Department of Mathematics
University of Notre Dame
Math 20580 – Spring 2012

Name: ____________________

Instructor: ______________
& Section

Exam I

February 16, 2012

This exam is in 2 parts on 7 pages and contains 12 problems worth a total of 96 points. An additional 4 points will be awarded for following the instructions. You have 1 hour and 15 minutes to work on it. No calculators, books, notes, or other aids are allowed. Be sure to write your name on this page and to put your initials at the top of every page in case pages become detached. Good luck!

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Place an $\times$ through your answer to each problem.

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MC.	________
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Multiple Choice

1. (6 pts.) Let $A = \begin{bmatrix} 1 & -1 & 0 & h \\ 2 & -1 & -1 & k \\ 0 & 1 & 1 & 1 \end{bmatrix}$ be the augmented matrix of a linear system. Only one of the statements below is correct. Which one?

(a) The system has infinitely many solutions for all values of h and k

(b) The system is inconsistent for any values of h and k.

(c) The system has a unique solution for all values of h and k

(d) If $h = k$ the system is inconsistent

(e) None of these

2. (6 pts.) Find the reduced echelon form of the matrix $\begin{bmatrix} 2 & 4 & 0 & 0 \\ 1 & -2 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 5 & -5 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. (6 pts.) Let A be an $p \times q$ matrix. Determine which of the following six statements are true.

I. $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^p$
II. $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^p$
III. $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^q$
IV. $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^q$
V. $\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A) = q$
VI. $\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A) = p$

(a) I, IV & VI (b) I, IV & V (c) III, IV & VI (d) II , III & VI (e) I, II & V

4. (6 pts.) Determine by inspection which of the following sets is linearly independent.

(a) all four sets are linearly dependent

(b) $\left\{ \begin{bmatrix} 3 \\ 5 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$

5. (6 pts.) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be linear transformations with

$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and

$S(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $S(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $S(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Which matrix below is the standard matrix of ST?

(a) $\begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

6. (6 pts.) Let A be the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & -2 & 1 \\ 3 & 6 & 9 & 0 \end{bmatrix}$. Which set below is a basis for $\text{Col}(A)$?

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 2 \\ 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \\ 0 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 9 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \\ 1 \end{bmatrix} \right\}$

7. (6 pts.) The determinant of $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & -5 \\ 9 & 6 & -14 \end{bmatrix}$ is

(a) 2 (b) 6 (c) -2 (d) -6 (e) 0

8. (6 pts.) Let $A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 2 & -4 & 6 & 3 \end{bmatrix}$. The *rank* of A is

(a) 2 (b) 1 (c) 3 (d) 0 (e) 4

9. (6 pts.) Let A be a $p \times q$ matrix. Consider the following four statements:

I. If $p < q$ then $\dim \mathrm{Nul}(A) > 0$
II. If $p < q$ then $\dim \mathrm{Nul}(A) = q - p$
III. $\dim \mathrm{Col}(A) = p$
IV. If A is invertible, then $\dim \mathrm{Nul}(A) = 0$

Which of these statements are always true and which of them can be false for some A.

(a) II & III are always true, I & IV can be false.

(b) I & IV are always true, II & III can be false.

(c) I, III & IV are always true, II can be false.

(d) I & II are always true, III & IV can be false.

(e) I & III are always true, II & IV can be false.

Partial Credit

You must show your work on the partial credit problems to receive credit!

10. (14 pts.) Express the solution set of

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & - & x_4 & = & 2 \\ & & & - & 2x_3 & + & x_4 & = & 12 \\ 2x_1 & - & 4x_2 & + & 6x_3 & + & 2x_4 & = & 12 \end{array}$$

in *Parametric Vector Form.*

11. (14 pts.) Let $\mathbf{v_1} = \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}$.

1. *(5 pts.)* Show that Span$\{\mathbf{v_1}, \mathbf{v_2}\}$ has a basis $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}\}$
2. *(4 pts.)* Find the $\mathcal{B}$-coordinates of $\mathbf{x}$.
3. *(5 pts.)* Let $\mathbf{y} \in \mathbb{R}^3$ have $\mathcal{B}$-coordinates $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\mathbf{y}$.

12. (14 pts.) Compute the inverse of the matrix $A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}$

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1. $\begin{bmatrix}1 & -1 & 0 & h\\ 2 & -1 & -1 & k\\ 0 & 1 & 1 & 1\end{bmatrix}$ $\begin{bmatrix}1 & -1 & 0 & h\\ 0 & 1 & -1 & k-2h\\ 0 & 1 & 1 & 1\end{bmatrix}$ $\begin{bmatrix}1 & -1 & 0 & h\\ 0 & 1 & -1 & k-2h\\ 0 & 0 & 2 & 1+2h-k\end{bmatrix}$ All the columns of the augment matrix are pivots so there is always a unique solution.

2. $\begin{bmatrix}2 & 4 & 0 & 0\\ 1 & -2 & 1 & -1\\ 0 & 1 & 1 & -1\end{bmatrix}$ $\begin{bmatrix}1 & 2 & 0 & 0\\ 1 & -2 & 1 & -1\\ 0 & 1 & 1 & -1\end{bmatrix}$ $\begin{bmatrix}1 & 2 & 0 & 0\\ 0 & -4 & 1 & -1\\ 0 & 1 & 1 & -1\end{bmatrix}$ $\begin{bmatrix}1 & 2 & 0 & 0\\ 0 & 1 & 1 & -1\\ 0 & -4 & 1 & -1\end{bmatrix}$ $\begin{bmatrix}1 & 2 & 0 & 0\\ 0 & 1 & 1 & -1\\ 0 & 0 & 5 & -5\end{bmatrix}$ $\begin{bmatrix}1 & 2 & 0 & 0\\ 0 & 1 & 1 & -1\\ 0 & 0 & 1 & -1\end{bmatrix}$ $\begin{bmatrix}1 & 2 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\end{bmatrix}$ $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\end{bmatrix}$

3. $A\colon \mathbb{R}^q \to \mathbb{R}^p$ so $\operatorname{Col}(A) \subset \mathbb{R}^p$ and $\operatorname{Nul}(A) \subset \mathbb{R}^q$. $\dim \operatorname{Nul}(A) + \operatorname{Col}(A) = q$. Hence I, IV and V are true.

4.

In (a), clearly one vector is not a multiple of the other so independent.
(b) too many vectors - dependent
(c) one vector is a multiple of the other- dependent
(d) zero vector - dependent

5. Compute $ST(\mathbf{e}_1) = S\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = S(\mathbf{e}_1+\mathbf{e}_2) = S(\mathbf{e}_1)+S(\mathbf{e}_2) = \begin{bmatrix}-1\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$ and $ST(\mathbf{e}_2) = S\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = S(\mathbf{e}_2+\mathbf{e}_3) = S(\mathbf{e}_2)+S(\mathbf{e}_3) = \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$ so the matrix is $\begin{bmatrix}ST(\mathbf{e}_1) & ST(\mathbf{e}_2)\end{bmatrix} = \begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}$

6. $\begin{bmatrix}1 & 2 & 3 & 0\\ 2 & 4 & -2 & 1\\ 3 & 6 & 9 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 2 & 3 & 0\\ 0 & 0 & -8 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$ so columns 1 and 3 of the original matrix are a basis.

7. $\begin{bmatrix}1 & 0 & -1\\ 3 & 2 & -5\\ 9 & 6 & -14\end{bmatrix}$ $\begin{bmatrix}1 & 0 & -1\\ 0 & 2 & -2\\ 0 & 6 & -5\end{bmatrix}$ $\begin{bmatrix}1 & 0 & -1\\ 0 & 2 & -2\\ 0 & 0 & 1\end{bmatrix}$ Since this last matrix is upper triangular, the determinant is $1\cdot 2\cdot 1 = 2$.

8. $\begin{bmatrix}1 & -2 & 3 & -1\\ 0 & 1 & -2 & 1\\ 2 & -4 & 6 & 3\end{bmatrix}$ $\begin{bmatrix}1 & -2 & 3 & -1\\ 0 & 1 & -2 & 1\\ 0 & 0 & 0 & 5\end{bmatrix}$ There are three pivot columns so the dimension of the column space is 3 so the rank is 3.

9. $A\colon \mathbb{R}^q \to \mathbb{R}^p$ I: Always true.

II: What is always true is that $\dim \operatorname{Nul}(A) \leqslant q-p$ but not always equal.
III: What is always true is that $\dim \operatorname{Col}(A) \leqslant p$ but not always equal.
IV. Always true.

10. $\begin{bmatrix} 1 & -2 & 3 & -1 & h \\ 0 & 0 & -2 & 1 & k \\ 2 & -4 & 6 & 2 & m \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 & -1 & h \\ 0 & 0 & -2 & 1 & k \\ 0 & 0 & 0 & 4 & m-2h \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 & -1 & h \\ -0 & -0 & 1 & -0.5 & -k/2 \\ 0 & 0 & 0 & 1 & (m-2h)/4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 1 & 0 & - \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 3 & 0 & \frac{m+2h}{4} \\ 0 & 0 & 1 & 0 & \frac{m-2h-4k}{8} \\ 0 & 0 & 0 & 1 & \frac{m-2h}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & \frac{m+2h}{4} - 3\frac{m-2h-4k}{8} \\ 0 & 0 & 1 & 0 & \frac{m-2h-4k}{8} \\ 0 & 0 & 0 & 1 & \frac{m-2h}{4} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 0 & \frac{-m+10h+12k}{8} \\ 0 & 0 & 1 & 0 & \frac{m-2h-4k}{8} \\ 0 & 0 & 0 & 1 & \frac{m-2h}{4} \end{bmatrix}$$

Hence a basis for the null space is $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and a particular solution is $\begin{bmatrix} \frac{-m+10h+12k}{8} \\ 0 \\ \frac{m-2h-4k}{8} \\ \frac{m-2h}{4} \end{bmatrix}$

Take $m = 12$, $h = 2$, $k = 12$, the particular solution is $\begin{bmatrix} 19 \\ 0 \\ -5 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 0 & -2 & 1 \\ 2 & -4 & 6 & 2 \end{bmatrix} \begin{bmatrix} 19 \\ 0 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 12 \end{bmatrix}$$

11. By inspection $\mathbf{x} = 3\mathbf{v}_1 + (-2)\mathbf{v}_2$ so $\mathbf{x} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$

Since $\mathbf{v}_1 \neq \mathbf{0}$, either $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent or else $\mathbf{v}_2 = c\mathbf{v}_1$ for some c. To get the first row right you need $c = -\frac{7}{3}$ but this does not work for row two or for row three. Therefore $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent and they certainly span their span.

From part 1 $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}_{\mathfrak{B}}$.

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix}_{\mathfrak{B}} = -2\mathbf{v}_1 + 3\mathbf{v}_2 = -2 \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 27 \\ -13 \\ 23 \end{bmatrix}.$$

12. FOR A DIFFERENT MATRIX $\begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ -4 & 0 & 2 & 0 & 1 & 0 \\ 4 & -2 & -9 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 8 & 2 & 1 & 0 \\ 0 & -4 & -15 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 8 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0.5 & 1.5 & 0.5 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0 & -2.5 & -3 & -1.5 \\ 0 & 1 & 0 & -7 & -7.5 & -4 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0.75 & 0.5 \\ 0 & 1 & 0 & -7 & -7.5 & -4 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix}$$

so $\begin{bmatrix} 1 & 0.75 & 0.5 \\ -7 & -7.5 & -4 \\ 2 & 2 & 1 \end{bmatrix}$ is the inverse.