Exam 1D solutions

Multiple choice. Exam 1D has all multiple choice answers (a).

(1) $\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -1 & 0 \\ -3 & -6 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(2) (a), clearly one vector is not a multiple of the other so LI.
(b): too many vectors - dependent.
(c): 3rd vector is a multiple of first- dependent.
(d): zero vector - dependent.
(e): false since LI in (a).

(3) The standard matrix of T is $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$. The standard matrix of S is $B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$. Hence the standard matrix of ST is the matrix product $BA = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

(4) $\begin{vmatrix} 0 & 2 & -3 \\ -2 & 6 & -12 \\ 1 & -2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & 3 \\ -2 & 6 & -12 \\ 0 & 2 & -3 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & 3 \\ 0 & 2 & -6 \\ 0 & 2 & -3 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & 3 \end{vmatrix}$
Since this last matrix is upper triangular, the determinant is $-1 \cdot 2 \cdot 3 = -6$.

(5) For a $p \times q$-matrix, $\text{rank}(A) + \dim(\text{null}(A)) = q$. Hence $\dim(\text{null}(A)) = q - \text{rank}(A) = 8 - 3 = 5$.

(6) We have to solve the linear system whose augmented matrix is first matrix following: $\begin{bmatrix} 1 & 1 & 2 & 4 \\ -2 & 0 & 2 & 2 \\ 1 & 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 2 & 6 & 10 \\ 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. The coordinate vector is $\begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}$.

(7) $Ax = b$ is inconsistent for some b, directly from the invertible matrix theorem.

(8) Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. Note A has determinant $2 \cdot 5 - 3 \cdot 4 = -2$ and so is invertible. The solution is $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} h \\ k \end{bmatrix}$. So $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$.

(9) Row reduce: $\begin{bmatrix} 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & -1 & -2 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 0 & 0 \\ 1 & 2 & -1 & -2 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & -2 & 4 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 3 & 6 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Partial credit

(10) $\begin{bmatrix} 1 & 1 & -1 & 2 & -6 \\ 1 & 0 & 1 & 1 & -3 \\ 1 & -1 & 3 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & -1 & 2 & -6 \\ 0 & -1 & 2 & -1 & 3 \\ 1 & -2 & 4 & -2 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & -1 & 2 & -6 \\ 0 & 1 & -2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 1 & -1 & -3 \\ 0 & 1 & -2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The equation is equivalent to

$$\begin{array}{ccccccccc} x_1 & & & + & x_3 & - & x_4 & = & -3 \\ & & x_2 & - & 2x_3 & + & x_4 & = & -3 \end{array}$$

The bound variables are x_1, x_2, and free variables are x_3, x_4. Rewriting with free variables on the right,

$$\begin{array}{cccccccc} x_1 & & = & - & 3 & -x_3 & - & x_4 \\ & x_2 & = & - & 3 & +2x_3 & - & x_4 \end{array}$$

or in vector parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} =$$

or writing $x_3 = a$, $x_4 = b$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 0 \\ 0 \end{bmatrix} + a \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(11) (a) The pivot columns of A and B are 1, 3, 5, so x_2 and x_4 are free variables. Writing the homogeneous equations from B with free variables on the right gives $\begin{array}{cccc} x_1 & = & 2x_2 - & x_4 \\ x_3 & = & & x_4 \\ x_5 & = & 0 & \end{array}$. The system has 2 basic solutions given by setting one free variable equal to 1 and the others equal to 2. Setting $x_2 = 1$ and $x_4 = 0$ gives the solution $v_1 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \end{bmatrix}^T$. Setting $x_2 = 0$ and

$x_4 = 1$ gives the solution $v_2 = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \end{bmatrix}^T$ (we write these using transpose T to save space). Then $\{ v_1, v_2 \}$ is a basis for null(A).

(b) Row operations don't change the solution space of the homogeneous equation or the linear dependences of columns of a matrix. The pivot columns (1st, 2rd, 5th) of B form a basis for col(B) so the pivot columns (1st, 2rd, 5th) of A form a basis for col(A). A basis of col(A) is given by $\{ w_1, w_2, w_3 \}$ where $w_1 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $w_3 = \begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix}$.

(c) Row span of a matrix is unchanged by ERO's, so row$(A) =$ row(B). Since B is in echelon form, its non-zero rows form a basis of row(B) and hence of row(A). So a basis of row(A) is given by $\{ u_1, u_2, u_3 \}$ where $u_1 = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \end{bmatrix}$ and $u_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

(12) Row-reduce: $\begin{bmatrix} 4 & 2 & 3 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 4 & 2 & 3 & 1 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & -2 \\ 0 & 2 & -1 & 1 & 0 & -4 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 1 & -1 & -2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 1 & -1 & -2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1/2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{bmatrix}$ so $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1/2 & -1 \\ -1 & 1 & 2 \end{bmatrix}$ is the inverse.