1. Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 3 & 2 \\ 4 & 5 & 1 \end{bmatrix}$. The rank of A is

(a) 2 (b) 3 (c) 0 (d) 4 (e) 1

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2. Let $\mathbf{P}_2 = \{a_0 + a_1 t + a_2 t^2\}$ where $\{a_0, a_1, a_2\}$ range over all real numbers, and let $T \colon \mathbf{P}_2 \to \mathbf{P}_2$ be a linear transformation dedifined by

$$T(a_0 + a_1 t + a_2 t^2) = a_1 + 9a_2 t$$

If $T\big(p(t)\big) = \lambda p(t)$ for some non-zero polynomial and some real number λ, then $p(t)$ is called an eigenvector corresponding to λ. The linear transformation T has an eigenvector:

(a) 100 (b) 0 (c) t^2 (d) $t + 9t^2$ (e) t

---

3. Let $\{\lambda_1, \lambda_2\}$ be two eigenvalues of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Then the product of two eigenvalues, $\lambda_1 \lambda_2$, is equal to

(a) 28 (b) -28 (c) 3 (d) 4 (e) 7

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4. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, Let $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

Let $\mathbf{x} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ and $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ Find $[\mathbf{x}]_{\mathbf{B}}$.

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (e) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

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5. The matrix $A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$ has a complex eigenvector:

(a) $\begin{bmatrix} 3 \\ 4i \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 3i \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ i \end{bmatrix}$

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6. Let $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. The eigenvalues of A are

(a) 1, 2, 3 (b) 1, ± 2 (c) 0, 1, 2 (d) 0, 1, -2 (e) 1, -2, -2

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7. Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and let $A = \begin{bmatrix} -2 & 99 \\ 0 & 1 \end{bmatrix}$. Compute the area of the images of S under the mapping $\mathbf{v} \mapsto A\mathbf{v}$.

(a) 3 (b) 99 (c) 2 (d) -2 (e) 5

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8. Let $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$. The complex eigenvalues of A are

(a) $1 \pm 3i$ (b) $3 \pm i$ (c) $\pm 3i$ (d) 4 (e) $\pm i$

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9. For what value(s) of h will $\mathbf{y}$ be in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, if $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 6 \\ 4 \\ h \end{bmatrix}$.

(a) 2 (b) 4 (c) 10 (d) 6 (e) 3

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10. Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathbf{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. If $P = \left[[\mathbf{b}_1]_\mathbf{C}, [\mathbf{b}_2]_\mathbf{C}\right]$ is the change-of-coordinates matrix from $\mathbf{B}$ to $\mathbf{C}$, then find P.

(a) $\frac{1}{4}\begin{bmatrix} 1 & -5 \\ 1 & 9 \end{bmatrix}$ (b) $\frac{1}{7}\begin{bmatrix} -5 & 3 \\ 4 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}$

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11. Find a matrix A such that $W = Col(A)$ where $W = \left\{ \begin{bmatrix} 9a - 8b \\ a + 2b \\ -5a \end{bmatrix} \right\}$ and $\{a, b\}$ range over all real numbers.

(a) $\begin{bmatrix} 9 & -8 \\ 1 & 2 \\ 0 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 9 & -8 \\ 1 & 2 \\ -5 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 9 & 1 & -5 \\ -8 & 2 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

---

12. Let $S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right\}$. Then the subset S

(a) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

(b) a basis of R^3

(c) spans R^3

(d) a linearly dependent subset

(e) a linearly independent subset

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13. Let $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$ Use Cramer's rule to solve $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

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14. Let $P_2 = \{a_0 + a_1 t + a_2 t^2\}$ where $\{a_0, a_1, a_2\}$ range over all real numbers, and let $T\colon P_2 \to P_1$ be a linear transformation given by $T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$. Suppose that $\mathbf{B} = \{1, t, t^2\}$ is a basis of P_2 and $\mathbf{C} = \{1, t\}$ is a basis of P_1.

(1) Find a matrix A such that $[T\mathbf{v}]_{\mathbf{C}} = A[\mathbf{x}]_{\mathbf{B}}$.

(2) Find $Nul(A)$ and $Col(A)$.

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15. Let $A = \frac{1}{5}\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find $\lim_{k\to\infty} A^k$. (Hint: Find the Diagonalization D of A use the formula $A^k = PD^kP^{-1}$.

Solutions

1.

Reduce to echelon form: $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 3 & 2 \\ 4 & 5 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & -1 \\ 4 & 5 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -7 \\ 0 & -7 & -7 \end{bmatrix}$ $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ There are two pivots so rank is 2

2.

With the basis $\mathcal{B} = \{1, t, t^2\}$, the matrix for T is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{bmatrix}$. Because the matrix is triangular, the characteristic polynomial is $-\lambda^3$ and hence the eigenvalue is 0 and the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so $100 = 100 + 0t + 0t^2$ is an eigenvector with eigenvalue 0.

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3. The product of the roots, with multiplicities, of any polynomial with leading coefficient 1 is $(-1)^n$ times the product of the roots where n is the degree: $(t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_n) = t^n + \cdots + (-1)^n(\lambda_1 \cdots \lambda_n)$. In this case $\lambda_1\lambda_2 = \det A$ so the answer is -28.

4.

We need to solve $A\mathbf{y} = \mathbf{x}$ where $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$. So reduce to reduced row echelon form $\begin{bmatrix} 1 & 1 & 5 \\ -1 & 2 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 5 \\ 0 & 3 & 9 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ Hence the solution is $[\mathbf{x}]_{\mathbf{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

5.

The characteristic equation is $\det \begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 + 9 = 16 - 8\lambda + \lambda^2 + 9 = \lambda^2 - 8\lambda + 25.$ The roots are $\lambda = \dfrac{8 \pm \sqrt{64 - 100}}{2} = 4 \pm 3i$. The eigenspace for $4 + 3i$ is the null space for $\begin{bmatrix} -3i & -3 \\ 3 & 3i \end{bmatrix}$ and an eigenvector is $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

6.

The characteristic equation is $\det\begin{vmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = (2-\lambda)\det\begin{vmatrix} -6-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 4\det\begin{vmatrix} -4 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3\det\begin{vmatrix} -4 & -6-\lambda \\ 3 & 3 \end{vmatrix} = (2-\lambda)\big((-6-\lambda)(1-\lambda)+9-\big(4(-4(1-\lambda)+9\big)+3\big(-12-3(-6-\lambda)\big)\big)$ and good luck to you!

OR

The sums of the five answers are all the different and the sum of the eigenvalues is the trace of the matrix $2+(-6)+1=-3$. Hence the only choice for the eigenvalues in the given answers is 1, -2, -2.

7.

The area is the area of S times the absolute value of the determinant of A, which is 2. The area of S is $\det[\mathbf{b}_1\mathbf{b}_2] = \det\begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} = 10-9=1$.

8.

The characteristic equation is $\lambda^2 - \mathrm{tr}A\lambda + \det A = \lambda^2 - 6\lambda + 10$ with roots $\lambda = \dfrac{6+\sqrt{36-40}}{2} = 3 \pm i$.

9.

We are looking for the h such that the equation $A\mathbf{x} = \mathbf{y}$ with $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 4 \end{bmatrix}$ so we need to reduce the following matrix to reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 1 & 1 & 4 \\ 3 & 3 & 4 & h \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -3 & -5 & -8 \\ 0 & -3 & -5 & h-18 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -3 & -5 & -8 \\ 0 & 0 & 0 & h-18+8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -3 & -5 & -8 \\ 0 & 0 & 0 & h-10 \end{bmatrix}.$$

To have a solution, $h-10=0$.

10.

We need to write $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix} = a\mathbf{c}_1 + b\mathbf{c}_2 = a\begin{bmatrix} 1 \\ -4 \end{bmatrix} + b\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ or solve $\begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}\mathbf{x} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$.

$\begin{bmatrix} 1 & 3 & -9 \\ -4 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -9 \\ 0 & 7 & -35 \end{bmatrix} \begin{bmatrix} 1 & 3 & -9 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -5 \end{bmatrix}$ so the first column of P is $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$.

$\begin{bmatrix} 1 & 3 & -5 \\ -4 & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -5 \\ 0 & 7 & -21 \end{bmatrix} \begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix}$ so the second column of P is $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

Hence $P = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$.

11.

$$\begin{bmatrix} 9a - 8b \\ a + 2b \\ -5a \end{bmatrix} = a \begin{bmatrix} 9 \\ 1 \\ -5 \end{bmatrix} + b \begin{bmatrix} -8 \\ 2 \\ 0 \end{bmatrix} \text{ so a matrix is } \begin{bmatrix} 9 & -8 \\ 1 & 2 \\ -5 & 0 \end{bmatrix}$$

12.

Since there are only two vectors they can not be a basis for R^3. They can not even span R^3. If the two are dependent, one is a multiple of the other and this is clearly not the case. Hence they are independent. The dot product is -1 so they are not orthogonal.

13.

To use Cramer's rule we need to compute four determinants:

$$\det \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = 2 \det \begin{vmatrix} -2 & 8 \\ -1 & 7 \end{vmatrix} - (-9) \det \begin{vmatrix} 1 & -4 \\ -1 & 7 \end{vmatrix} + 0 = -12 + 27 = 15$$

$$\det \begin{vmatrix} 0 & -4 & 2 \\ 1 & 8 & -9 \\ 0 & 7 & 0 \end{vmatrix} = 0 - 1 \det \begin{vmatrix} -4 & 2 \\ 7 & 0 \end{vmatrix} + 0 = 14$$

$$\det \begin{vmatrix} 1 & 0 & 2 \\ -2 & 1 & -9 \\ -1 & 0 & 0 \end{vmatrix} = -0 + 1 \det \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} - 0 = 2$$

and

$$\det \begin{vmatrix} 1 & -4 & 0 \\ -2 & 8 & 1 \\ -1 & 7 & 0 \end{vmatrix} = 0 - 1 \det \begin{vmatrix} 1 & -4 \\ -1 & 7 \end{vmatrix} + 0 = -3$$

Hence the solution vector is $\mathbf{x} = \begin{bmatrix} \frac{14}{15} \\ \frac{2}{15} \\ -\frac{3}{15} \end{bmatrix}$.

14.

A is a 2×3 matrix and the i^{th} column is found by working out T on the i^{th} basis element of $\mathbf{B}$.

Column 1: $T(1) = 0$ so $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Column 2: $T(t) = 1$ so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Column 3: $T(t^2) = 0$ so $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Hence $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

The matrix A is already in echelon form and there are two pivot positions. Hence $Col(A)$ has dimension 2 and so is all of P_1. The space $Nul(A)$ has dimension 1, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is clearly a non-zero vector in the null space so it is a basis for it.

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15. The characteristic equation for A is $\lambda^2 - \text{trace}(A)\lambda + \det(A) = \lambda^2 - \frac{8}{5}\lambda + \frac{15}{25} = (\lambda - \frac{3}{5})(\lambda - 1)$. Hence the eigenvalues are $\dfrac{3}{5}$ and 1.

The eigenspace for $\dfrac{3}{5}$ is the null space for $\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ -\frac{4}{5} & -\frac{2}{5} \end{bmatrix}$ and this is spanned by $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$.

The eigenspace for 1 is the null space for $\begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ -\frac{4}{5} & -\frac{4}{5} \end{bmatrix}$ and this is spanned by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

If $P = \begin{bmatrix} 2 & 1 \\ -4 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} \frac{3}{5} & 0 \\ 0 & 1 \end{bmatrix}$, then $A = PDP^{-1}$. Note $\lim_{k \to \infty} D^k = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so

$$\lim_{k \to \infty} A^k = \begin{bmatrix} 2 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & -1 \\ 4 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 4 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}.$$

1.	(•)	(b)	(c)	(d)	(e)
2.	(•)	(b)	(c)	(d)	(e)
3.	(a)	(•)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(•)	(e)
5.	(a)	(b)	(c)	(d)	(•)
6.	(a)	(b)	(c)	(d)	(•)
7.	(a)	(b)	(•)	(d)	(e)
8.	(a)	(•)	(c)	(d)	(e)
9.	(a)	(b)	(•)	(d)	(e)
10.	(a)	(b)	(c)	(•)	(e)
11.	(a)	(b)	(•)	(d)	(e)
12.	(a)	(b)	(c)	(d)	(•)