Homework 9: NP-completeness

CSE 30151 Spring 2016

Due 2016/04/26

Instructions

Please note that you will **lose one point** if you don't follow these instructions.

- You can prepare your solutions however you like, but you must submit them as a single PDF file.
- Please name your PDF `netid-hw9.pdf`, where `netid` is replaced with your NetID, or `netid-hw9-1234.pdf`, where `1234` is replaced with the problems you are submitting.
- If you use the same name twice, only the most recent version will be graded!
- Submit your PDF file in Sakai. Don't forget to click the Submit (or Resubmit) button!

Problems

A *two-dimensional TM* (2dTM) has, in place of a tape, a sheet of graph paper that has a top and left edge but extends infinitely to the right and down. Initially the head is at the upper left corner. The transitions are as in a standard TM, except that there are four directions: L (left), R (right), U (up), and D (down).

1. Describe how to simulate a 2dTM using a TM that has one or more one-dimensional tapes, in three parts:

   (a) Describe how a two-dimensional sheet with a head can be represented as one or more one-dimensional tapes with heads. You can use the construction we sketched in class or your own.

   (b) Simulating reading and writing symbols is presumably trivial, but if not, give implementation-level descriptions.

   (c) Give implementation-level descriptions of how to simulate moving left, right, up, and down.

2. Let n be the length of the input string. Choose a k and show that if a computation takes $t(n)$ steps in a 2dTM, it takes, in the worst case, $O((t(n)^k)$ steps in a standard TM (single one-way infinite tape). Your algorithm doesn't have to be optimal and your analysis doesn't have to be tight.

3. [Problem 7.39 in 2nd ed., 7.41 in 3rd ed.] In the proof of the Cook-Levin theorem, a window is a 2×3 rectangle of cells. Show why the proof would have failed if we had used 2×2 windows instead. Your answer should include:

   (a) An example NTM N.

   (b) Two configurations of N (call them upper and lower) such that all of the following hold:

   - The upper configuration is legal.
   - The lower configuration either is illegal or doesn't follow the upper configuration according to N's rules.
   - Every window of the two configurations (when the upper is stacked on top of the lower) is legal according to N's rules.

4. In class we saw a puzzle called TetraVex that is NP-complete.[1] In this problem, we'll prove the NP-completeness of a simpler version of the puzzle:

   - You are given a set of tiles, each with a north, south, east, and west label. You can make as many copies as you want of each tile. You can't rotate the tiles.
   - You are also given a rectangular frame with labels.
   - The object is to fill the frame with tiles such that all abutting labels match.

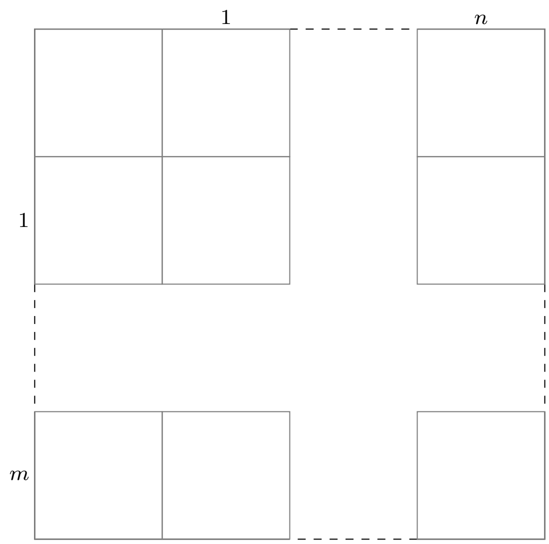
   The proof is by reduction from 3SAT. Let $x_1, \ldots, x_n$ be a set of variables, and let ϕ be a formula in 3CNF,

   $$\phi = (\phi_{11} \vee \phi_{12} \vee \vee \phi_{13}) \wedge \cdots \wedge (\phi_{m1} \vee \phi_{m2} \vee \phi_{m3})$$

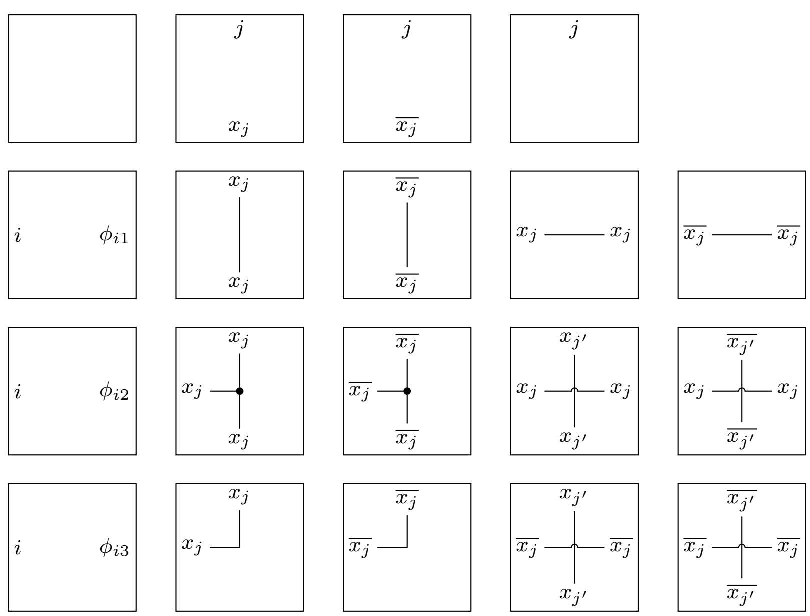
   where each $\phi_{i1}, \phi_{i2}, \phi_{i3}$ is either x_j or $\overline{x_j}$ for some j.

[1] `http://arxiv.org/pdf/0903.1147v1.pdf`

Define a function f that converts ϕ into an instance of the puzzle as follows. The frame is:



The tiles are, for all $i \in \{1, \ldots, m\}$ and $j, j' \in \{1, \ldots n\}$:



Notes:

- Each $\phi_{i1}, \phi_{i2}, \phi_{i3}$ is equal to either x_j or $\overline{x_j}$ for some j.
- Blank edges only match other blank edges.
- The "wires" drawn in the interior of some tiles are merely suggestive; they don't affect matching at all.

Prove the following statements. Your answers don't have to be very formal, but try to make them clear.

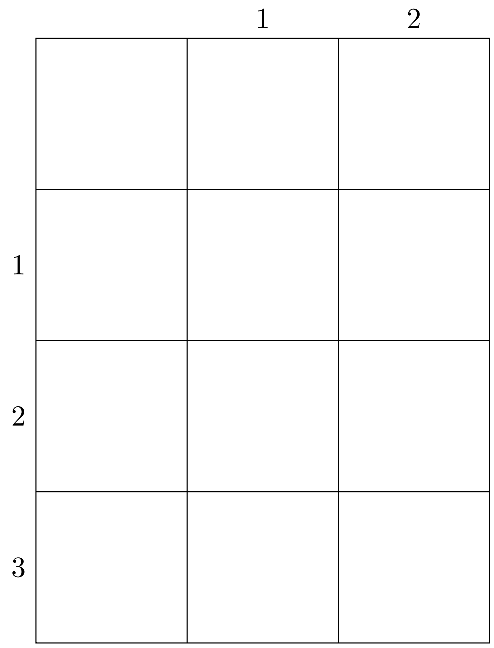
(a) The set of solvable puzzles is in NP.

(b) f is computable in polynomial time.

(c) If ϕ is satisfiable, then the corresponding puzzle $f(\phi)$ is solvable.

(d) If the puzzle $f(\phi)$ is solvable, then ϕ is satisfiable.

For the formula $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$, the frame would be



And the tiles would be