Math 30530: Introduction to Probability, Fall 2011

Final Exam

Monday December 12

Name: ____________________________ **Instructors:** David Galvin & Chuck Stanton

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This exam contains 10 problems on 12 pages (including the front cover, and a standard normal table at the back). Calculators may be used, but no books or notes. **Show all your work** on the paper provided. The honor code is in effect for this exam.

Scores

Question	Score	Out of
1		15
2		15
3		15
4		15
5		15
6		15
7		15
8		15
9		15
10		15
Total		150

GOOD LUCK !!!

1. There are twelve parking spaces in a row outside the main building - reserved (in no particular order) for Provost Burrish and his eleven associate provosts.

   (a) One day, six of the twelve drive to work, and when each one arrives they choose a random empty parking spot to park in. When the sixth person arrives, what is the probability that he finds that the spots at both ends are free?

   (b) On another day, eight of the twelve drive, and again they each choose a random empty parking spot to park in. What is the probability that after all eight have parked, there are four consecutive free parking spaces?

2. I buy three bags of holiday themed maltballs from the South Bend Chocolate Company. By the time I get home with them, all that's left in bag I is 3 red maltballs and 2 green ones; all that's left in bag II is 2 red and 1 green, and all that's left in bag III is 1 red and 3 green. I give the three bags to my wife.

   (a) If she selects one maltball from each bag, what is the probability that she selects three red balls?

   (b) If she instead selects one bag at random, and selects a ball at random from it, what is the probability that she selects a red ball?

   (c) It turns out that she did select a red ball in part b) above. Given that information, what is the probability that she made her selection from bag 1?

3. There are three hiking trials that lead up to the summit of Hoosier hill, Indiana's highest point. On a typical Saturday afternoon in summer, hikers arrive at the summit from the (easy) Colfax trail at a rate of one every 12 minutes, from the (moderate) LaSalle trail on average once an hour, and from the (strenuous) Marquette trail on average once every 90 minutes. Hikers in Indiana are rugged individualists, and trek independently of each other. One Saturday at noon I take myself up to the summit to watch the hikers arrive.

   (a) Using an appropriate random variable to model the situation, calculate the probability that over the course of an hour I see no more than 8 hikers arrive from the Colfax trail. (You may leave your answer in the form of a sum.)

   (b) What is the probability that over the course of two hours, I see no more than 8 hikers reaching the summit in total (from all three trails)? (You may leave your answer in the form of a sum.)

   (c) What is the probability that in exactly three of the five hours that I am at the summit (noon to 1pm, 1 to 2pm, etc.), I see at least one hiker arriving at the summit from the Marquette trail? (You need not fully simplify your answer, but for full credit it should not contain a summation.)

4. Let X be an exponential random variable with parameter λ. Let $Y = \sqrt{X}$.

   (a) Compute the distribution function of Y.

   (b) Compute the density function of Y.

5. A new life form, ET2, has been discovered in a remote part of the galaxy. The lifetime of a randomly chosen ET2 is an exponentially distributed random variable X with parameter λ.

   (a) Compute the cumulative distribution function F of X.

   (b) The distribution function satisfies the equation $F(1000) = 1/2$. (We say that 1000 is the *median* of the distribution.) Find λ.

   (c) How does $E(X)$ compare to the median, 1000?

   (d) Let $a > 0$. Find $b > a$ such that

$$P(X > b | X > a) = \frac{1}{2}.$$

6. A joint probability density function of random variables X and Y is given by the formula

$$f(x,y) = \begin{cases} \frac{2}{3}(2x+y) & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal density $f_Y(y)$.

(b) Write down (but don't evaluate!) an integral whose value equals $P(3Y > X + 1)$.

(c) Write down (but don't evaluate!) an integral whose value equals $E((X - Y)^2 \ln Y)$.

7. In Flatland, a three-sided die (marked with the numbers 1, 2, 3, one on each side) is a very popular toy. Two such dice are rolled. Let X be the number rolled on the first dice, and Y the maximum of the two numbers rolled.

   (a) Find the joint probability mass function of X and Y.

$X \diagdown Y$	**1**	**2**	**3**
1			
2			
3			

   (b) Find $E(Y)$.

   (c) Find the mass function of the random variable XY.

   (d) Find $\mathrm{Cov}(X, Y)$.

8. The veterinarian Dr. Logue weighs my cat Paris, using a scale that has an error that is normally distributed with mean .5 lbs, standard deviation .2 lbs (here, positive error means that on average the scale gives an answer that is .5 lbs greater than the actual weight). Independently, Dr. Niemann weighs my dog Casey, using a scale that has an error that is normally distributed with mean $-.4$ lbs (so on average the scale gives an answer that is .4 lbs less than the actual weight), with standard deviation .1 lbs.

   (a) If Paris weighs in at 11.4 lbs, what is the probability that she weighs more than 10.5 lbs?

   (b) If Casey weighs in at 11 lbs, what is the probability that Casey is heavier than Paris?

9. Let N be a positive integer, and let X be a discrete random variable with probability mass function given by

$$p(k) = \begin{cases} \frac{1}{N} & \text{if } k \in \{1, \ldots, N\} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the moment generating function of X. (You may leave it in summation form).

(b) Use the moment generating function to find $E(X)$. (You may leave your answer in summation form, but there's a bonus point for writing it in closed form).

(c) Let $X_1, \ldots, X_n$ be independent random variables, each having the same mass function p (as given above). What is the moment generating function of $X_1 + \ldots + X_n$?

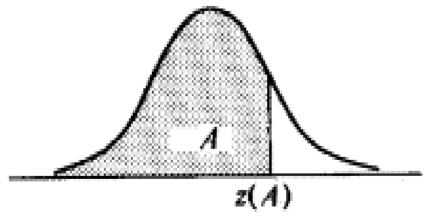
10. When I make a phone call to one of my friends in Ireland, the length of the call (measured in minutes) is a random variable X with density function

$$f(x) = \begin{cases} \frac{3}{x^4} & \text{if } x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute $E(X)$, $E(X^2)$ and $\text{Var}(X)$.

(b) I make 25 calls in a row. Assuming that call lengths are independent, use the central limit theorem to estimate the probability that I am on the phone for more than 42 minutes.

Entry is area A under the standard normal curve from $-\infty$ to $z(A)$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998