ACMS 70860 - Stochastic Analysis

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Class Location: DeBartolo Hall, Room 319
Class Schedule: Tuesday, Thursday, 3:30PM - 4:45PM
Office Hours: TBD or by appointment
Class Website: http://www3.nd.edu/~dschiava/ACMS40760-01_spring19.html

Course Description - The analysis of systems affected by randomness or with uncertain parameters/forcing has countless applications in a variety of fields, from generalizations of deterministic differential systems in the presence of uncertainty, to the analysis of financial products, to biological and physiological applications. This course addresses the main theoretical results for Stochastic Differential Equations (SDEs). After an introduction on stochastic processes and Brownian motion, the main results of Itô calculus and Itô integration will be introduced and applied to the solution of SDEs. We will review stochastic methods for the solution of deterministic PDEs, numerical solution and filtering (Kalman filter) of SDEs.

Course Objectives - By the end of this course, the students should be able to:

- ✓ Acquire familiarity with stochastic processes, particularly Brownian motion.
- ✓ Understand Ito's calculus and the solution techniques for SDEs.
- ✓ Write code to numerically approximate stochastic processes and solve SDEs.
- ✓ Compute the solution of elliptic PDEs using Monte Carlo methods.
- ✓ Implement a simple Kalman-Bucy filter for the localization of a dynamical system.

Pre-requisite - Applied Probability (ACMS 60850).

Academic Calendar - Spring Semester 2019 - Classes start on 01/15. Spring break 03/09-17. Easter holiday 04/19-22. End of classes 05/01. Final exams 05/06-10.

Textbook and other references - There is **no suggested textbook** for the class. However, material from the following sources will be discussed.

- ✓ Evans Lawrence C., *An introduction to stochastic differential equations*, Notes from a course in stochastic differential equations at UC Berkeley, 2012.
- ✓ Øksendal, B., *Stochastic differential equations*, Springer, 2003.
- ✓ Gardiner C., *Stochastic methods: a handbook for the natural and social sciences 4th ed.*, Springer, 2009.

Required Work and Grading Criteria - The required work consists of homework problems, one midterm exam, and one final project. The breakdown of marks is:

- ✓ **Homeworks:** 30%.
- ✓ **Midterm:** 30%.
- ✓ **Final Project:** 40%.

Homework Assignments - Homework assignments will be based on the material presented in class. Most of the homeworks will require to develop code (Python preferred) and to answer questions from the theory.

Midterm Exam - There will be a take-home midterm exam, to be scheduled.

Final Project - During March-April, students will be asked to propose a project that combines their research activities with some of the topics discussed in class. This may involve, for example, reproducing (and critically reviewing) the results of a paper in the literature. Alternatively, a project will be assigned by the instructor. Each project may be performed individually or in groups, and will be presented to the rest of the class.

Honor Code - All students must familiarize themselves with the Honor Code on the University's website and pledge to observe its provisions in all written and oral work, including oral presentations, quizzes and exams, and drafts and final versions of essays. While discussion in small groups in doing homework is permitted (and strongly encouraged) in this course, **the work should be your own**. Exams are closed book and are to be done completely by yourself with no help from others.

Tentative program

Week n.	Tentative Content
Week 1	**Intro and rudiments of measure theoretic probability.** Deterministic and stochastic differential equations. Intro to stochastic differentials. Itô stochastic integration. Bertrand's Paradox, σ-algebras, probability measure, Borel σ-algebra, Dirac point mass and Buffon's needle problem.
Week 2	**Intro and rudiments of measure theoretic probability.** Borel sets, definition of random variable, indicator function, simple function, generated σ-algebra. Stochastic Processes. Integration w.r.t. a measure, vector random variables, distribution function, joint distribution function, density function. Integration with respect to the density function, moments. **Conditional probability.** Intuition on conditional probability, definition for events, σ-algebras, r.v.s and functions of r.v.s. Factorization of distribution and density functions. Expected value and variance of independent r.v.s. **Some Probabilistic Methods, LLN, CLT.** Chebyshev inequality.
Week 3	**Some Probabilistic Methods, LLN, CLT.** Borel-Cantelli lemma, characteristic function of a r.v., i.i.d. r.v.s, strong law of large numbers, average and variance of simple Bernoulli trials, Laplace De-Moivre theorem, central limit theorem. **Conditional Expectation.** Conditional expectation for events, r.v.s and σ-algebras.
Week 4	**Conditional Expectation.** Conditional expectation as a projection operator. **Martingales.** Definition, history of a stochastic process. Martingale, sub martingale, Brownian motion. Martingale inequalities. **Brownian motion.** Intro and analogy to diffusion.
Week 5	**Brownian motion.** Joint probability. Statistical moments. Non differentiability and relationship to white noise. **Construction of Brownian motion.** Expansion of white noise and BM with Haar and Schauder functions. Uniform convergence of Schauder series, existence.
Week 6	**Sample Path Properties.** Hölder continuity and Kolmogorov theorem. Continuity and nowhere differentiability of Brownian sample paths. Markov property. **Preliminaries on Itô calculus.** Integrability w.r.t the Brownian motion. Paley-Wiener-Zygmund stochastic integral.
Week 7	**Preliminaries on Itô calculus.** Reimann sum approximation. Quadratic variations. Itô and Stratonovich stochastic integrals. **Itô Integral.** Filtration, Approximation by a step process. Properties of the Itô integral, representation through Reimann sums. Martingale property of the indefinite Itô integral. **Itô chain and product rules.** Stochastic differential. Itô's chain rule with examples.
Week 8	**Itô chain and product rules.** Itô's product rule. Itô's integral in higher dimensions. **Stochastic differential equations.** Solution of Itô's SDE. Reduction of higher order SDEs in Itô's form. Applications: drift, stock prices, Brownian bridge.
Midterm	

Week n.	Tentative Content
Week 10	**Stochastic differential equations.** Langevin's equation, Ornstein-Uhlenbeck process, Random harmonic oscillator. **Existence and Uniqueness of the solution.** Solution of SDE by successive approximations. Change of variables. Existence and uniqueness of a solution. **Properties of the solution of an SDE.** Estimates on higher moments and application to sample path properties. Dependence of parameters and application to a case with small noise. Solution of linear SDE.
Week 11	**Numerical methods for SDE.** One-variable Taylor expansion. Euler methods. Higher order schemes and multiple stochastic integrals. Strong and weak order of convergence for the Euler algorithm. Strong and weak order of convergence for the Milstein scheme. Stability of explicit, fully implicit and semi-implicit numerical schemes for SDEs. Consistency.
Week 12	**Stopping times.** Definition. Minimum and maximum hitting times. Stochastic integral and stopping times. Itô's chain rule and stopping times. Laplacian generator for Brownian motion. **Applications to PDEs.** Solution of diffusion PDEs as expected hitting times. Application to Laplace equation and harmonic functions. Feynman-Kac formula.
Week 13	**The Stratonovich integral.** White noise approximation and stability of SDE. Stratonovich integral as a Reimann sum. Difference from Itô's integral and conversion formula. Stratonovich chain rule. Conversion between Itô and Stratonovich SDEs. **Filtering of SDEs.** Intro and motivation. The 1D linear filtering problem. Summary of the solution steps. Example with discrete r.v.s. Best Z-linear and Z-measurable estimates.
Week 14	**Filtering of SDEs.** Innovations and Brownian motion. Explicit formula for the projection operator. The deterministic Riccati equation. The Kalman-Bucy filter. Noisy observations of a constant process. Noisy observations of a Brownian motion. Estimation of a parameter. Noisy observations of a population growth.
Week 15	**Student project presentations**