $\dim \mathcal{SRT}_N = \dim \mathcal{SE}_N$. Moreover, it is well known that $\dim \mathcal{SO}_N = \frac{1}{2}N(N-1)$, and thus

$$\dim \mathcal{SRT}_N = \dim \mathcal{SE}_N = \dim \mathcal{SO}_N + N = \frac{1}{2}N(N-1) + N = \frac{1}{2}(N+1)N = \dim \mathcal{SO}_{N+1}.$$

We now turn to degrees and assume that $(x, y, M) \in \mathcal{SRT}_N$ and $\mathcal{P} \in \mathcal{SO}_{N+1}$ are generic. Define

$$\mathcal{M} = \begin{bmatrix} M & y \\ x^T & 1 \end{bmatrix}.$$

Let $\mathcal{M}_{1:N}$ and $\mathcal{P}_{1:N}$ be the $N \times (N+1)$ matrices corresponding to the first N rows of $\mathcal{M}$ and $\mathcal{P}$, respectively, and let $\mathcal{M}_{N+1}$ and $\mathcal{P}_{N+1}$ be the last rows of $\mathcal{M}$ and $\mathcal{P}$, respectively. We know that the row spans of $\mathcal{M}_{1:N}$ and $\mathcal{P}_{1:N}$ are both an N-dimensional linear space in $\mathbb{C}^{N+1}$ with $\mathcal{M}_{1:N} \cdot \mathcal{M}_{N+1}^T = \mathcal{P}_{1:N} \cdot \mathcal{P}_{N+1}^T = 0$. In particular, it follows that $\mathcal{M}_{N+1}$ and $\mathcal{P}_{N+1}$ are each uniquely defined given $\mathcal{M}_{1:N}$ and $\mathcal{P}_{1:N}$, respectively.

We now consider the vector in Plücker coordinates corresponding with $\mathcal{M}_{1:N}$ and $\mathcal{P}_{1:N}$, namely $v_{\mathcal{M}}$ and $v_{\mathcal{P}}$, respectively. Since $\det M = 1$, we can write

$$v_{\mathcal{M}} = [1 \ v_1 \ \cdots \ v_N]^T.$$

Since $\mathcal{P}_{1:N} \cdot \mathcal{P}_{1:N}^T = I$, we have $v_{\mathcal{P}}^T v_{\mathcal{P}} = 1$. Hence, the first N rows of $\mathcal{M}$ correspond with vectors in the Plücker embedding defined on the affine coordinate patch where the first coordinate is 1. In addition, the first N rows of $\mathcal{P}$ correspond with vectors in the Plücker embedding for which $v^T v = 1$. In particular, due to the selection of coordinate patches (linear versus quadratic), there is a generically 1-to-2 relationship between $\mathcal{M}_{1:N}$ and $\mathcal{P}_{1:N}$ which yields a generically 1-to-2 relationship between $\mathcal{M}$ and $\mathcal{P}$ so that

$$\deg \mathcal{SRT}_N = \frac{1}{2} \cdot \deg \mathcal{SO}_{N+1}.$$

By (5.2) and (5.3), we see that $\mathcal{SE}_N$ is obtained by intersecting $\mathbb{C} \times \mathcal{SRT}_N$ with the degree 2 hypersurface defined by $2r + x^T x = 0$. Hence, Bézout's theorem yields

$$\deg \mathcal{SE}_N \leq 2 \cdot \deg \mathcal{SRT}_N = \deg \mathcal{SO}_{N+1}. \tag{5.6}$$

We will show that this is an equality for odd N and a strict inequality for even N.

Assume that $N = 2k+1$. Following Example 5.4, we show that $x^T x$ is generically nonzero at infinity for $\mathcal{SRT}_N$ so that the intersection of $\mathbb{C} \times \mathcal{SRT}_N$ with the hypersurface $2r + x^T x = 0$ is transverse, i.e., equality holds in (5.6). For every $M \in \mathcal{SO}_N$, there exists $O \in \mathbb{C}^{N \times N}$ with $O^T O = I$ such that

$$O^T M O = \begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \ddots & & \\ & & & A_k & \\ & & & & 1 \end{bmatrix},$$

where each $A_i \in \mathcal{SO}_2$. In particular, the points at infinity of $\mathcal{SO}_N$ lie in the Zariski closure of matrices of the form

$$P = O \begin{bmatrix} \gamma_1 \cdot B_1 & & & & \\ & \gamma_2 \cdot B_2 & & & \\ & & \ddots & & \\ & & & \gamma_k \cdot B_k & \\ & & & & 0 \end{bmatrix} O^T,$$

where $\gamma_j \in \mathbb{C}$ and $B_j \in \{B, \overline{B}\}$ with B and $\overline{B}$ as in (5.5). By Lemma 5.6(3), we can assume without loss of generality that $B_j = B$ so that the set of points at infinity for $\mathcal{SO}_N$ is irreducible.

Affinely, belonging to $\mathcal{SRT}_N$ implies $Mx+y = 0$. Thus, if we write $x = O\cdot\widetilde{x}$ and $y = O\cdot\widetilde{y}$, then $\widetilde{x}_N + \widetilde{y}_N = 0$, which must also hold at infinity. Since we also need $Px = 0$, $P^T y = 0$, and $x^T x = y^T y$ at infinity, we must have $x = O \cdot (\alpha_1 \cdot x_1 + \cdots + \alpha_k \cdot x_k + \alpha_{k+1} \cdot e_N)$ and $y = O \cdot (\beta_1 \cdot y_1 + \cdots + \beta_k \cdot y_k - \alpha_{k+1} \cdot e_N)$ for any $\alpha_j, \beta_j \in \mathbb{C}$, where

$$x_1 = [u^T, 0, \ldots, 0]^T, \ldots, x_k = [0, \ldots, 0, u^T, 0]^T, \; y_1 = [\overline{u}^T, 0, \ldots, 0]^T, \ldots, y_k = [0, \ldots, 0, \overline{u}^T, 0]^T,$$

where u and $\overline{u}$ as in (5.5). In particular, we know that $x^T x = y^T y = \alpha_{k+1}^2$ is generically nonzero as required.

All that remains to be shown for the odd case is that no other components for $\mathcal{SRT}_N$ can arise at infinity of projective dimension equal to $\dim \mathcal{SRT}_N - 1$ by taking some $\gamma_j = 0$, e.g., as in Example 5.5. If, without loss of generality, $\gamma_k = 0$, then suppose $x = O \cdot (\alpha_1 \cdot x_1 + \cdots + \alpha_{k-1} \cdot x_{k-1} + \alpha_k \cdot e_{N-2} + \alpha_{k+1} \cdot e_{N-1} + \alpha_{k+2} \cdot e_N)$ and $y = O \cdot (\beta_1 \cdot y_1 + \cdots + \beta_{k-1} \cdot y_{k-1} + \beta_k \cdot e_{N-2} + \beta_{k+1} \cdot e_{N-1} + \beta_{k+2} \cdot e_N)$ with $\alpha_k^2 + \alpha_{k+1}^2 + \alpha_{k+2}^2 = \beta_k^2 + \beta_{k+1}^2 + \beta_{k+2}^2$. Hence, the total affine degrees of freedom for this setup is $(k+2) + (k+2) - 1 = 2k + 3 = N + 2$, meaning that the best case scenario is a gain of two degrees of freedom over the case above. However, the codimension of matrices P where $\gamma_k = 0$ is 3, showing that the total affine dimension is strictly smaller than $\dim \mathcal{SRT}_N$.

Now we turn to the case that $N = 2k$. Following Example 5.5, we show that there exists at least one irreducible component at infinity for $\mathcal{SRT}_N$ for which $x^T x = 0$, showing that the intersection of $\mathbb{C} \times \mathcal{SRT}_N$ with the hypersurface $2r + x^T x = 0$ is not transverse, i.e., the inequality in (5.6) is strict. For every $M \in \mathcal{SO}_N$, there exists $O \in \mathbb{C}^{N\times N}$ with $O^T O = I$ such that

$$O^T M O = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{bmatrix},$$

where each $A_i \in \mathcal{SO}_2$. In particular, the points at infinity in $\mathcal{SO}_N$ lie in the Zariski closure of matrices of the form

$$P = O \begin{bmatrix} \gamma_1 \cdot B_1 & & & \\ & \gamma_2 \cdot B_2 & & \\ & & \ddots & \\ & & & \gamma_k \cdot B_k \end{bmatrix} O^T, \tag{5.7}$$

where $\gamma_j \in \mathbb{C}$ and $B_j \in \{B, \overline{B}\}$ with B and $\overline{B}$ as in (5.5). Hence, by Lemma 5.6, the set of points at infinity for $\mathcal{SO}_N$ is reducible. To focus on one irreducible component, we take $B_j = B$ for all j, which affinely has dimension equal to the dimension of $\mathcal{SO}_N$. We can then produce a family of points at infinity for $\mathcal{SRT}_N$ by taking $x = O \cdot (\alpha_1 \cdot x_1 + \cdots + \alpha_k \cdot x_k)$ and $y = O \cdot (\beta_1 \cdot y_1 + \cdots + \beta_k \cdot y_k)$, where $\alpha_j, \beta_j \in \mathbb{C}$ and

$$x_1 = [u^T, 0, \ldots, 0]^T, \ldots, x_k = [0, \ldots, 0, u^T]^T, \; y_1 = [\overline{u}^T, 0, \ldots, 0]^T, \ldots, y_k = [0, \ldots, 0, \overline{u}^T]^T,$$

with u and $\overline{u}$ as in (5.5). Hence, since $x^T x = y^T y = 0$ for such points, the only item that remains is to show that the Zariski closure of the collection of such points has projective dimension equal to $\dim \mathcal{SRT}_N - 1$. Since there are k degrees of freedom in both x and y, this collection of points has affine dimension equal to $\dim \mathcal{SO}_N + 2k = \dim \mathcal{SO}_N + N = \dim \mathcal{SRT}_N$ so that the projective dimension is $\dim \mathcal{SRT}_N - 1$ as required.

It is interesting to note that, in the even case, the set of all P with, for example, $\gamma_k = 0$, has codimension 1 with respect to when γ_k is general. Moreover, replacing $\alpha_k \cdot x_k$ and $\beta_k \cdot y_k$ with $\alpha_k \cdot e_{N-1} + \alpha_{k+1} \cdot e_N$ and $\beta_k \cdot e_{N-1} + \beta_{k+1} \cdot e_N$, respectively, where $\alpha_k^2 + \alpha_{k+1}^2 = \beta_k^2 + \beta_{k+1}^2$, yields a gain in one degree of freedom. Hence, the even cases have at least one irreducible component at infinity where $x^T x \neq 0$ generically.

6. Conclusion. The relationship between leg constraints of a Stewart–Gough platform and the Segre embedding yields a geometric class of exceptional Stewart–Gough platforms via linear dependency of point-pairs under the Segre embedding. We considered the planar-planar, planar-spatial, and spatial-spatial cases for Stewart–Gough platforms, the planar-planar and planar-spatial Stewart–Gough platforms with two degrees of freedom, and the spatial-spatial Stewart–Gough platforms that move in 4-D space. The complete classification of all exceptional Stewart–Gough platforms remains an open problem.

Since the generic number of assembly configurations of a Stewart–Gough platform is equal to the degree of a representation of the special Euclidean group in 3-D space, we also considered the degree of the special Euclidean group in other dimensions. We show that the degree of a representation of the special Euclidean group in odd dimensions is equal to the degree of the special orthogonal group in one higher dimension. For the corresponding case of special Euclidean groups in even dimensions, we give a inequality, but we leave as an open problem the determination of a sharp result for $\mathcal{SE}_N$, even $N \geq 8$.

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