Problem Solving in Math (Math 43900) Fall 2013

Week one (August 27) problems — a grab-bag

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Most weeks' handouts will be a themed collection of problems — all involving inequalities of one form or another, for example, or all involving linear algebra. The Putnam Competition itself has no such theme, and so every so often we'll have a handout that includes the realistic challenge of figuring out what (possibly different) approach to take for each problem. This introductory problem set is one such.

Look over the problems, pick out some that you feel good about, and tackle them! You'll do best if you engage your conscious brain fully on a single problem, rather than hopping back-and-forth between problems every few minutes (but it's also a good idea to read all the problems before tackling one, to allow your subconscious brain to mull over the whole set).

Remember that problem solving is a full-contact sport: throw everything you know at the problem you are tackling! Sometimes, the solution can come from an unexpected quarter.

The point of Math 43900 is to introduce you to (or reacquaint you with) a variety of tricks and tools that tend to be frequently useful in the solving of competition puzzles; but even before that process starts, there are lots of common-sense things that you can do to make problem-solving fun, productive and rewarding. Whole books are devoted to these strategies (such as Larson's *Problem solving through problems* and Pólya's *How to solve it*).

Without writing a book on the subject, here (adapted from a list by Ravi Vakil) are some slogans to keep in mind when solving problems:

- Try small cases!
- Plug in small numbers!
- Do examples!
- Look for patterns!
- Draw pictures!
- Write lots!
- Talk it out!
- Choose good notation!
- Look for symmetry!
- Break into cases!
- Work backwards!

- Argue by contradiction!
- Consider extreme cases!
- Modify the problem!
- Make a generalization!
- Don't give up after five minutes!
- Don't be afraid of a little algebra!
- Take a break!
- Sleep on it!
- Ask questions!

And above all:

- Enjoy!

The problems

1. A locker room has 100 lockers, numbered 1 to 100, all closed. I run through the locker room, and open every locker. Then I run through the room, and close the lockers numbered 2, 4, 6, etc. (all the even numbered lockers). Next I run through the room, and change the status of the lockers numbered 3, 6, 9, etc. (opening the closed ones, and closing the open ones). I keep going in this manner (on the ith run through the room, I change the status of lockers numbered i, $2i$, $3i$, etc.), until on my 100th run through the room I change the status of locker number 100 only. At the end of all this, which lockers are open?

2. Fix an integer $k \geq 1$. Let $f(x) = 1/(x^k - 1)$. The nth derivative of $f(x)$ may be written as

$$f^{(n)}(x) = \frac{g_n(x)}{(x^k - 1)^{n+1}}$$

for some function $g_n(x)$. Find the value (in terms of n and k) of $g_n(1)$.

3. (a) Is there a sequence $(a_n)_{n\geq 1}$ of positive terms, such that both of the sums

$$\sum_{n=1}^{\infty} \frac{a_n}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{1}{a_n}$$

converge?

(b) Is there a sequence $(a_n)_{n\geq 1}$ of positive terms, such that both of the sums

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{a_n}$$

converge?

4. Prove the following statement: for every even number $n \geq 2$, the numbers 1 up to n can be paired off (into $n/2$ pairs) in such a way that the sum of each pair is a prime number (for example, if $n = 8$ the pairing scheme $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, $\{7, 8\}$ doesn't work, because $7 + 8 = 15$ is not prime; but the pairing scheme $\{1, 4\}$, $\{2, 3\}$, $\{5, 8\}$, $\{6, 7\}$ does).

5. Find the sum of the digits, of the sum of the digits, of the sum of the digits, of the number 4444^{4444}.

6. Find polynomials $f(x)$, $g(x)$ and $h(x)$ such that

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1, \\ 3x + 2 & \text{if } -1 \leq x \leq 0, \\ -2x + 2 & \text{if } x > 0, \end{cases}$$

   OR show that no such polynomials can be found.

7. Consider the following game played with a deck of $2n$ cards, numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players, A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game.

   Assuming optimal strategy by both A and B, who wins?

8. Let f be a non-constant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.