2.1) Perturbed Infinite Potential Well

Solve the following problem from Griffiths (Introduction to Quantum Mechanics).

Suppose we put a delta-function bump in the center of the infinite square well of width $a$:

$$W(x) = \alpha \delta(x - a/2),$$

where $\alpha$ is the strength of the bump in units of J·m.

a) Find the first-order and second-order correction to the allowed energies. Explain why the energies are not perturbed for even $n$.

b) Find the first order correction to the ground state wave-function.

2.2) Rayleigh-Schrodinger Perturbation theory: Stark Effect

Apply the Rayleigh-Schrodinger theory to evaluate the first two eigenvalues of a particle in a box, when it is perturbed by a potential of a constant electric field $V(x) = Fx$. Retain terms till the second order perturbation. Do the eigenvalues go up or down if $F > 0$? This is the Stark-effect, or the quantum-confined Stark effect (QCSE): if there are optical transitions between say the ground state and the excited states, the energy of transition is shifted by the field.

Then find the perturbed eigenstates of the ground state, to as many orders as you can. Make sketches to illustrate what you find.

2.3) Degenerate perturbation theory: An anharmonic oscillator

Degenerate perturbation theory involves solving the Hamiltonian matrix for eigenvalues and eigenstates. Solve the following problem from Kroemer’s QM text in Fig 1. You are free to use Mathematica for analytical evaluation, or of course you can always do it by hand.

2.4) Time-Dependent Perturbation Theory of 2-Level Systems

Solve the ‘exactly solvable’ problem from Sakurai (Modern Quantum Mechanics) in Fig 2.
A one-dimensional harmonic oscillator is perturbed by a small perturbation that is proportional to $x^3$ which may always be written in the form
\[ W(x) = \alpha \cdot \hbar \omega \cdot (x/L)^3, \tag{14\cdot3\cdot32} \]
where $\alpha (\ll 1)$ is a numerical parameter characterizing the strength of the perturbation, and $\omega$ and $L$ are the natural frequency and the natural unit of length of the oscillator.

(a) With the help of a sketch of the perturbed potential, discuss why this is, strictly speaking, no longer a bound-state problem at all, but a problem of quasi-bound states, in the sense of section 2.6. Up to which quantum number $n = n_{\text{max}}$ would you expect perturbation theory to give reasonable values for the energy of the quasi-bound states?

(b) Use two-state degenerate perturbation theory to estimate the effect of the perturbation on the energy of the two lowest states of the oscillator, as a function of $\alpha$. Whatever the validity of the two-state procedure for the perturbation of the ground-state energy, why is this procedure a very poor one for the perturbation of the second state, even for small values of $\alpha$?

(c) Now do a three-state perturbation treatment, assuming $\alpha = 1/10$, solving the cubic secular equation numerically. Compare the result with the analytical result obtained under (b), for the same value of $\alpha$.

5.30 Consider a two-level system with $E_1 < E_2$. There is a time-dependent potential that connects the two levels as follows:
\[ V_{11} = V_{22} = 0, \quad V_{12} = \gamma e^{i\omega t}, \quad V_{21} = \gamma e^{-i\omega t} \quad (\gamma \text{ real}). \]
At $t = 0$, it is known that only the lower level is populated—that is, $c_1(0) = 1$, $c_2(0) = 0$.

(a) Find $|c_1(t)|^2$ and $|c_2(t)|^2$ for $t > 0$ by exactly solving the coupled differential equation
\[ i\hbar \dot{c}_k = \sum_{n=1}^{2} V_{kn}(t) e^{i\omega_{kn} t} c_n, \quad (k = 1, 2). \]

(b) Do the same problem using time-dependent perturbation theory to lowest non-vanishing order. Compare the two approaches for small values of $\gamma$. Treat the following two cases separately: (i) $\omega$ very different from $\omega_{21}$ and (ii) $\omega$ close to $\omega_{21}$.

Answer for (a): (Rabi’s formula)
\[
|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4} \sin^2 \left\{ \frac{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4}{4} t \right\},
\]
\[ |c_1(t)|^2 = 1 - |c_2(t)|^2. \]

2.5) Tunneling escape times by Fermi’s golden rule

Figure 3 shows a 1-dimensional potential for an electron, which is in the state with energy $E_0$ at $t = 0$. Since there is a lower potential for $x > L_{w} + L_{b}$, the state $|E_0\rangle$ is a quasi-bound state. The
electron is destined to leak out.

a) Using WKB tunneling probability, and combining semi-classical arguments, find an analytical formula that estimates the time it takes for the electron to leak out. Find a value of this lifetime for $L_b \sim 3$ nm, $L_w \sim 2$ nm, $V_0 \sim 1$ eV, $E_0 \sim 2$ eV, and $E_b \sim 5$ eV. How many years does it take?

b) This feature is at the heart of flash memory, which you use in computers and cell phones. Find an analytical expression that describes how the lifetime changes if a voltage $V_a$ is applied across the insulator. Estimate the new lifetime for $V_a \sim 2.8$ V. This is the readout of the memory.

c) Now try solving the same problem using Fermi’s golden rule. Model the problem carefully so that you can apply Fermi’s golden rule. Discuss your approximations and their validity.

2.6) Higher-order time-dependent perturbation theory: Dyson series

In class, we used the interaction representation to write the perturbed quantum state at time $t$ as $|\psi_t\rangle = e^{-\frac{iH_0}{\hbar}t}|\psi(0)\rangle$, where $H_0$ is the unperturbed Hamiltonian operator. This step helped us recast the time-dependent Schrödinger equation $i\hbar \frac{\partial}{\partial t}|\psi_t\rangle = (H_0 + V_t)|\psi_t\rangle$ to the simpler form $i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = V(t)|\psi(t)\rangle$, where $V(t) = e^{\frac{iH_0}{\hbar}t}V_t e^{-\frac{iH_0}{\hbar}t}$ is the time-evolution operator. This equation was integrated over time to yield the Dyson series

$$|\psi(t)\rangle = |\psi(t)\rangle^{(0)} + \frac{1}{i\hbar} \int_{t_0}^t dt' V(t') |\psi(t)\rangle^{(1)} + \frac{1}{(i\hbar)^2} \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' V(t') V(t'') |\psi(t)\rangle^{(2)} + \frac{1}{(i\hbar)^3} \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \int_{t_0}^{t''} dt''' V(t') V(t'') V(t''') |\psi(t)\rangle^{(3)} + \ldots$$

Figure 3: Escape and field-emission by tunneling.
where \(|\psi(t_0)\rangle = |0\rangle\) is the initial state. Restricting the Dyson series to the 1st order term in \(V\) for a perturbation of the form \(V_t = e^{\eta t}V\), we derived Fermi’s golden rule for the transition rate 
\[
\Gamma^{(1)}_{0\rightarrow n} = \frac{2\pi}{\hbar}|\langle n|V|0\rangle|^2\delta(\epsilon_0 - \epsilon_n). 
\]
We used the relation \(\lim_{\eta \to 0^+} \frac{2\eta}{x^2 + \eta^2} = 2\pi \delta(x)\) in this process.

(a) Show that the second and third order terms in \(V\) in the Dyson series lead to a modified golden rule result
\[
\Gamma_{0\rightarrow n} = \frac{2\pi}{\hbar}|\langle n|V|0\rangle| + \sum_m \frac{\langle n|V|m\rangle\langle m|V|0\rangle}{\epsilon_0 - \epsilon_m + i\eta \hbar} + \sum_{k,l} \frac{\langle n|V|k\rangle\langle k|V|l\rangle\langle l|V|0\rangle}{(\epsilon_0 - \epsilon_k + 2i\eta \hbar)(\epsilon_0 - \epsilon_l + i\eta \hbar)} + \cdots |\rangle^2 \delta(\epsilon_0 - \epsilon_n),
\]
where in the end we take \(\eta \to 0^+\). We identify the Green’s function propagators of the form 
\[
G = \sum_m \frac{|m\rangle\langle m|}{\epsilon_0 - \epsilon_m + i\eta \hbar}. 
\]
Thus, the result to higher orders may be written in the compact form
\[
\Gamma_{0\rightarrow n} = \frac{2\pi}{\hbar}|\langle n|V + VGV + VGGV + \cdots |0\rangle|^2 \delta(\epsilon_0 - \epsilon_n).
\]

2.7) Application of 1st and higher order perturbation theories

5.23 A one-dimensional harmonic oscillator is in its ground state for \(t < 0\). For \(t \geq 0\) it is subjected to a time-dependent but spatially uniform force (not potential!) in the \(x\)-direction, 
\[
F(t) = F_0 e^{-t/\tau}.
\]

(a) Using time-dependent perturbation theory to first order, obtain the probability of finding the oscillator in its first excited state for \(t > 0\). Show that the \(t \to \infty\) (\(\tau\) finite) limit of your expression is independent of time. Is this reasonable or surprising?

(b) Can we find higher excited states? You may use
\[
\langle n'|x|n\rangle = \sqrt{\hbar/2m\omega}(\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1}).
\]

Figure 4: Harmonic oscillator perturbed by a time-dependent field.

Solve this problem from Sakurai (Modern Quantum Mechanics). Note that for part (b), you will need to invoke higher-order perturbation terms as discussed in Problem 2.6.