More examples of closure proofs

CSE 30151

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1 Symbol substitution

We started the proof of closure under string homomorphism but (as predicted) it was quite messy and we didn't finish. Here we show that regular languages are closed under a one-for-one symbol substitution, defined as follows.

If w is a string, define $\mathsf{SUBST}_{a\to b}(w)$ to be a copy of w with all occurrences of a replaced with b. More formally,

$$\begin{aligned}
\mathsf{SUBST}_{a\to b}(\varepsilon) &= \varepsilon \\
\mathsf{SUBST}_{a\to b}(a) &= b \\
\mathsf{SUBST}_{a\to b}(\sigma) &= \sigma \qquad \text{if } \sigma \neq a \\
\mathsf{SUBST}_{a\to b}(u \circ v) &= \mathsf{SUBST}_{a\to b}(u) \circ \mathsf{SUBST}_{a\to b}(v).
\end{aligned}$$

Claim If L is a regular language, then $\mathsf{SUBST}_{a\to b}(L)$ is a regular language.

Proof idea For every transition on symbol a, replace it with a transition on symbol b.

Proof Let $M = (Q, \Sigma, \delta, q_0, F)$ be a NFA that recognizes L. Construct $M' = (Q, \Sigma', \delta', q_0, F)$, where

$$\begin{aligned}
\Sigma' &= \Sigma \setminus \{a\} \cup \{b\} \\
\delta'(q,\sigma) &= \begin{cases} \delta(q,\sigma) \cup \delta(q,a) & \text{if } \sigma = b \\ \delta(q,\sigma) & \text{otherwise} \end{cases} \\
\delta'(q,\varepsilon) &= \delta(q,\varepsilon).
\end{aligned}$$

Clearly, M' recognizes $\mathsf{SUBST}_{a\to b}(L)$, so $\mathsf{SUBST}_{a\to b}(L)$ is regular.

2 Single accept state

Claim Every NFA can be converted into an equivalent NFA with exactly one accept state.

Proof idea Add a new accept state and add ε-transitions from the original accept states to the new accept state.

Proof Construct $M' = (Q \cup \{q_{\text{accept}}\}, \Sigma, \delta', q_0, \{q_{\text{accept}}\})$, where

$$\delta'(q, \sigma) = \delta(q, \sigma)$$
$$\delta'(q, \varepsilon) = \begin{cases} \delta(q, \varepsilon) \cup \{q_{\text{accept}}\} & \text{if } q \in F \\ \delta(q, \varepsilon) & \text{otherwise.} \end{cases}$$

Clearly, M' recognizes exactly the same strings that M does, and has exactly one accept state q_{accept}.

3 Reversal

Claim If L is a regular language, so is L^R.

Proof idea Reverse the direction of every transition. Add a new start state with ε-transitions to the original accept states. Make the original start state into an accept state.

Proof Construct $M' = (Q \cup \{q_{\text{start}}\}, \Sigma, \delta', q_{\text{start}}, \{q_0\})$, where

$$\delta'(q, \sigma) = \{r \in Q \mid q \in \delta(r, \sigma)\}$$
$$\delta'(q, \varepsilon) = \begin{cases} \{r \in Q \mid q \in \delta(r, \varepsilon)\} & \text{if } q \in Q \\ F & \text{if } q = q_{\text{start}}. \end{cases}$$

Clearly, M' recognizes L^R, so L^R is regular.