1.(5pts) Let $B = \begin{bmatrix} 3 & -1 & 1 \\ 5 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$. Compute $\det(B)$.

(a) -10 (b) 2 (c) 16 (d) 4 (e) -6

2.(5pts) Determine which statement is *not* always true for $n \times n$ matrices A and B.

(a) If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$.

(b) If a multiple of one row of A is added to another row to produce B then $\det(B) = \det(A)$.

(c) $\det(A + B) = \det(A) + \det(B)$

(d) If one row of A is multiplied by k to produce B, then $\det(B) = k \det(A)$.

(e) $\det(AB) = \det(A)\det(B)$

3.(5pts) Determine which set is *not* a subspace of $\mathbb{P}_n$, the space of polynomials of degree at most n.

(a) The set of $p(t)$ in $\mathbb{P}_n$ such that $p(0) = p(1)$.

(b) The set of $p(t)$ in $\mathbb{P}_n$ such that $p(-t) = -p(t)$.

(c) The set of $p(t)$ in $\mathbb{P}_n$ such that $p(0) = 1$.

(d) The set of $p(t)$ in $\mathbb{P}_n$ of the form $p(t) = at + at^2$ where a is in $\mathbb{R}$.

(e) The set of $p(t)$ in $\mathbb{P}_n$ of the form $p(t) = t + at$ where a is in $\mathbb{R}$.

4.(5pts) Determine which statement is always true.

(a) The rows of a matrix A containing the pivot positions form a basis for Row A.

(b) If $H = \text{Span}\,\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for H.

(c) A linearly independent set in a subspace H is a basis for H.

(d) If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis of V.

(e) A basis is a spanning set that is as large as possible.

5.(5pts) The set $\mathcal{B} = \{1-t, t-t^2, t^2-t^3, t^3\}$ is a basis for $\mathbb{P}_3$, the space of polynomials of degree at most 3. Find the coordinate vector of $p(t) = 4-t+5t^2-2t^3$ relative to the basis $\mathcal{B}$.

(a) $\begin{bmatrix} 4 \\ -1 \\ 5 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 3 \\ 8 \\ 6 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 4 \\ 3 \\ -2 \end{bmatrix}$ (d) $\begin{bmatrix} 4 \\ 3 \\ 4 \\ 3 \end{bmatrix}$ (e) $\begin{bmatrix} 3 \\ 8 \\ 9 \\ 4 \end{bmatrix}$

6.(5pts) Let $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 2 & 6 \\ 5 & 10 & 3 & 9 \\ 6 & 12 & 4 & 12 \end{bmatrix}$. Find a basis for $\mathrm{Row}\,(A)$.

(a) $(1,2,0,0)$, $(0,0,1,3)$

(b) $(1,2,1,3)$, $(2,4,2,6)$

(c) $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$

(d) $(1,0,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$

(e) $(1,2,5,6)$, $(1,2,3,4)$

7.(5pts) Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 8 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$. Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

(a) $\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 7 \\ -1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} -.3 & .7 \\ -.1 & .1 \end{bmatrix}$

8.(5pts) The eigenvalues of a certain 3×3 matrix A are -2, 0, and 1. Determine which statement is true.

(a) $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbb{R}^3$.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) $\mathrm{rank}\,(A) = 3$

(d) A is not diagonalizable.

(e) $\det(A) = -2$

9.(5pts) Determine which vector is an eigenvector of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

10.(5pts) Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^2$. Find the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \to A\mathbf{x}$.

(a) $\begin{bmatrix} -6 & -5 \\ 18 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} 15 & -6 \\ -6 & 10 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ (e) $\begin{bmatrix} 9 & 16 \\ -5 & -9 \end{bmatrix}$

11.(10pts) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$. Use cofactors to compute $\operatorname{adj}(A)$.

12.(10pts) Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

13.(10pts) Find the complex eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$.
The transformation $\mathbf{x} \to A\mathbf{x}$ is the composition of a rotation and a scaling.
Give the angle θ of the rotation and the scale factor r.

Solutions

1.

$$0\begin{vmatrix}-1 & 1\\ 2 & 4\end{vmatrix} - 1\begin{vmatrix}3 & 1\\ 5 & 4\end{vmatrix} + 1\begin{vmatrix}3 & -1\\ 5 & 2\end{vmatrix} = (-1)(12-5) + 1(6-(-5)) = 11 - 7 = 4$$

---

2. All but $\det(A+B) = \det(A) + \det(B)$ are standard rules for manipulating determinants. Note $\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \neq \det\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \det\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$

---

3. Since $1+1 \neq 1$, the set of $p(t)$ in $\mathbb{P}_n$ such that $p(0) = 1$ is not closed under addition. The others are. Similarly, all but the set of $p(t)$ in $\mathbb{P}_n$ such that $p(0) = 1$ are closed under scalar multiplication.

---

4. (d) is a theorem in the book and so is always true. The others are approximates statements of results from the book but none of the others is quite right.

---

5. $p(t) = 4(1-t)+3t+5t^2-2t^3 = 4(1-t)+3(t-t^2)+8t^2-2t^3 = 4(1-t)+3(t-t^2)+8(t^2-t^3)+6t^3$ so $\begin{bmatrix}4\\3\\8\\6\end{bmatrix}$.

---

6. Reduce to echelon form. $\begin{bmatrix}1 & 2 & 1 & 3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 3\\ 0 & 0 & -2 & -6\end{bmatrix}$ $\begin{bmatrix}1 & 2 & 1 & 3\\ 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ The notation for the column vectors in the book uses parentheses so (a) is correct.

---

7. Write $\begin{bmatrix}2\\-4\end{bmatrix} = c_1\begin{bmatrix}1\\-1\end{bmatrix} + c_2\begin{bmatrix}-3\\5\end{bmatrix}$. Either solve or by inspection $c_1 = c_2 = -1$ so first column is $\begin{bmatrix}-1\\-1\end{bmatrix}$. (b) is the only choice with this column but if you want ot work out the second column solve $\begin{bmatrix}-2\\8\end{bmatrix} = c_1\begin{bmatrix}1\\-1\end{bmatrix} + c_2\begin{bmatrix}-3\\5\end{bmatrix}$ to get $c_1 = 3$, $c_2 = 7$.

8. Since 0 is an eigenvalue, the null space has dimension at least 1 and since there are two more distinct eigenvalues, all three eigenspaces have dimension exactly 1. Hence the rank is 2 as is the dimension of the column space. Therefore (a) is true. To see the others note that the determinant is $(-2) \cdot 0 \cdot 1 = 0$. The matrix is diagonalizable since there are three distinct eigenvalues a

---

9. Since two rows are equal, the matrix has a non-trivial null space and either by row reduction or inspection $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue 0.

This is not a good multiple choice problem since the quickest way to get the answer is to evaluate the matrix on the answers. To work out the characteristic polynomial, find the root(s) and then the eigenvectors is a waste a time.

10.
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 9 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + (-5) \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ so the first column is $\begin{bmatrix} 9 \\ -5 \end{bmatrix}$.
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 16 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + (-9) \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ so the first column is $\begin{bmatrix} 16 \\ -9 \end{bmatrix}$.

The $\mathcal{B}$ matrix is $\begin{bmatrix} 9 & 16 \\ -5 & -9 \end{bmatrix}$.

11.

There are 9 cofactors

$$C_{1,1} = (-1)^{1+1} \det \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} = -3$$

$$C_{1,2} = (-1)^{1+2} \det \begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{1,3} = (-1)^{1+3} \det \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2$$

$$C_{2,1} = (-1)^{2+1} \det \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$C_{2,2} = (-1)^{2+2} \det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{2,3} = (-1)^{2+3} \det \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

$$C_{3,1} = (-1)^{3+1} \det \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3$$

$C_{3,2} = (-1)^{3+2} \det \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = -3$

$C_{3,3} = (-1)^{3+3} \det \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2$

So the i, $j^{|rmth}$ entry in the adjunct of A is $C_{j,i}$ so the adjunct is $\begin{bmatrix} -3 & 0 & 3 \\ 0 & 0 & -3 \\ 2 & -1 & -2 \end{bmatrix}$

12.

The characteristic polynomial is $\det \begin{vmatrix} 1-\lambda & 2 & 4 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & 4-\lambda \end{vmatrix}$ and since this matrix is upper triangular, the determinant is $(1-\lambda)(2-\lambda)(4-\lambda)$ so the eigenvalues are 1, 2 and 4 and all eigenspaces are 1 dimensional.

The eigenspace for 1 is the null space for $\begin{bmatrix} 0 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$. One vector in this null space is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ so it is a basis.

The eigenspace for 2 is the null space for $\begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix}$. One vector in this null space is $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ so it is a basis.

The eigenspace for 4 is the null space for $\begin{bmatrix} -3 & 2 & 4 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$. One vector in this null space is $\begin{bmatrix} 8 \\ 6 \\ 3 \end{bmatrix}$ so it is a basis.

Hence $P = \begin{bmatrix} 1 & 2 & 8 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. You need do no further work since the results in the book tell you that these choices will work out.

13.

The characteristic equation is $\det \begin{vmatrix} 2-\lambda & 2 \\ -2 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 4 = \lambda^2 - 4\lambda + 8$. The roots are $\lambda = \dfrac{4 \pm \sqrt{16-32}}{2} = 2 \pm 2i$.

The eigenspace for $2+2i$ is the null space for $\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}$ and $\begin{bmatrix} i \\ -1 \end{bmatrix}$ is a basis for the eigenspace.

The eigenspace for the conjugate $2 - 2i$ is the null space for $\begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix}$ and $\begin{bmatrix} -i \\ -1 \end{bmatrix}$ is a basis for the eigenspace.

To work out the angle and the scaling, results in the book say that the scaling is by $|\lambda| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ and the angle of rotation counterclockwise is $-\theta$ where $2 + 2i = \sqrt{2}(\cos\theta + i\sin\theta)$. Hence θ is 45 degrees or $\frac{\pi}{4}$ radians.

1.	(a)	(b)	(c)	(•)	(e)
2.	(a)	(b)	(•)	(d)	(e)
3.	(a)	(b)	(•)	(d)	(e)
4.	(a)	(b)	(c)	(•)	(e)
5.	(a)	(•)	(c)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)
7.	(a)	(•)	(c)	(d)	(e)
8.	(•)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(•)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(•)