

Independent sets

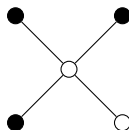
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An extremal question for independent sets

Independent set Set of pairwise non-adjacent vertices



- $i(G)$: Number of independent sets in G
- $i_t(G)$: Number of independent sets of size t

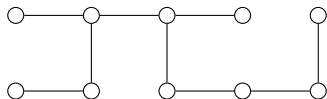
Question

Fix a family \mathcal{G} of graphs.

- What is the maximum of $i(G)$ as G ranges over \mathcal{G} ?
- What about the maximum of $i_t(G)$ for each t ?

Trees of fixed order

$\mathcal{T}(n)$: trees on n vertices



Theorem (Prodinger, Tichy, 1982)

For $T \in \mathcal{T}(n)$,

- $i(G)$ maximized by the star $K_{1,n-1}$

Theorem (Wingard, 1995)

For $T \in \mathcal{T}(n)$, and all t ,

- $i_t(G)$ maximized by $K_{1,n-1}$

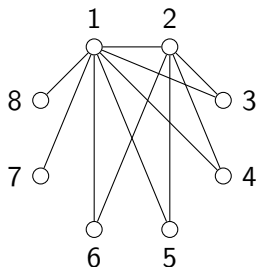
Graphs with fixed order and number of edges

$\mathcal{H}(n, m)$: graphs on n vertices with m edges

Theorem (Cutler, Radcliffe, 2011)

For $G \in \mathcal{H}(n, m)$,

- $i(G)$ maximized by the lex graph $L(n, m)$
- for all t , $i_t(G)$ maximized by $L(n, m)$



The lex graph $L(8, 11)$

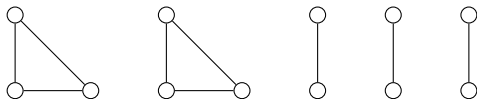
Graphs with fixed order and independence number

$\mathcal{I}(n, \alpha)$: graphs on n vertices with $\alpha(G) = \alpha$

Theorem (Roman, 1976)

For $G \in \mathcal{I}(n, \alpha)$,

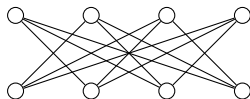
- $i(G)$ maximized by union of α almost-equal-sized cliques
- for all t , $i_t(G)$ maximized by same graph



The case $n = 12$, $\alpha = 5$

Graph of fixed order that are regular of fixed degree

$\mathcal{R}(n, d)$: d -regular graphs on n vertices



Theorem (Kahn, 2001; (Yufei) Zhao, 2011)

For $G \in \mathcal{R}(n, d)$,

- $i(G)$ maximized by $\frac{n}{2d} K_{d,d}$, union of $n/2d$ copies of $K_{d,d}$

Conjecture (Kahn, 2001)

For $G \in \mathcal{R}(n, d)$, and all t ,

- $i_t(G)$ maximized by $\frac{n}{2d} K_{d,d}$

Asymptotic evidence given by Carroll, G., Tetali, and by Zhao

Graphs of fixed order with fixed minimum degree

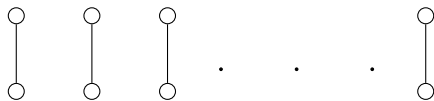
$\mathcal{G}(n, \delta)$: graphs on n vertices with minimum degree δ

Speculation

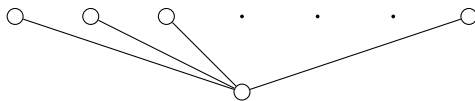
Removing edges increases independent set count, so maybe

- $i(G)$ maximized by $\frac{n}{2\delta} K_{\delta, \delta}$

Not true, even for $\delta = 1$



$$i(nK_2) = 3^{n/2}$$



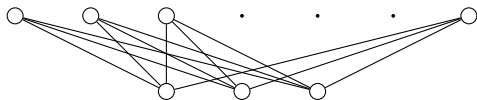
$$i(K_{1, n-1}) = 2^{n-1} - 1$$

An unbalanced maximizer

Theorem (G., 2011)

For $n \geq 8\delta^2$ and $G \in \mathcal{G}(n, \delta)$,

- $i(G)$ uniquely maximized by $K_{\delta, n-\delta}$.



$K_{3, n-3}$

Conjecture (G., 2011)

For $G \in \mathcal{G}(n, \delta)$,

- for $n \geq 2\delta$, $i(G)$ maximized by $K_{\delta, n-\delta}$
- for smaller n , $i(G)$ maximized by $K_{n-\delta, n-\delta, \dots, n-\delta, x}$ ($x \leq n - \delta$)

Fixed size independent sets in $\mathcal{G}(n, \delta)$

$i_2(G)$ = number of non-edges, so $K_{\delta, n-\delta}$ definitely *not* the maximizer

Conjecture (G., 2011)

For $n \geq 2\delta$, $t \geq 3$ and $G \in \mathcal{G}(n, \delta)$,

- $i_t(G)$ maximized by $K_{\delta, n-\delta}$

Theorem (Alexander, Cutler, Mink, 2011+)

- Conjecture true for bipartite G

Theorem (Engbers, G., 2011+)

Full conjecture is true for

- $\delta = 1, 2, 3$
- $t \geq \delta + 1$, $n \geq 3.2\delta$
- $t \geq 2\delta + 1$, $n \geq 3\delta + 1$

Proof for $t \geq 2\delta + 1$, $n \geq 3\delta + 1$ (I)

Observation

- Suffices to consider $t = 2\delta + 1$

Proof Suppose we know that for some $t > \delta$,

$$i_t(G) \leq i_t(K_{\delta, n-\delta}) = \binom{n-\delta}{t}$$

Then

$$\#(\text{ordered independent } t\text{-sets}) \leq (n-\delta)^t$$

Once t vertices chosen, at least $\delta + t$ ruled out, so

$$\#(\text{ordered independent } (t+1)\text{-sets}) \leq (n-\delta)^t (n - (\delta + t)) = (n-\delta)^{t+1}$$

and so

$$i_{t+1}(G) \leq \binom{n-\delta}{t+1} = i_{t+1}(K_{\delta, n-\delta})$$

Proof for $t \geq 2\delta + 1$, $n \geq 3\delta + 1$ (II)

Proof strategy

- Prove $t = 2\delta + 1$ case by induction on n

Base case $n = 3\delta + 1$ is trivial

Induction, case 1 There is $x \in V(G)$ with $\delta(G - x) = \delta$

$$\begin{aligned}i_t(G) &= i_t(G - x) + i_{t-1}(G - x - N(x)) \\ &\leq \binom{(n-1) - \delta}{t} \text{ (induction)} + \binom{n - (\delta + 1)}{t-1} \text{ (trivial)} \\ &\leq \binom{n - \delta}{t} \text{ (Pascal)}\end{aligned}$$

Proof for $t \geq 2\delta + 1$, $n \geq 3\delta + 1$ (III)

Induction, case 2 There is no $x \in V(G)$ with $\delta(G - x) = \delta$

Ordered independent t -sets starting with vertex of degree $> \delta$:

$$N_{>\delta} \leq k(n - (\delta + 2))(n - (\delta + 3)) \dots (n - (\delta + t))$$

where k = number of vertices of degree $> \delta$

Ordered independent t -sets starting with vertex of degree $= \delta$:

$$N_{=\delta} \leq (n - k)(n - (\delta + 1))(n - (\delta + 2)) \dots (n - 2\delta) \\ (n - \widehat{(2\delta + 2)})(n - (2\delta + 2)) \dots (n - (\delta + t))$$

Why the missing term?

- Worst case: each new vertex shares δ neighbors of first choice
- This *can't* happen $\delta + 1$ times (or we're in case 1)
- $(\delta + 1)$ st choice (at worst) removes a new vertex

Proof for $t \geq 2\delta + 1$, $n \geq 3\delta + 1$ (IV)

Have

$$N_{>\delta} \leq k(n - (\delta + 2))(n - (\delta + 3)) \dots (n - (\delta + t))$$

and

$$N_{=\delta} \leq (n - k)(n - (\delta + 1))(n - (\delta + 2)) \dots (n - 2\delta) \\ (n - \widehat{(2\delta + 2)})(n - (2\delta + 2)) \dots (n - (\delta + t))$$

Worst case $k = n$, giving bound

$$i_t(G) \leq \frac{n(n - (\delta + 2))(n - (\delta + 3)) \dots (n - (\delta + t))}{t!} \\ < \binom{n - \delta}{t}$$

Last inequality uses $t = 2\delta + 1$

Final comments

- Improve result by considering first, second, third ... choices more carefully, and optimizing a linear program
- $\delta = 2, 3$ requires messy case analysis, structural results for δ -critical graphs, with $\delta = 4$ hopeless

Open questions

- $i_t(G)$ for all t and n -vertex, d -regular G
- $i(G)$ for $n \leq 8\delta^2$ for n -vertex G with min. degree δ
- $i_t(G)$ for $3 \leq t \leq \delta$ and $n \leq 3.2\delta$ for n -vertex G with min. degree δ
- ...
- e.g., \mathcal{G} the family of n -vertex, triangle-free, average degree t graphs

THANK YOU!

Slides at <http://nd.edu/~dgalvin1>