Math 10120, Spring 2016. Exam 1 solutions

Multiple Choice

1. (5 pts.) Let $\mathcal{U}$ be the universal set
$$\mathcal{U} = \{0, 1, 2, 3, 4, 5, 6\}.$$
If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{2, 4, 5\}$, then $(A \cup B) \cap C' =$

(a)   ANS: $\{1, 3\}$
(b) $\{2, 4\}$
(c) $\{3\}$

(d) $\{1, 2, 3, 4\}$
(e) $\{5\}$

Solution: $A \cup B = \{1, 2, 3, 4\}$ and $C' = \{0, 1, 3, 6\}$, so $(A \cup B) \cap C' = \{1, 3\}$.

2. (5 pts.) A code consists of AT LEAST 4 symbols without repetition from the set of symbols
$$\mathcal{S} = \{\$, \circ, \#, \&, !, \%\}.$$
The total number of codes possible is

(a)   ANS: $\mathbf{P}(6, 4) + \mathbf{P}(6, 5) + \mathbf{P}(6, 6)$
(b) $\mathbf{C}(6, 4)$

(c) $\mathbf{P}(6, 4)$
(d) $\mathbf{C}(6, 4) + \mathbf{C}(6, 5) + \mathbf{C}(6, 6)$

(e) $\mathbf{P}(6, 4) \cdot \mathbf{P}(6, 5) \cdot \mathbf{P}(6, 6)$

Solution: Order matters in code, so problem is about permutations. There are $\mathbf{P}(6, 4)$ permutations of the six symbols taken 4 at a time, $\mathbf{P}(6, 5)$ permutations of the six symbols taken 5 at a time and $\mathbf{P}(6, 6)$ permutations of the six symbols taken 6 at a time. We choose one of these options, so by addition principle total number of possibilities is $\mathbf{P}(6, 4) + \mathbf{P}(6, 5) + \mathbf{P}(6, 6)$.

3. (5 pts.) At Tubway Sandwich shop there are 4 kinds of bread, 3 kinds of meat, 3 kinds of cheese and 5 kinds of vegetable. You can make your own sandwich by choosing 1 bread, 1 type of meat, 1 kind of cheese and 2 kinds of vegetable. How many different sandwiches are possible?

(a)   ANS: 360
(b)   720
(c)   180

(d)   90
(e)   20

Solution: First choose a bread, THEN (whatever bread was chosen) choose a meat, THEN a cheese, THEN veggies. Multiplication principle tells us the multiply the number of possibilities at each stage. Already know 4 kinds of bread, 3 kinds of meat, 3 kinds of cheese. For veggies, we must choose 2 from 5, order not mattering, so number of possibilities is $\mathbf{C}(5, 2) = 10$. Total is then $4 \cdot 3 \cdot 3 \cdot 10 = 360$.

4. (5 pts.) How many 3 digit numbers greater than 200 can be made from the set of numbers
$$\{1, 2, 3, 7, 8\},$$
where numbers are NOT allowed to repeat?

(a)   ANS: 48   (b)   11   (c)   36

(d)   125   (e)   100

Solution: To make sure number is greater than 200, must choose one of 2,3,7,8 first digit (4 options). Whichever we choose, any of the remaining 4 are good for the second digit, and then any of the remaining 3 are good for the third digit. By multiplication principle, total number of possibilities is $4 \cdot 4 \cdot 3 = 48$.

5. (5 pts.) Compute $\mathbf{C}(10, 6) \cdot 4!$

(a)   ANS: 5,040   (b)   840   (c)   604,800

(d)   3,628,800   (e)   120,960

Solution:
$$\mathbf{C}(10, 6) \cdot 4! = \frac{10!}{6!4!} \cdot 4! = \frac{10!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7 = 5040.$$

6. (5 pts.) A committee of 3 people needs to be chosen from among 4 men and 5 women. How many different committees can be formed that include exactly 1 women?

(a)   ANS: 30   (b)   60   (c)   12

(d)   80   (e)   100

Solution: 5 options for the women on the committee. Whichever woman we pick, there are 2 slots left to fill from among the 4 men, which can be done in $\mathbf{C}(4, 2) = 6$ ways. By multiplication principle, total number of possibilities is $5 \cdot 6 = 30$.

7. (5 pts.)   In the cellar of the Snite museum there are seven recently discovered Monet paintings, six Van Goghs, and ten Picassos. In how many ways can the museum curator choose a set of three paintings to display upstairs, if all three should be by the same artist?

(a)   ANS: $\mathbf{C}(7, 3) + \mathbf{C}(6, 3) + \mathbf{C}(10, 3)$ (b) $\mathbf{C}(23, 3) \cdot 3!$

(c) $\mathbf{P}(7, 3) + \mathbf{P}(6, 3) + \mathbf{P}(10, 3)$ (d) $\mathbf{P}(7, 3) \cdot \mathbf{P}(6, 3) \cdot \mathbf{P}(10, 3)$

(e) $\mathbf{C}(7, 3) \cdot \mathbf{C}(6, 3) \cdot \mathbf{C}(10, 3)$

Solution: Order]bf does not matter here (we are selecting a **set** of paintings), so problem is about combinations. There are $\mathbf{C}(7, 3)$ combinations of the seven Monet paintings taken 3 at a time, $\mathbf{C}(6, 3)$ combinations of the six Van Gogh paintings taken 3 at a time, and $\mathbf{C}(10, 3)$ combinations of the ten Picasso paintings taken 3 at a time. We must choose one of these options, so by addition principle total number of possibilities is $\mathbf{C}(7, 3) + \mathbf{C}(6, 3) + \mathbf{C}(10, 3)$.

8. (5 pts.) There are 15 American League baseball teams and 15 National League teams. During a month of inter-league play, each team from one league should play all the teams from the other league exactly twice. How many inter-league games are played in all?

(a)   ANS: 450   (b)   225   (c)   60

(d)   120   (e)   435

    Solution: To specify a match-up between an American League team and a National League team, we first select one of the 15 American League teams and then one of the 15 National League teams. By multiplication principle there are $15 \cdot 15 = 225$ such match-ups. Each mathch-up leads to two games, so $2 \cdot 225 = 450$ games in all.

9. (5 pts.) I have a combinatorics class with twelve students. I have four different final projects in mind for the class, and I want to split the class into four groups, each of size three, one group to do the first project, one to do the second, one to do the third, and one to do the fourth. I also want to select a group leader for each group. In how many ways can I do all this?

(a)   ANS: $\binom{12}{3,3,3,3} \cdot 3^4$ (b) $\binom{12}{3,3,3,3}/4!$ (c) $\binom{12}{3,3,3,3} \cdot 4!$

(d) $4\frac{12!}{3!3!3!3!}$ (e) $\frac{12!}{3!3!3!3!}$

    Solution: This begins as an ordered partition problem: I want to split the class into four groups, each of size 3, with specific tasks in mind for each group, so an order of the parts of the partition. There are $\binom{12}{3,3,3,3}$ options for this part of the process. There are 3 ways to choose a group leader for group 1, 3 for group 2, 3 for group 3 and 3 for group 4. These are done in sequence (I do all of them, one after another, rather than choosing just one of them to do), so the number of options for this part of the process is, by multiplication principle, $3 \cdot 3 \cdot 3 \cdot 3 = 3^4$. Finally, again by multiplication principle, the number of options for the whole process is $\binom{12}{3,3,3,3} \cdot 3^4$.

10. (5 pts.) 24 couples go to a dance (so 48 people in total). At some point the DJ wants to choose 6 people to help him plan the next few tracks on his playlist, but he doesn't want any two people who are a couple to be among the six. In how many different ways can he choose the six people?

(a)   ANS: $\mathbf{C}(24, 6) \cdot 2^6$ (b) $\mathbf{C}(24, 6) \cdot 2^{24}$ (c) $\mathbf{C}(24, 6) \cdot 6^2$

(d) $\mathbf{C}(48, 6) - 2^6$ (e) $\mathbf{C}(48, 6)/24$

    Solution: One way the DJ can make his selection is by first choosing six of the couples ($\mathbf{C}(24, 6)$ options) and then choosing one person from each of the chosen couples (by multiplication principle, $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$ options). Again by multiplication principle, the total number of options is $\mathbf{C}(24, 6) \cdot 2^6$

Partial Credit

You must show **all of your work** on the partial credit problems to receive full credit! Make sure that your answer is **clearly** indicated. You're more likely to get partial credit for a wrong answer if you explain your reasoning.

11. (10 pts.) In a survey where 100 students reported which subjects they like, 32 students in total liked Mathematics, 38 students liked Business and 30 students liked Literature. Moreover, 10 students liked both Mathematics and Business, 8 students liked both Business and Literature, and 7 students liked both Mathematics and Literature. 5 students liked all three subjects.
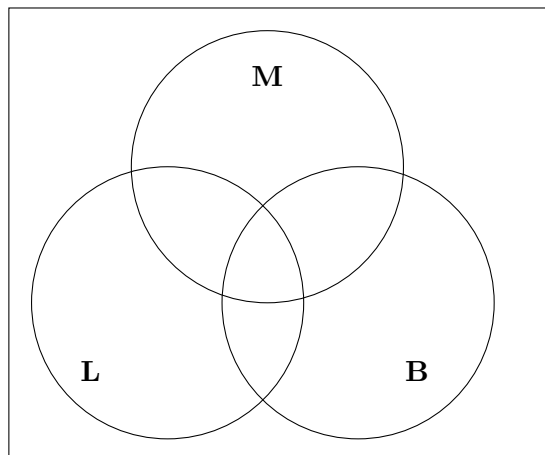
(a) Find the number of people who liked exactly 1 subject.

Solution: 5 liked all three, so that number goes into the center chamber of the Venn diagram.

7 liked both M and L, so "2" (= 7 − 5) goes into the chamber immediately above and left of the center.

8 liked both B and L, so "3" (= 8 − 5) goes into the chamber immediately below the center.

10 liked both M and B, so "5" (= 10 − 5) goes into the chamber immediately above and right of the center.

32 liked M, of which 5 + 2 + 5 = 12 have already been accounted for, so "20" (= 32 − 12) goes into the chamber that has the letter M.

38 liked B, of which 5 + 3 + 5 = 13 have already been accounted for, so "25" (= 38 − 13) goes into the chamber that has the letter B.

30 liked L, of which 5 + 2 + 3 = 10 have already been accounted for, so "20" (= 30 − 10) goes into the chamber that has the letter L.

This accounts for 5 + 2 + 3 + 5 + 20 + 25 + 20 = 80 of the 100 students; so the remaining 20 belong outside all three sets in the center of the Venn diagram.

The number that liked exactly one subject is 20 (the number who liked M only) + 25 (the number who liked B only) + 20 (the number who liked L only) = 65.

(b) How many students did not like any of the subjects?

Solution: As determined above, 20 students liked none of the three.

12. (10 pts.) Bob, John, Andrew, Jessica and Valentina wants to take a photo of themselves where they stand side-by-side.

(a) How many different photos are possible?

Solution: Permutation problem (order matters). Answer is **P**(5, 5) or 5! or 120.

(b) Jessica and Valentina are BFFs, and they want to stand next to each other in the photo. How many different photos are possible with Jessica and Valentina standing next to each other?

Solution: Treating Jessica and Valentina as a single unit (Jessantina), there are now 4 friends to be arranged, so **P**(4, 4) ways. But once the 4 have been arranged, there are a further 2 ways to arrange

Jessantina (Jessica to the left, Valentina to the right, or vice-versa). By the multiplication principle the number of arrangements is $\mathbf{P}(4,4) \cdot 2 = 48$.

13. (10 pts.) For the following problem you do not need to simply your answers, i.e. you may express your answers using the symbols for permutations ($\mathbf{P}(n,r)$), combinations ($\mathbf{C}(n,r)$) or factorials ($n!$). The following is part of the city map of Gridville, Iowa.
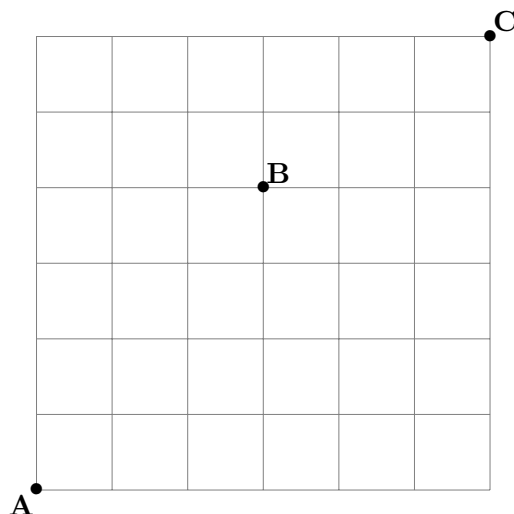
(a) If one only travels east (i.e. to the right) or north (i.e. up), how many paths are there from $\mathbf{A}$ to $\mathbf{C}$?

  Solution: Need to take 6 steps east (E) and 6 steps north (N), so the number of ways is the number of re-arrangements of the letters E,E,E,E,E,E,N,N,N,N,N,N. This is 12! (for arranging the letters as if they were distinguishable) divided by 6!6! (the overcount factor caused by the fact that the 6 E's in any re-arrangement can be themselves re-arranged in 6! ways without changing the re-arrangement, while at the same time the 6 S's can also be re-arranged in 6! ways), or

$$\frac{12!}{6!6!} = \mathbf{C}(12,6).$$

(b) How many paths from $\mathbf{A}$ to $\mathbf{C}$ (again only traveling east or north) **do not** go through $\mathbf{B}$?

  Solution: It's easier to first count how many paths from $\mathbf{A}$ to $\mathbf{C}$ **do** go through $\mathbf{B}$. To count this note that to go from $\mathbf{A}$ to $\mathbf{C}$ through $\mathbf{B}$ we must first go from A to B (7!/3!4! ways, by the same reasoning as in the last part), and then go from B to C (5!/2!3! ways, by the same reasoning as in the last part), so by the multiplication principle the number of paths from $\mathbf{A}$ to $\mathbf{C}$ through $\mathbf{B}$ is the product of these numbers.

  To find how many paths from $\mathbf{A}$ to $\mathbf{C}$ **not** through $\mathbf{B}$, we subtract what we have just calculated from the total number of paths, calculated in the first part, to get a final answer of

$$\frac{12!}{6!6!} - \frac{7!}{3!4!}\frac{5!}{3!2!}.$$

14. (10 pts.) I have an ordinary coin, that comes up either Heads or Tails each time I toss it. I plan to toss it 5 times in a row, and record the sequence of heads and tails that I get.
(a) How many possible sequences are there?

Solution: There are two choices for the first entry in the sequence (H or T), and then two for the second (again, H or T), and then two for the third, and then two for the fourth, and then two for the fifth. By the multiplication principle, the total number of options for the sequence is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$.

(b) In how many of those sequences counted in part (a) are the last three tosses all Heads?

Solution: Now there are still two choices for the first entry in the sequence (H or T), and then still two for the second (again, H or T), but then only one for the third (H), and then only one for the fourth (H), and then only one for the fifth (H). By the multiplication principle, the total number of options for the sequence is $2 \cdot 2 \cdot 1 \cdot 1 \cdot 1 = 4$.

(c) In how many of those sequences counted in part (a) are EITHER the first three tosses all Heads OR the last three tosses all Heads?

Solution: Let A be the set of all sequences in which the first three tosses are all Heads, and let B be the set of all sequences in which the last three tosses are all Heads. We have calculated $n(B) = 4$ and by the same reasoning $n(A) = 4$. We want to know what is $n(A \cup B)$ ($A \cup B$ is the set of sequences with EITHER the first three tosses all Heads OR the last three tosses all Heads). By inclusion-exclusion,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Now $A \cap B$ is the set of sequences with BOTH the first three tosses all Heads AND the last three tosses all Heads. There is only one such sequence — HHHHH. So $n(A \cap B) = 1$ and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 4 + 4 - 1 = 7.$$

15. (10 pts.) 12 teams enter a volleyball tournament. In the first round, the teams are paired off in six pairs to play each other.

(a) How many different pairings are possible if neither the order of the pairings nor the order of teams within each pairing matters? (for this part, calculate out your answer as a number)

Solution: The task here is to take a set of size 12 (the 12 teams) and partition it up into 6 blocks, each of size 2 (these will be the six pairs of teams that play each other). We are told that the order of the blocks doesn't matter, and nor does the order within the blocks. So this is an **unordered partition problem**, and the solution is

$$\frac{1}{6!} \binom{12}{2, 2, 2, 2, 2, 2} = 10,395.$$

The $\binom{12}{2,2,2,2,2,2}$ here counts the **ordered** partition problem of selecting a first pair of teams, then a second pair, etc.; dividing by 6! (the number of ways or re-arranging the 6 pairs) removes the order.

(b) Assume that in each matchup, one team is assigned to be the home team and the other is assigned to be the away team (this is, the order within each pairing matters), and also the six first round matches are to be run one after another. How many different pairings are possible, now that **both** the order within each pairing, **and** the order of the pairings, matters? (for this part, an answer involving permutation, combination, partition and/or factorial symbols is fine)

Solution: Because we now care about the order in which the pairs of teams are selected, we are dealing with an **ordered partition problem**. There are

$$\binom{12}{2, 2, 2, 2, 2, 2}$$

ordered partitions of the 12 twelve teams into 6 pairs. But we must also choose, inside each pair, one team to be the home team. We have 6 choices to make, one after another, each from a set of size 2, so

by the multiplication principle there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$ possible choices. Finally, again by the multiplication principle, the total number of possibilities is the product of these two numbers:

$$\binom{12}{2,2,2,2,2,2} \cdot 2^6.$$